Mathematica Slovaca

Vladimír Haluška

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Mathematica Slovaca, Vol. 41 (1991), No. 1, 89--99

Persistent URL: http://dml.cz/dmlcz/128591

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ON THE EXISTENCE OF A SOLUTION OF A VECTOR PERIODIC BOUNDARY VALUE PROBLEM

VLADIMÍR HALUŠKA

ABSTRACT. A nonlinear vector periodic boundary value problem for the third order is studied. By means of the estimates for the derivatives of scalar functions with respect to the Green function the existence of a solution for that problem is established.

In the paper a nonlinear vector periodic boudary value problem for the third order system is studied. The methods of the papers [5], [6] are used. Existence theorems for that problem are obtained by means of the estimates for derivatives of scalar functions which are given in the paper [7] and by introducing an admissible system of functions with respect to the Green function. The obtained results extend some theorems proved in [7].

In the paper the following vector boudary value problem will be considered

$$x''' + x + F(t, x, x', x'') = e(t) \tag{1}$$

$$x(0) = x\left(\frac{2\pi}{\sqrt{3}}\right), \; x'(0) = x'\left(\frac{2\pi}{\sqrt{3}}\right), \; x''(0) = x''\left(\frac{2\pi}{\sqrt{3}}\right) \tag{2}$$

where

$$F \in C(D, \mathbf{R}^d), \; e \in C\left(\left[0, \frac{2\pi}{\sqrt{3}}\right], \mathbf{R}^d\right), \; d \geqq 1$$

and

$$D = \left[0, \frac{2\pi}{\sqrt{3}}\right] \times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d.$$

The scalar case ($d = 1$) has been studied in the paper [7].

AMS Subject Classification (1985): Primary 34B15 Secondary 34B27
Key words: Banach space, Green function, Continuous operator

Preliminaries

If $x = (x_1, \dots, x_d)^T$ is a column vector, then we denote $|x| = (|x_1|, \dots, |x_i|)^T$. A partial ordering in $\mathbf{R}^d$ will be introduced by the relation:

If $x = (x_1, \dots, x_d)^T$, $y = (y_1, \dots, y_d)^T$ belong to $\mathbf{R}^d$, then $x \leqq y$ iff $x_j \leqq y_j$ for $j = 1, \dots, d$. Further, $u_d = (1, \dots, 1)^T \in \mathbf{R}^d$.

The set of all real $d \times d$ matrices will be denoted as $M_{d \times d}$. Similarly as in the case of vectors, if $L = (l_{ij})$, then $|L| = (|l_{ij}|)$, $i, j = 1, \dots, d$. Further, $L \leqq \bar{L}$ iff $l_{ij} \leqq \bar{l}_{ij}$ for $i, j = 1, \dots, d$, and $L = (l_{ij})$, $\bar{L} = (\bar{l}_{ij})$.
$U_d(O_d)$ will mean the matrix from $M_{d \times d}$, all elements of which are 1 (0). E_α will denote the unit matrix. As usual, the spectral radius $\varrho\ (L)$ of the matrix $L \in M_{d \times d}$ means $\max |\lambda_i|$, where $|\lambda_i|$ are all eigenvalues of L.

Denote by $G = G(t, s)$ the Green function of the corresponding homogeneous scalar problem

$$y''' + y = 0 \tag{1'}$$

$$y(0) = y\left(\frac{2\pi}{\sqrt{3}}\right),\ y'(0) = y'\left(\frac{2\pi}{\sqrt{3}}\right),\ y''(0) = y''\left(\frac{2\pi}{\sqrt{3}}\right). \tag{2'}$$

Then the folowing result holds (Lemma 7 in [7], p. 352)

Lemma 1.

$$\max_{0 \leqq t \leqq \frac{2\pi}{\sqrt{3}}} \int_0^{\frac{2\pi}{\sqrt{3}}} |G(t, s)|\, \mathrm{d}s = K_0 = 1$$

$$\max_{0 \leqq t \leqq \frac{2\pi}{\sqrt{3}}} \int_0^{\frac{2\pi}{\sqrt{3}}} \left|\frac{\partial G(t, s)}{\partial t}\right| \mathrm{d}s = K_1 \leqq 1.04 \cdot \frac{2\pi}{\sqrt{3}} = 3{,}77270 \tag{3}$$

$$\max_{0 \leqq t \leqq \frac{2\pi}{\sqrt{3}}} \int_0^{\frac{2\pi}{\sqrt{3}}} \left|\frac{\partial^2 G(t, s)}{\partial t^2}\right| \mathrm{d}s = K_2 \leqq 1.04 \cdot \frac{2\pi}{\sqrt{3}} = 3{,}77270$$

Similarly as in [5] we shall use the concept of a generalized norm. Let us mention the fundamental properties of the generalized normed space. If $\mathbf{E}$ is a real vector space, then the generalized norm for $\mathbf{E}$ is a mapping $\|\,.\,\|_G\colon \mathbf{E} \to \mathbf{R}^d$ denoted by

$$\|x\|_G = (a_1(x), \dots, a_d(x))^T.$$

such that

a) $\| x \|_G = 0$ i.e. $a_j(x) = 0$ for $j = 1 \dots d$, $x \in \mathbf{E}$;
b) $\| x \|_G = 0$ iff $x = 0$,
c) $\| cx \| = c \cdot \| x \|_G$, $c \in \mathbf{R}$, $x \in \mathbf{E}$;
d) $\| x + y \|_G \leqq \| x \|_G + \| y \|_G$, $x, y \in \mathbf{E}$.

Then the couple $(\mathbf{E}, \| . \|_G)$ is called a generalized normed space. The Banach fixed point theorem (Lemma 1 in [5], p. 78) has the following formulation in the generalized Banach space (a complete generalized normed space).

Lemma 2. *Let* $(\mathbf{E}, \| . \|_G)$ *be a generalized Banach space and let* T: $\mathbf{E} \to \mathbf{E}$ *be such that for all* $x, y \in \mathbf{E}$ *and for some positive integer* p

$$\| T_x^p - T_y^p \|_G \leqq L \| x - y \|_G ,$$

where $L \in M_{d \times d}$ *is a nonnegative matrix with* $\varrho(L) < 1$ *and* T^p *is the p-th iterate of* T. *Then* T *has a unique fixed point.*

Admisible system of functions and associated system of constants

Let G be the Green function of the scalar problem (1′), (2). Then the functions

$$\Phi_j(t) = \int_0^{2\pi/\sqrt{3}} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \mathrm{d}s, \; 0 \leqq t \leqq 2\pi/\sqrt{3}, j = 0, 1, 2 \tag{4}$$

are continuous in $[0, 2\pi/\sqrt{3}]$.

Definition 1. *The system of nonnegative continuous scalar functions* φ_j *in* $[0, 2\pi/\sqrt{3}]$, $j = 0, 1, 2$, *is called admissible* (*with repect to the Green function* G: *if there exist positive constants* k_j, $j = 0, 2, 1$, *such that*

$$\Phi_j(t) \leqq k_j \varphi_j(t), \; 0 \leqq t \leqq 2\pi/\sqrt{3}, j = 0, 1, 2 . \tag{5}$$

If such a situation arises, then in view of the boudedness of the functions φ_j, $j = 0, 1, 2$, there exist positive constatns $\bar{k}_{l,j}$, $l, j = 0, 1, 2$, such that

$$\int_0^{2\pi/\sqrt{3}} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \varphi_l(s) \, \mathrm{d}s \leqq \bar{k}_{l,j} \cdot \varphi_j(t), \; 0 \leqq t \leqq 2\pi/\sqrt{3}, \; l, j = 0, 1, 2 . \tag{6}$$

Let $k_{l,j} = \inf \bar{k}_{l,j}$, $l, j = 0, 1, 2$.

Denote

$$\varkappa_l = \max (k_{l0}, k_{l1}, k_{l2}), \; l = 0, 1, 2 . \tag{7}$$

Hence

$$\int_0^{2\pi/\sqrt{3}} \left|\frac{\partial^j G(t, s)}{\partial t^j}\right| \varphi_l(s)\,\mathrm{d}s = \varkappa_l \varphi_j(t),\ 0 \leqq t \leqq 2\pi/\sqrt{3},\ l, j = 0, 1, 2. \quad (8)$$

By the definition of $\varkappa_l$, for a constant $\bar{\varkappa}_l < \varkappa_l$, the inequality (8) cannot hold for all $t \in [0, 2\pi/\sqrt{3}]$, and $j = 0, 1, 2$.

Definition 2. *The system of the smallest nonnegative constants* $\varkappa_j$, $j = 0, 1, 2$, *such that* (8) *are true for all* $t \in [0, 2\pi/\sqrt{3}]$, $l, j = 0, 1, 2$, *will be called the associated system of constants to the admissible system* φ_j, $j = 0, 1, 2$.

By means of the last two notions we shall prove the following theorem.

Theorem 1. *Let* φ_j, $j = 0, 1, 2$, *be an admissible system and* $\varkappa_j$, $j = 0, 1, 2$, *the associated system of constants to that system. Let the function F satisfy the Lipschitz condition*

$$|F(t, u_0, u_1, u_2) - F(t, v_0, v_1, v_2)| \leqq \sum_{k=0}^{2} L_k |u_k - - v_k| \quad (9)$$

with nonnegative matrices $L_k \in M_{d \times d}$, $k = 0, 1, 2$. *Then there exists a unique solution to* (1), (2) *provided the spectral radius*

$$\varrho\left(\sum_{k=0}^{2} \varkappa_k L_k\right) < 1 \quad (10)$$

Proof. The problem (1), (2) is eqivalent to the equation

$$x(t) = \int_0^{2\pi/\sqrt{3}} G(t, s) e(s)\,\mathrm{d}s - \int_0^{2\pi/\sqrt{3}} G(t, s) F[s, x(s), x'(s), x''(s)]\,\mathrm{d}s =$$

$$= w(t) - \int_0^{2\pi/\sqrt{3}} G(t, s) F[s, x(s), x'(s), x''(s)]\,\mathrm{d}s,\ 0 \leqq t \leqq 2\pi/\sqrt{3}.$$

Now the define the operator T on $S = C^2([0, 2\pi/\sqrt{3}, \mathbf{R}^d)$ by

$$Tx(t) = w(t) - \int_0^{2\pi/\sqrt{3}} G(t, s) F[s, x(s), c'(s), x''(s)]\,\mathrm{d}s,\ 0 \leqq t \leqq 2\pi/\sqrt{3} \quad (11)$$

Clearly $T\colon S \to S$.

The space S will be provided by the generalized norm

$$\|x\| = \max\left(\max_{0 \leqq t \leqq 2\pi/\sqrt{3}} |x(t)|,\ \max_{0 \leqq t \leqq 2\pi/\sqrt{3}} |x'(t)|,\ \max_{0 \leqq t \leqq 2\pi/\sqrt{3}} |x''(t)|\right)$$

whereby $\max(x_1, x_2, x_3)$ for $x_1, x_2, x_3 \in {}^d$ is defined componentwise, i.e.

if $x_i = (x_{1i}, \ldots, x_{di})^{\mathrm{T}}$, $i = 1, 2, 3$, then $\max(x_1, x_2, x_3) =$

$$= \left(\max_{i=1,2,3} (x_{1,i}, \ldots, \max_{i=1,2,3} x_{d,i} \right)^{\mathrm{T}}$$

$(S, \| . \|)$ is a generalized Banach space.
Denote

$$K = \max_{j=0,1,2} k_j, \tag{12}$$

where $k_j, j = 0, 1, 2$ are arbitrary but fixed numbers satisfying (5).

Let $u, v \in S$ and let $j \in \{0, 1, 2\}$. Then, with respect to (11), (9), (5) and (12) we obtain the following inequalites. First

$$|T^{(j)}(u)(t) - T^{(j)}(v)(t)| \leqq$$
$$\leqq \int_0^{2\pi/\sqrt{3}} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \left(\sum_{k=0}^{2} L_k |v^{(k)}(s) - v^{(k)}(s)| \right) \mathrm{d}s \leqq$$
$$\leqq K\varphi_j(t) \sum_{k=0}^{2} L_k \|u - v\|, \; 0 \leqq t \leqq 2\pi/\sqrt{3}.$$

Suppose that for a natural p the inequality

$$|(T^p)^{(j)}(u)(t) - (T^p)^{(j)}(v)(t)| \leqq K\varphi_j(t) \left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p-1} \sum_{k=0}^{2} L_k \|u - v\|,$$
$$0 \leqq t \leqq 2\pi/\sqrt{3}, \tag{13}$$

is true. Then using (11), (9), (13), (6), (7) we obtain the inequalities

$$|(T^{p+1})^{(j)}(u)(t) - (T^{p+1})^{(j)}(v)(t)| \leqq$$
$$\leqq \int_0^{2\pi/\sqrt{3}} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \left(\sum_{k=0}^{2} L_k K\varphi_k(s) \right) \left[\left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p-1} \cdot \sum_{k=0}^{2} L_k \|u - v\| \right] \mathrm{d}s =$$
$$= K\varphi_j(t) \left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p} \sum_{k=0}^{2} L_k \|u - v\|, \; 0 \leqq t \leqq 2\pi/\sqrt{3}.$$

Hence, by induction, we get that (13) is true for all natural p. The inequality (13) implies that

$$\| T^p(u) - T^p(v) \| \leqq K \left[\max_{j=0,1,2} \left(\max_{0 \leqq t \leqq 2\pi/\sqrt{3}} (t) \right) \right] \left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p-1} \cdot$$
$$\cdot \sum_{k=0}^{2} L_k \|u - v\|.$$

By (10), $\lim_{p \to \infty} \left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p-1} = O_d$ and hence there exists a p_0 such that for all $p \geqq p_0$

$$\varrho \left(K \left[\max_{j=0,1,2} \left(\max_{0 \leqq t \leqq 2\pi/\sqrt{3}} \varphi_j(t) \right) \right] \left(\sum_{k=0}^{2} \varkappa_k L_k \right)^{p-1} \sum_{k=0}^{2} L_k \right) < 1 .$$

Lemma 2 then implies that the operator T has a unique fixed point S which gives the statement of the theorem.

Corollary 1. *Let the function f satisfy the Lipschitz condition* (9) *with non-negative matrices $L_k \in M_{d \times d}$, $k = 0, 1, 2$. Let*

$$K_k = \max_{0 \leqq t \leqq 2\pi/\sqrt{3}} \int_0^{2\pi/\sqrt{3}} \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| \mathrm{d}s, \ k = 0, 1, 2 \tag{14}$$

Then there exists a unique solution to (1), (2) *provided*

$$\varrho \left(\sum_{k=0}^{2} K_k L_k \right) < 1 . \tag{15}$$

Proof. Clearly the functions Φ_j, $j = 0, 1, 2$, given by (4), form an admissible system of functions. As

$$\int_0^{2\pi\sqrt{3}} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \Phi_l(s) \, \mathrm{d}s \leqq K_l \varphi_j(t), \ 0 \leqq t \leqq 2\pi/\sqrt{3}, \ l, j = 0, 1, 2$$

the associated system of constants $\varkappa_l$, $l = 0, 1, 2$, to that admissible system of functions fulfils the relation

$$\varkappa_l \leqq K_l, \ l = 0, 1, 2 .$$

Thus $\varrho \left(\sum_{k=0}^{2} \varkappa_k L_k \right) = \varrho \left(\sum_{k=0}^{2} K_k L_k \right) < 1$ and, by Theorem 1, the statement of the corollary follows.

Remark. Corollary 1 extends Theorem 2 in [7], p. 356, to the vector periodic boudary value problem.

Optimal values of the associated system of constants

We have seen that for each admissible system of functions $(\varphi_0, \varphi_1, \varphi_2)$ there exists a unique associated system of constants $(\varkappa_0, \varkappa_1, \varkappa_2)$. We shall show that for each $k \in \{0, 1, 2\}$ there exists the smallest value $\varkappa_k$. To that aim we consider the Banach space $\mathbf{E} = C([0, 2\pi/\sqrt{s}], \mathbf{R})$ with the sup-norm, partially ordered by the relation $x \leqq y$ iff $x(t) \leqq y(t)$ for all $t \in [0, 2\pi/\sqrt{3}]$.

Then $(\mathbf{E}, \leqq)$ is an ordered Banach space with positive cone $\mathbf{P} = \{x \to \mathbf{E}: x(t) \geqq 0, 0 \leqq t \leqq 2\pi/\sqrt{3}\}$. $\mathbf{P}$ is normal, i.e. every order interval $[x, y] = \{z \in \mathbf{E}: x \leqq z \leqq y\}$ is bounded in the norm, and $\mathbf{P}$ is generating, i. e. $\mathbf{E} = \mathbf{P} - \mathbf{P}$.

Let $k \in \{0, 1, 2\}$ and let $G = G(t, s)$ be the Green function of the scalar problem (1′), (2). Define the operator

A_k: $\mathbf{E} \to \mathbf{E}$ by

$$A_k(x(t) = \int_0^{2\pi/\sqrt{3}} \left|\frac{\partial^k G(t, s)}{\partial t^k}\right| x(s)\, \mathrm{d}s, \quad 0 \leqq t \leqq 2\pi/\sqrt{3}. \tag{16}$$

A_k is a positive and completely continuous operator. If $\varphi_0, \varphi_1, \varphi_2$, is an admissible system of functions, then for any $x \in \mathbf{P}$, $x(t) \neq 0$ in $[0, 2\pi/\sqrt{3}]$ there exists a constant $C = C(x) > 0$ such that

$$A_k x(t) = \int_0^{2\pi/\sqrt{3}} \left|\frac{\partial^k G(t, s)}{\partial t^k}\right| x(s)\, \mathrm{d}s \leqq \max_{0 \leqq t \leqq 2\pi/\sqrt{3}} x(t) \cdot \Phi_k(t) \leqq$$

$$\leqq \left(\max_{0 \leqq t \leqq 2\pi/\sqrt{3}} x(t)\right) \cdot k_k \varrho_k(t) = C(x)\varphi_k(t), \quad 0 \leqq t \leqq 2\pi/\sqrt{3}.$$

Hence A_k is φ_k — bounded from above as well as Φ_k — bounded from above ([3], p. 78). Since the operator A_k, the space $\mathbf{P}$, satisfy all asumptions of Lemma 2 in [5], pp 83—84, by that lemma we get the following result.

Lemma 3. *Let $\varphi_0, \varphi_1, \varphi_2$ be an admissible system of functions for the Green function G and let $(\varkappa_0, \varkappa_1, \varkappa_2)$ be the associated system of constants to that admissible system.*

Let A_k be the operator given by (16). *Then*

$$\varkappa_j \geqq \varrho(A_j), \quad j = 0, 1, 2,$$

where $\varrho(A_j)$, is the spectral radius of the operator A_j, $j = 0, 1, 2$.

With respect to this lema, the values $\varkappa_j = \varrho(A_j)$, $j = 0, 1, 2$, are optimal, provided that they are from the associated system of constants. We shall show that for each $k \in \{0, 1, 2, \}$ there is an admissible system of functions such that the constant $\varkappa_k$ from the associated system of constants is equal to $\varrho(A_k)$. First we apply a reslult from [4] p. 72.

Definition 3 ([4], p. 72.) *A sequence of points $(t_1, t_2), (t_2, t_3), \ldots, (t_{n-1}, t_n), (t_n, t_1)$, where $t_1, \ldots t_n$ are inner points of Q, will be called a path of regularity for the function $H = H(t, s): Q \times Q \to \mathbf{R}$ in the operator A, defined by*

$$Ax(t) = \int H(t, s) x(s)\, \mathrm{d}s, \tag{17}$$

where Q is the closure of a bounded region in $\mathbf{R}^l$, if the kernel H is continuous and different from zero at each point of that sequence.
E.g. if the kernel $H(t)$ is continuous and different from 0 at a point (t_0, t_0) and t_0 is an inner point of Q, or if $H(t, s)$ is continuous and different from 0 at two inner points (t_0, t_0), (s_0, s_0) of Q.

The meaning of the path of regularity is given by the following lemma. (Theorem 9.9 in [4] p. 72).

Lemma 4. *If the operator* (17) *is completely continuous, if the kernel $H(t, s)$ is nonnegative and there exists a path of regularity for that kernel, then the operator A has a nonnegative eigenfunction which corresponds to the positive eigenvalue $\varrho(A)$.*

Lemma 5. *Let $k \in \{0, 1, 2\}$. Then the operator defined by* (16) *has a path regularity. More exactly, each point (t_0, t_0) with $0 < t_0 < \frac{2\pi}{\sqrt{3}}$ is a path of regularity for the operators A_0, A_1 and the points $\left(t_0, t_0 - \frac{\pi}{\sqrt{3}}\right)$, $\left(t_0 - \frac{\pi}{\sqrt{3}}, t_0\right)$ for each t_0 such that $\frac{\pi}{\sqrt{3}} < t_0 < \frac{2\pi}{\sqrt{3}}$ form a path of regularity for the operator A_2.*

Proof. By Lemma 3 and the calculations on p. 351 in [7],

$$|G(t_0, t_0)| = \frac{\alpha}{3(1-\alpha)} + \frac{\beta}{3(1-\alpha)} + \frac{\beta}{3(1+\beta)} > 0,$$

$$\left|\frac{\partial G(t_0, t_0)}{\partial T}\right| = \frac{\alpha}{3(1-\alpha)} + \frac{\beta}{3(1+\beta)} \emptyset$$

$$\text{for } 0 < t_0 < \frac{2\pi}{\sqrt{3}} \text{ and } \left|\frac{\partial^2 G\left(t_0, t_0 - \frac{\pi}{\sqrt{3}}\right)}{\partial t^2}\right| = \frac{1}{3(1-\alpha)} e^{-\frac{\pi}{\sqrt{3}}} > 0,$$

$$\left|\frac{\partial^2 G\left(t_0 - \frac{\pi}{\sqrt{3}}, t_0\right)}{\partial t^2}\right| = \frac{\alpha}{3(1-\alpha)} e^{\frac{\pi}{\sqrt{3}}} > 0 \text{ for } \frac{\pi}{3} < t_0 < \frac{2\pi}{\sqrt{3}},$$

$$\text{since } 0 < \alpha = e^{-\frac{2\pi}{\sqrt{3}}} < 1,\ \beta = e^{\frac{\pi}{\sqrt{3}}} > 0 \text{ ([7], p. 346)}.$$

In view of the last lemma. Lemma 4 guarantees that for each $k \in \{0, 1, 2\}$ the operator A_k defined by (16) has a nonnegative eigenfunction φ_k which corresponds to the positive eigenvalue $\lambda_k > 0$. By Lemma 4 and Lemma 5, we get the following lemma.

Lemma 6. *Let $k \in \{0, 1, 2\}$. Then the operator A_k defined by* (16) *has a positive spectral radius $\varrho(A_k)$ and thus, $\varrho(A_k)$ is an eigenvalue of A_k with an eigenvector in* $\mathbf{P}$.

In the next lemma we need the following definition.

Definition 3. ([3], p. 77) *The position operator $A: \mathbf{E} \to \mathbf{e}$ is called u_0 — positive if $u_0 \in \mathbf{P}$, $u_0 \neq$ and for each $x \in \mathbf{P}$ there exist positive numbers $\alpha(x)$, $\beta(x)$ such that*

$$\alpha(x)\, u_0 \leqq Ax \leqq \beta(x)\, u_0 .$$

Lemma 7. (a) *The operator A_k given by* (16) *is Φ_k — positive where Φ_j have been defined by* (4).

(b) *There exists a unique, up to a multiplicative positive constant, eigenfunction $\tilde{\varphi}_k$ of A_k belonging to P. $\tilde{\varphi}_k$ corresponds to $\varrho(A_k)$ and A_k is $\tilde{\varphi}_k$ positive.*

(c) $\tilde{\varphi}_k(t) > 0$ *on* $[0, 2\pi/\sqrt{3}]$.

Proof. (a). We have already seen that A_k is Φ_k — bounded from above. To prove that this operator is Φ_k — bounded from below, that is to show that for each nonzero element $x \in \mathbf{P}$ there exists a constant $\alpha(x) > 0$ such that

$$\alpha(x)\Phi_k \leqq A_k x, \tag{18}$$

it suffices to guarantee that for such an element x. $A_k x(t) > 0$ in $\left(0, \dfrac{2\pi}{\sqrt{3}}\right)$. This will be, true if $\dfrac{\partial^k G(t_0, s)}{\partial t^k} \equiv 0$ does not hold in s on any subinteral $[a, b] \subset [0, 2\pi/\sqrt{3}]$ for any $t_0 \in [a, b]$.

By Lemma 4 in [7], p. 349, $G(t, s) > 0$ on $[0, 2\pi/\sqrt{3}] \times [0, 2\pi/\sqrt{3}]$. As to the function $\dfrac{\partial G}{\partial t}$, we have from the proof of Lemma 6, ([7], p. 351) that for $0 \leqq \leqq t_0 < s < 2\pi/\sqrt{3}$.

$$\frac{\partial G}{\partial t}(t_0, s) = E\,\mathrm{e}^s + F\,\mathrm{e}^{-\frac{s}{2}}\left[-\cos\frac{\sqrt{3}}{2}(s - t_0) + \sqrt{3}\sin\frac{\sqrt{3}}{2}(s - t_0)\right] = \tilde{A}(s),$$

where $E = \dfrac{-\alpha\,\mathrm{e}^{-t_0}}{3(1-\alpha)}, \quad F = \dfrac{\beta\,\mathrm{e}^{\frac{t_0}{2}}}{3(1+\beta)}$.

If $\dfrac{\partial G}{\partial t}(t_0, s) \equiv 0$ on an interval $[a, b] \subset [0, 2\pi/\sqrt{3}]$, then the same holds about the function $\tilde{A}(s)$, which is

$$\tilde{A}'(s) = E\,\mathrm{e}^s + 2F\,\mathrm{e}^{-\frac{s}{2}}\cos\frac{\sqrt{3}}{2}(s - t_0),$$

and hence

$$\tilde{A}'(s) - \tilde{A}(s) = F\,\mathrm{e}^{-\frac{s}{2}}\left[3\cos\frac{\sqrt{3}}{2}(s-t_0) - \sqrt{3}\sin\frac{\sqrt{3}}{2}(s-t_0)\right].$$

The last expression cannot be identically zero on any subinterval in $[0, 2\pi/\sqrt{3})$. Therefore the function A has the same property. Similarly we obtain tha $\frac{\partial G}{\partial t}(t_0, s) \equiv 0$ has the same property. Similarly we obtain that $\frac{\partial G}{\partial T}(t_0, s) \equiv 0$ cannot hold on any subinterval $(a, b) \subset [0, 2\pi/\sqrt{3}]$ in the case when $0 < s < t_0 \leqq \leqq 2\pi/\sqrt{3}$. By analogous considerations we get that $\frac{\partial^2 G}{\partial t^2}(t_0, s) \equiv 0$ does not hold on any subinterval $[a, b]$ of $[0, 2\pi/\sqrt{3}]$ for any $t_0 \in [0, 2\pi/\sqrt{3}]$.
This implies that $A_k x(t) > 0$ in $[0, 2\pi/\sqrt{3}]$ for any nonzero element $x \in \mathbf{P}$ and (18) is true.

b) The first part of this statement follows from Lemma 6 in ([5], p. 85). Hence there exists a unique, up to a multiplicative positive constant, eigenfunction $\tilde{\varphi}_k$ of A_k, belonging to $\mathbf{P}$, $\tilde{\varphi}_k$ corresponds to $\varrho(A_k)$.

(c) As $A_k\tilde{\varphi}_k = \varrho(A_k)\tilde{\varphi}_k$ and $\tilde{\varphi}_k(t) \not\equiv 0$ in $[0, 2\pi/\sqrt{3}]$, by the property of $\frac{\partial^k G}{\partial t^k}$ which has been proved in part (a), $A_k\tilde{\varphi}_k(t) > 0$ as well as $\tilde{\varphi}_k(t) > 0$ *in* $[0, 2\pi/\sqrt{3}]$.
Now we prove the following theorem.

Theorem 2. *Let $k \in \{0, 1, 2\}$ and let $\tilde{\varphi}_k \in \mathbf{P}$ be the eignefunction of the operator A_k defined by* (16). *Then the function*

$$\varphi_j(t) = \frac{1}{\varrho(A_k)}\int_0^{2\pi/\sqrt{3}} \left|\frac{\partial^j G(t, s)}{\partial t^j}\right| \tilde{\varphi}_k(s)\,\mathrm{d}s \quad 0 \leqq t \leqq 2\pi/\sqrt{3}, j = 0, 1, 2 \quad (19)$$

forms an admissible system of functions with respect to G such that for the associated system of constants $\tilde{\varkappa}_j$, $j = 0, 1, 2$,

$$\tilde{\varkappa}_k = \varrho(A_k) \quad (20)$$

is true.

Proof. The functions φ_j determined by (19) are all continouous and positive in $[0, 2\pi/\sqrt{3}]$ and $\varphi_k = \tilde{\varphi}_k$. Now we show that the functions $\varphi_j j = 0, 2, 1$, form an admissible system of functions with respect to G. In view of (c) in Lemma 7, there is a $C_k > 0$ such that $C_k \leqq \tilde{\varphi}_k(t)$ on $[0, 2\pi/\sqrt{3}]$ and hence $1 \leqq \frac{\tilde{\varphi}_k(t)}{C_k}$ which, on the basis of (16), (19) for each $j = 0, 1, 2$ implies that

$$\Phi_j(t) = A_j(1)(t) \leqq \frac{1}{C_k} A_j(\tilde{\varphi}_k)(t) = \frac{\varrho(A_k)}{C_k} \varphi_j(t),\ 0 \leqq t \leqq 2\pi/\sqrt{3}.$$

Finally we prove (20). By Lemma 3, $\tilde{\varkappa}_k \geqq \varrho(A_k)$. On the other hand, by Definition 2 and (19), $\tilde{\varkappa}_k \leqq \varrho(A_k)$, hence (20) is true.

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Received December, 9, 1988

Katedra matematiky
Chemickotechnologická fakulta SVŠT
Radlinského 9
812 37 Bratislava