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Mathematica Slovaca, Vol. 26 (1976), No. 1, 3--18

Persistent URL: http://dml.cz/dmlcz/128597

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ON SOME PBOPERTIES OF THIRD-ORDER LINEAR DIFFERENTIAL EQUATION

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The present paper is a study ot the oscillation and other properties of solutions of the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0,$$

where $p(x)$, $q'(x)$, $r(x)$ are continuous for $x \in \langle a, \infty)$, $(-\infty < a)$. This differential equation will be considered in the form

$$\text{(R)} \qquad y''' + p(x)y'' + 2A(x)y' + (A'(x) + b(x))y = 0,$$

where $2A(x) = q(x)$, $A'(x) + b(x) = r(x)$. We shall assume throughout that $p(x)$, $A(x)$, $A'(x) + b(x)$ and $b(x) - A(x)p(x)$ do not change the sign on the interval $\langle a, \infty)$. A solution of (R) will be said to be *oscillatory* if it changes the sign for arbitrarily large values of x.

In the present paper we shall generalize some results of the second and third part of A. C. Lazer [1] which concern the existence, uniqueness (with the exception of constant multiples) and asymptotic behaviour of nontrivial, nonoscillatory solutions, and criteria for the existence of oscillatory solutions in terms of the behaviour of nonoscillatory solutions. This generalization is not only in that $p(x) \not\equiv 0$ in $\langle a, \infty)$, but also in the following:

1. Generally the differential equation (R) cannot be transformed into the differential equation of the form

$$\text{(L)} \qquad y''' + P(x)y' + Q(x)y = 0$$

(for example, if $p(x)$ is not differentiable in any number $x \in \langle a, \infty)$).

2. If the differential equation (R) can be transformed into the differential equation (L), the coefficients $P(x)$, $Q(x)$ need not have the properties as in [1] (for example, the same sign is not necessary).

For example, the differential equation

$$(*) \qquad y''' + \frac{1 + \sin e^x}{x^2} y'' + 2 \sin^2 x\, y' + \left(1 + \frac{1}{x^2}\right) y = 0,$$

$x \in (1, \infty)$, by Theorem 2.1 of this paper is oscillatory. By transformation

$$y = z \exp\left(-\frac{1}{3}\int_1^x \frac{1 + \sin e^t}{t^2}\, dt\right), \quad x > 1$$

we obtain the oscillatory equation of the form (L)

$$(^{**}) \qquad z''' + P(x)z' + Q(x)z = 0, \quad x > 1,$$

where

$$P(x) = 2\sin^2 x - \frac{e^x \cos e^x}{x^2} + \frac{2 \sin e^x}{x^3} + \frac{2}{x^3},$$

$$Q(x) = 1 + \frac{1}{x^2} - \frac{2}{x^4} - \frac{2\sin^2 x(1 + \sin e^x)}{3x^2} + \frac{2(1 + \sin e^x)^3}{27x^6} + $$
$$+ \frac{e^x \cos e^x(4 - x)}{3x^2} + \frac{\sin e^x}{3x^2}\left(e^{2x} - \frac{6}{x^2}\right).$$

It is easy to see that $P(x)$, $Q(x)$, $2Q(x) - P'(x)$ are not of the same sign in any (τ, ∞); $\tau > 1$. Therefore it cannot be determined whether the differential equation $(^{**})$ is oscillatory or not according to Theorem 3.1 in [1].

This paper is divided into three parts in which we will apply the identity

$$\text{(I)} \qquad F[y(x), c] \equiv (y'^2(x) - 2y(x)y''(x) - 2A(x)y^2(x)) \exp\left(\int_c^x p(\eta)\, d\eta\right)$$
$$= F[y(c), c] + \int_c^x p(t)y'^2(t) \exp\left(\int_c^t p(\eta)\, d\eta\right) dt +$$
$$+ 2\int_c^x (b(t) - A(t)p(t))y^2(t) \exp\left(\int_c^t p(\eta)\, d\eta\right) dt,$$

which holds for any solution $y(x)$ of (R) and for every $c \geqq a$. It may be verified through differentiation.

A similar problem was studied among others by L. Moravský in [6].

1. Lemma 1.1. *If* $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$ *and* $y(x)$ *is any solution of* (R) *satisfying the initial conditions*

$$y(x_0) \geqq 0, \quad y'(x_0) \geqq 0, \quad y''(x_0) > 0$$

($x_0 \in \langle a, \infty)$ *arbitrary*), then

$$y(x) > 0, \quad y'(x) > 0, \quad y''(x) > 0 \quad \textit{for} \quad x > x_0$$

and

$$\lim_{x\to+\infty} y(x) = +\infty$$

and furthermore if

$$\int_{x_0}^{\infty} \exp\left(-\int_{x_0}^{t} p(\eta)\,\mathrm{d}\eta\right) \mathrm{d}t = +\infty,$$

then also

$$\lim_{x\to+\infty} y'(x) = +\infty.$$

Proof. We assert that $y''(x) > 0$ for $x \geqq x_0$.

If $y''(x)$ vanished for some value of x greater than x_0, there would be a smallest number $x_1 > x_0$ such that $y''(x_1) = 0$. Since $y(x_0) \geqq 0$, $y'(x_0) \geqq 0$, $y''(x_0) > 0$, we would have $y(x) > 0$, $y'(x) > 0$ for $x \in (x_0, x_1)$. Moreover, since $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$ it would follow that

$$\left(y''(x) \exp\left(\int_{x_0}^{x} p(\eta)\,\mathrm{d}\eta\right)\right)' = -2A(x)y'(x) \exp\left(\int_{x_0}^{x} p(\eta)\,\mathrm{d}\eta\right) -$$

$$-(A'(x) + b(x))y(x) \exp\left(\int_{x_0}^{x} p(\eta)\,\mathrm{d}\eta\right) \geqq 0$$

for $x \in \langle x_0, x_1\rangle$. However, by integrating the above inequality between x_0 and x_1 we would obtain the impossible inequality

$$0 = y''(x_0) + \int_{x_0}^{x_1} [-2A(t)y'(t) - (A'(t) + b(t))y(t)] \exp\left(\int_{x_0}^{t} p(\eta)\,\mathrm{d}\eta\right) \mathrm{d}t > 0.$$

Thus $y''(x) > 0$ for $x \geqq x_0$ and since $y(x_0) \geqq 0$, $y'(x_0) \geqq 0$, we see that $y(x) > 0$, $y'(x) > 0$ and $y''(x) > 0$ for $x > x_0$. From the above inequalities it follows easily that

$$\lim_{x\to+\infty} y(x) = +\infty.$$

Furthermore, since

$$y''(x) \exp\left(\int_{x_0}^{x} p(\eta)\,\mathrm{d}\eta\right) = y''(x_0) - 2\int_{x_0}^{x} A(t)y'(t) \exp\left(\int_{x_0}^{t} p(\eta)\,\mathrm{d}\eta\right) \mathrm{d}t -$$

$$-\int_{x_0}^{x} (A'(t) + b(t))y(t) \exp\left(\int_{x_0}^{t} p(\eta)\,\mathrm{d}\eta\right) \mathrm{d}t > 0$$

it follows that

$$y''(x) \exp\left(\int_{x_0}^{x} p(\eta)\,\mathrm{d}\eta\right) \geqq y''(x_0)$$

and

$$y'(x) \geqq y'(x_0) + y''(x_0) \int_{x_0}^{x} \exp\left(-\int_{x_0}^{t} p(\eta)\, d\eta\right) dt .$$

Hence

$$\lim y'(x) = +\infty \quad \text{if} \quad \int_{x_0}^{\infty} \exp\left(-\int_{x_0}^{t} p(\eta)\, d\eta\right) dt = +\infty$$

and the proof is complete.

Lemma 1.1′. *If* $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$ *and* $u(x) \not\equiv 0$ *is any nonoscillatory solution of* (R), *then there exists a number* $c \in \langle a, \infty)$ *such that either*

$$u(x)u'(x) > 0 \quad \textit{for} \quad x \geqq c,$$

or

$$u(x)u'(x) \leqq 0 \quad \textit{for} \quad x \geqq c.$$

Proof. If $u(x)$ is any nontrivial, nonoscillatory solution of (R), then by Lemma 1.1 it follows that $u(x)$ can have at most one double-zero. Without loss of generality we may suppose that $u(x) > 0$ for $x \geqq b$. To prove the lemma it is sufficient to show that $u'(x)$ can change from negative to positive values at most once in the interval $\langle b, \infty)$. In fact, then there exists a point c such that $u(c) > 0$, $u'(c) > 0$ and $u''(c) > 0$. By Lemma 1.1, $u(x) > 0$ and $u'(x) > 0$ for $x > c$ and the proof is complete.

Theorem 1.1. *If* $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$ *and* $\int_{a}^{\infty} \exp\left(-\int_{a}^{t} p(\eta)\, d\eta\right) dt = +\infty$, *then a necessary and sufficient condition for* (R) *to have oscillatory solutions is that for every nonoscillatory solution* $u(x) \not\equiv 0$, *there exists a number* $c \in \langle a, \infty)$ *such that*

$$\operatorname{sgn} u(x) = \operatorname{sgn} u'(x) = \operatorname{sgn} u''(x) \neq 0 \tag{2}$$

for $x \geqq c$, *and*

$$\lim_{x \to +\infty} |u(x)| = \lim_{x \to +\infty} |u'(x)| = +\infty . \tag{3}$$

Proof. If $u(x) \not\equiv 0$ is any nonoscillatory solution, then by Lemma 1.1′ there exists a number $b \in \langle a, \infty)$ such that either $u(x)u'(x) > 0$, or $u(x)u'(x) \leqq 0$, for $x \geqq b$. Thus, $\lim_{x \to +\infty} u(x)$ exists as finite or infinite. Let $v(x)$ be an oscillatory solution of (R) and consider the Wronskian $W(v(x), u(x)) = v(x)u'(x) - v'(x)u(x)$. $W(v(x), u(x))$ must certainly vanish for some values of x in the interval $\langle a, \infty)$, otherwise the zeros of $v(x)$ and $u(x)$ would separate and $u(x)$ would be oscillatory.

If d is a zero of $W(v(x), u(x))$, there exists constants c_1, and c_2, $c_1^2 + c_2^2 \neq 0$, such that

$$c_1 v(d) + c_2 u(d) = 0$$

$$c_1 v'(d) + c_2 u'(d) = 0$$

and
$$c_1 v''(d) + c_2 u''(d) > 0.$$

We now consider the solution

$$z(x) = c_1 v(x) + c_2 u(x).$$

Since $z(d) = z'(d) = 0$, and $z''(d) > 0$, it follows from Lemma 1.1 that

$$\lim_{x \to \infty} z(x) = \lim_{x \to +\infty} z'(x) = +\infty. \tag{4}$$

If $\lim\limits_{x \to \infty} u(x)$ were finite, we would have

$$\lim_{x \to +\infty} c_1 v(x) = \lim_{x \to +\infty} (z(x) - c_2 u(x)) = +\infty$$

and $v(x)$ could not be oscillatory. Thus $\lim\limits_{x \to +\infty} u(x) = \pm\infty$ and from Lemma 1.1′ we see that there must exist a number $c \in \langle a, \infty)$ such that $u(x)u'(x) > 0$ for $x \geqq c$. Without loss of generality let us suppose that $u(x) > 0$ and $u'(x) > 0$ for $x \geqq c$.

We will now show that $u''(x) > 0$ for $x \geqq d$. It is sufficient to show that

1. $u''(x)$ can change from negative to positive values at most once in the interval $\langle c, \infty)$. In fact, if $d \geqq c$ is a point such that $u(d) > 0$, $u'(d) > 0$, $u''(d) > 0$, then by Lemma 1.1 $u(x) > 0$, $u'(x) > 0$, $u''(x) > 0$ for $x \geqq d$.

2. If $u''(x) \leqq 0$ for $x \geqq d$, then since $u'(x) > 0$ for $x \geqq d \geqq c$ $\lim\limits_{x \to +\infty} u'(x)$ would be finite and by (4)

$$\lim_{x \to +\infty} c_1 v'(x) = \lim_{x \to +\infty} (z'(x) - c_2 u'(x)) = +\infty$$

and $v(x)$ could not oscillate.

Hence

$$\operatorname{sgn} u(x) = \operatorname{sgn} u'(x) = \operatorname{sgn} u''(x) \neq 0$$

for $x \geqq c$, and

$$\lim_{x \to +\infty} |u(x)| = \lim_{x \to +\infty} |u'(x)| = +\infty$$

by Lemma 1.1.

The proof of the sufficient condition is similar to that of Theorem 1 in [2] and will be omitted.

Lemma 1.2. If $p(x) \leqq 0$, $A(x) \leqq 0$, $b(x) - A(x)p(x) \leqq 0$ *and*

$$\int_a^\infty p(\eta)\,\mathrm{d}\eta > -\infty,$$

then the derivative of any oscillatory solution of (R) *is bounded on* $\langle a, \infty)$.

Proof. Let us suppose that $y(x)$ is an oscillatory solution of (R) and that $c \in \langle a, \infty)$ is a zero of $y''(x)$. Since the function $F[y(x), a]$ is nonincreasing and $A(x) \leqq 0$, we see that

$$y'^2(c) \exp\left(\int_a^c p(\eta)\,\mathrm{d}\eta\right) \leqq (y'^2(c) - 2A(c)y^2(c)) \exp\left(\int_a^c p(\eta)\,\mathrm{d}\eta\right) =$$

$$= F[y(c), a] \leqq F[y(a), a],$$

hence

$$y'^2(c) \leqq \exp\left(-\int_a^c p(\eta)\,\mathrm{d}\eta\right) F[y(a), a].$$

Thus the values of the function $y'(x)$ are bounded at its relative maxima and minima and furthermore, since $y(x)$ is oscillatory, $y'(x)$ vanishes for arbitrary large values of x. From these two conditions we see at once that $y'(x)$ is bounded on $\langle a, \infty)$.

Theorem 1.3. *If* $p(x) \leqq 0$, $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$, $b(x) - A(x)p(x) \leqq 0$ *and*

$$\int_a^\infty p(\eta)\,\mathrm{d}\eta > -\infty,$$

then the zeros of any two linearly independent oscillatory solutions of (R) *separate on* $\langle a, \infty)$.

Proof. It is sufficient to show that if $u(x)$ and $v(x)$ are any two linearly independent oscillatory solutions of (R), then their Wronskian $W(u(x), v(x))$ does not vanish for any $x \in \langle a, \infty)$. If we assumed on the contrary that $W(u(c), v(c)) = u(c)v'(c) - u'(c)v(c) = 0$ for some $c \in \langle a, \infty)$, then there would exist constants c_1 and c_2, $c_1^2 + c_2^2 \neq 0$, such that

$$c_1u(c) + c_2v(c) = 0,$$

$$c_1u'(c) + c_2v'(c) = 0$$

and

$$c_1u''(c) + c_2v''(c) > 0.$$

We consider the solution $z(x) = c_1u(x) + c_2v(x)$. It would follow from Lemma 1.1 that

$$\lim_{x\to+\infty} z(x) = \lim_{x\to+\infty} z'(x) = +\infty .$$

On the other hand, since $u(x)$ and $v(x)$ are oscillatory, it would imply, by Lemma 1.2, that both $u'(x)$ and $v'(x)$ and hence $z'(x) = c_1u'(x) + c_2v'(x)$ are bounded as x tends to infinity. From this contradiction it follows that $W(u(x), v(x)) \neq 0$ for all $x \in \langle a, \infty)$.

Theorem 1.4, [2]. *If* $p(x) \leqq 0$, $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$ *and there exists one oscillatory solution of* (R), *then there exist two linearly independent oscillatory solutions* u *and* v *of* (R) *such that any nontrivial linear combination of* u *and* v *is also oscillatory and the zeros of* u *and* v *separate, i.e. between every two consecutive zeros of* u *there is precisely one zero of* v.

Theorem 1.4'. *If* $p(x) \leqq 0, A(x) \leqq 0, A'(x) + b(x) \leqq 0, b(x) - A(x)p(x) \leqq 0$, $\int_a^\infty p(\eta)\,\mathrm{d}\eta > -\infty$ *for* $x \in \langle a, \infty)$ *and* (R) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions* u *and* v *whose zeros separate and such that a solution of* (R) *is oscillatory if and only if it is a nontrivial linear combination of* u *and* v. *If* w *is a nontrivial solution of* (R) *which is not a linear combination of* u *and* v, *then*

$$\lim_{x\to+\infty} |w(x)| = \lim_{x\to+\infty} |w'(x)| = +\infty .$$

Proof. Since the conditions of Theorem 1.4 are satisfied, there exist two linearly independent oscillatory solutions u and v of (R) such that the zeros of u and v separate and any nontrivial linear combination of u and v is also oscillatory. Moreover, since the assumptions of Lemma 1.2 are satisfied, u' and v' are bounded on $\langle a, \infty)$.

Let z be a solution of (R) which satisfies the initial conditions $z(a) = z'(a) = 0$, $z''(a) = 1$. By Lemma 1.1, $z(x) > 0$, $z'(x) > 0$, $z''(x) > 0$ and

$$\lim_{x\to+\infty} z(x) = \lim_{x\to+\infty} z'(x) = +\infty . \tag{5}$$

Since z is nonoscillatory, z is not a linear combination of u and v which implies that u, v and z are linearly independent. Hence any solution of (R) is a linear combination of u, v and z. If w is a solution such that $w = c_1u + c_2v + c_3z$, at which $c_3 \neq 0$, then by (5) and from the boundedness of u', v' it follows

$$\lim_{x\to+\infty} |w(x)| = \lim_{x\to+\infty} |w'(x)| = +\infty ,$$

i. e. w is nonoscillatory.

Theorem 1.5. *If* $p(x) \leqq 0$, $A(x) \leqq 0$, $A'(x) + b(x) \leqq 0$, $(A(x)p(x) - b(x))\exp\left(\int_a^x p(\eta)\,\mathrm{d}\eta\right) \geqq m > 0$, $\int_a^\infty p(\eta)\,\mathrm{d}\eta > -\infty$ *and* $u(x)$ *is any oscillatory solution of* (R), *then* $u(x) \in L^2\langle a, \infty)$ *and* $\lim_{x\to+\infty} u(x) = 0$.

Proof. Since $u(x)$ is an oscillatory solution, the function $F[u(x), a]$ is nonnegative for arbitrarily large values of x, i. e., those values of x for which $u(x)$ vanishes. Since the conditions of this theorem include those of Lemma 1.2, it follows that $u'(x)$ is bounded. Thus,

$$0 \leqq F[u(a), a] + M\left(\exp\left(\int_a^x p(\eta)\,\mathrm{d}\eta\right) - 1\right) + $$
$$+ 2\int_a^x (b(t) - A(t)p(t))u^2(t)\exp\left(\int_a^t p(\eta)\,\mathrm{d}\eta\right)\mathrm{d}t,$$

where $u'^2(x) \leqq M$ for all $x \in \langle a, \infty)$, and hence we obtain

$$\int_a^x u^2(t)\,\mathrm{d}t \leqq \frac{1}{m}\int_a^x (A(t)p(t) - b(t))u^2(t)\exp\left(\int_a^t p(\eta)\,\mathrm{d}\eta\right)\mathrm{d}t <$$

$$< \frac{1}{2m}\left\{F[u(a), a] + M\left(\exp\left(\int_a^x p(\eta)\,\mathrm{d}\eta\right) - 1\right)\right\}$$

for all $x \in \langle a, \infty)$.

Hence

$$\int_a^\infty u^2(t)\,\mathrm{d}t < +\infty.$$

Therefore, since $u(x) \in L^2\langle a, \infty)$, it is easy to see that

$$\lim_{x\to+\infty} u(x) = 0.$$

Lemma 1.6. *If* $A'(x) + b(x) \in C^2(\langle a, \infty))$, $A'(x) + b(x) > 0$ (< 0) *and*

$$\frac{4A(x)}{A'(x) + b(x)} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}(A'(x) + b(x))^{-1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p(x)}{A'(x) + b(x)}\right) \leqq 0\ (\geqq 0),$$

then the absolute values of a solution at its successive maxima and minima form a nondecreasing (nonincreasing) sequence.

Proof. If $u(x)$ is any solution of (R), then as can be verified through differentiation, we have the identity

$$G[u(x), a] = u^2(x) + \frac{2u'(x)u''(x)}{A'(x) + b(x)} + \frac{(A'(x)+b(x))'u'^2(x)}{(A'(x) + b(x))^2} + \frac{p(x)u'^2(x)}{A'(x) + b(x)}$$

$$= G[u(a), a] + 2\int_a^x \frac{u''^2(t)}{A'(t) + b(t)}\,dt - \int_a^x \left[\frac{4A(t)}{A'(t) + b(t)} + \right.$$

$$\left. + \frac{d^2}{dt^2}(A'(t) + b(t))^{-1} - \frac{d}{dt}\left(\frac{p(t)}{A'(t) + b(t)}\right)\right] u'^2(t)\,dt.$$

By the conditions of the theorem $G[u(x), a]$ is a nondecreasing (nonincreasing) function of x. At a maximum or minimum point of $u(x)$, where $u'(x) = 0$, $G[u(x), a] = u^2(x)$; hence the squares of the maxima and minima of $u(x)$, and hence the corresponding values of $|u(x)|$ form a nondecreasing (nonincreasing) sequence.

Theorem 1.6. *If* $A'(x) + b(x) \in C^2(\langle a, \infty))$, $A'(x) + b(x) < 0$,

$$4\frac{A(x)}{A'(x) + b(x)} + \frac{d^2}{dx^2}(A'(x) + b(x))^{-1} - \frac{d}{dx}\left(-\frac{p(x)}{A'(x) + b(x)}\right) \geqq 0,$$

then the zeros of any two linearly independent oscillatory solutions of (R) *separate.*

Proof. If $u(x)$ and $v(x)$ are two linearly independent oscillatory solutions of (R), then by the above conditions and Lemma 1.6 the absolute values of $u(x)$ and $v(x)$ at their successive maxima and minima points form nonincreasing sequences. Since $u(x)$ and $v(x)$ vanish for arbitrarily large values of x, it is easy to see that both $u(x)$ and $v(x)$ are bounded on $\langle a, \infty)$. If the Wronskian $W(u(x), v(x))$ vanished at a point $b \in \langle a, \infty)$, then by the same argument as used in the proof of Theorem 1.3, there would exist constants c_1 and c_2 such that

$$\lim_{x \to +\infty} (c_1 u(x) + c_2 v(x)) = +\infty.$$

But this is impossible if both $u(x)$ and $v(x)$ are bounded. This contradiction shows that $W(u(x), v(x)) \neq 0$ for all $x \in \langle a, \infty)$, and hence, the zeros of $u(x)$ and $v(x)$ separate.

2. Lemma 2.1. *If* $p(x) \geqq 0$, $A(x) \geqq 0$, $A'(x) + b(x) \geqq 0$, *and* $b(x) - A(x)p(x) \geqq 0$ *and not identically zero in any subinterval of* $\langle a, \infty)$, $\int_a^\infty p(\eta)\,d\eta < +\infty$ *and* $y(x) \not\equiv 0$ *is a nonoscillatory solution of* (R), *which is eventually nonnegative with*

$$0 \leqq F[y(c), c] = y'^2(c) - 2y(c)y''(c) - 2A(c)y^2(c)$$

($c \in \langle a, \infty)$ *arbitrary*), *then there exists a number* $d \geqq c$ *such that*

$$y(x) > 0,\ y'(x) > 0,\ y''(x) \geqq 0 \quad \text{and} \quad y'''(x) \leqq 0, \quad \text{for} \quad x \geqq d.$$

Proof. Since $F[y(x), c]$ is positive and increasing function in (c, ∞), $\lim F[y(x), c]$ exists and is positive and every nontrivial solution of (R) has at most one double-zero. Let $y(x) \not\equiv 0$ be a solution of (R). Then there exists a number $c_1 \geqq c$ such that $y(x) \neq 0$ in $\langle c_1, \infty)$. Without loss of generality, we may suppose $y(x) > 0$ for $x \geqq c_1$. We will now show that there exists a number $c_2 \geqq c_1$ such that $y'(x) \neq 0$ for $x \geqq c_2$.

Since $A(x) \geqq 0$ and $F[y(x), c] > 0$ for $x > c_1$, it follows that

$$0 < y'^2(x) - 2y(x)y''(x) - 2A(x)y^2(x) \leqq y'^2(x) - 2y(x)y''(x) \leqq$$
$$\leqq 2(y'^2(x) - y(x)y''(x))$$

and hence

$$y(x)y''(x) - y'^2(x) < 0 \quad \text{for} \quad x > c_1.$$

From the last inequality it follows that

$$\left(\frac{y'(x)}{y(x)}\right)' < 0 \quad \text{for} \quad x > c_1.$$

Hence there exists a number $c_2 > c_1$ such that $y'(x) \neq 0$ for $x \geqq c_2$. We will now show that $y'(x) > 0$ for $x \geqq c_2$. Suppose on the contrary that $y'(x) < 0$ for $x \geqq c_2$. There are three possibilities for $y''(x)$:

1. If $y''(x) \leqq 0$ for $x \geqq b \geqq c_2$, then $y'(x) \leqq y'(b) < 0$ for $x \geqq b$ so that $y(x)$ would eventually become negative in $\langle b, \infty)$, which is a contradiction.
2. If $y''(x) \geqq 0$ for $x \geqq b \geqq c_2$, then since $y'(x) < 0$ for $x \geqq b$, we would have $\lim\limits_{x\to+\infty} y'(x) = 0$, and consequently

$$\lim_{x\to+\infty} F[y(x), c] = \lim_{x\to+\infty} (y'^2(x) - 2y(x)y''(x) - 2A(x)y^2(x)) \exp\left(\int_c^x p(\eta)\, d\eta\right)$$

$$\lim_{x\to+\infty} (-2y(x)y''(x) - 2A(x)y^2(x)) \exp\left(\int_c^x p(\eta)\, d\eta\right) \leqq 0$$

for $x \geqq b$, which would contradict the fact that $\lim\limits_{x\to+\infty} F[y(x), c] > 0$. Finally, suppose that

3. $y''(x)$ changed signs for arbitrarily large values of x. Then there would exist an increasing sequence of points $\{x_n\}$, $c_2 \leqq x_1$, with the following properties:

$$\lim_{n\to\infty} x_n = \infty, \quad -\varepsilon < y'(x_n) < 0, \quad y''(x_n) = 0,$$

where $\varepsilon > 0$ is arbitrary.

The existence of such a sequence $\{x_n\}$ is clear since $y'(x) < 0$ and $\limsup_{x \to +\infty} y'(x) = 0$.

We would then have

$$F[y(x_n), c] \leqq (\varepsilon^2 - 2A(x_n)y^2(x_n)) \exp \left(\int_c^{x_n} p(\eta)\, d\eta \right) \leqq \varepsilon^2 \exp \left(\int_c^{\infty} p(\eta)\, d\eta \right) \leqq \varepsilon^2 K ,$$

where

$$\exp \left(\int_c^{\infty} p(\eta)\, d\eta \right) \leqq K < +\infty,$$

for arbitrarily large values of x_n, which would imply that

$$\lim_{n \to \infty} F[y(x_n), c] \leqq 0.$$

This, as in the above, would be a contradiction. Thus, since the three mutually exclusive and exhaustive cases all lead to a contradiction when we assume $y'(x) < 0$ for $x \geqq c_2$, we must have $y'(x) > 0$ for $x \geqq c_2$.

We will now show that $y''(x) \geqq 0$ for $x \geqq c$. Suppose on the contrary: Let there exist a number $d \geqq c$, such that $y''(d) < 0$, since y is a solution of (R)

$$\left(y''(x) \exp \left(\int_d^x p(\eta)\, d\eta\right)\right)' = -2A(x)y'(x) \exp \left(\int_d^x p(\eta)\, d\eta\right) -$$

$$- (A'(x) + b(x))y(x) \exp \left(\int_d^x p(\eta)\, d\eta\right)$$

so that we have

$$\left(y''(x) \exp \left(\int_d^x p(\eta)\, d\eta\right)\right)' \leqq 0.$$

By integrating the above inequality from d to x, we have

$$y''(x) \exp \left(\int_d^x p(\eta)\, d\eta\right) \leqq y''(d) \quad \text{for} \quad x \geqq d,$$

hence

$$y'(x) \leqq y'(d) + y''(d) \int_d^x \exp \left(- \int_d^t p(\eta)\, d\eta\right) dt$$

and

$$\lim_{x \to +\infty} y'(x) = -\infty.$$

This is a contradiction to the assumption that $y'(x) > 0$. Hence there exists a number $d \geqq c$, such that $y(x) > 0$, $y'(x) > 0$, $y''(x) \geqq 0$ for $x \geqq d$. From this it follows that $y'''(x) = -p(x)y''(x) - 2A(x)y'(x) - (A'(x) + b(x))y(x) \leqq 0$ for $x \geqq d$.

Lemma 2.1', [1]. *If $y(x) \in C^3(\langle a, \infty))$, and $y(x) > 0$, $y'(x) > 0$, $y'''(x) \leqq 0$, for $x \geqq a$, then*

$$\liminf_{x \to +\infty} \frac{y(x)}{xy'(x)} \geqq \frac{1}{2}.$$

By means of the two preceding lemmas and the classical Sturm comparison theorem we shall derive an oscillation condition for (R).

Theorem 2.1. *If $p(x) \geqq 0$, $\int_a^\infty p(\eta)\,\mathrm{d}\eta < +\infty$, $A(x) \geqq 0$, $A'(x) + b(x) \geqq 0$, and $b(x) - A(x)p(x) \geqq 0$ and not identically zero in any interval, and there exists a number $m < 1/2$ such that the second-order differential equation*

$$w'' + pw' + [2A(x) + (A'(x) + b(x))mx]w = 0$$

is oscillatory, then (R) *has oscillatory solutions. In fact if $y(x)$ is any nonzero solution of* (R) *with*

$$0 \leqq F[y(c), a] = (y'^2(c) - 2y(c)y''(c) - 2A(c)y^2(c)) \exp\left(\int_a^c p(\eta)\,\mathrm{d}\eta\right)$$

$c \in \langle a, \infty)$,
then $y(x)$ is oscillatory.

Proof. Suppose that $u(x) \not\equiv 0$ were a nonoscillatory solution of (R) with $F[u(c), a] \geqq 0$. Without loss of generality, we could assume $u(x)$ to be eventually nonnegative. By Lemma 2.1, there would exist a number $d \geqq c$ such that

$$u(x) > 0,\ u'(x) > 0,\ u''(x) \geqq 0 \quad \text{and} \quad u'''(x) \leqq 0$$

for $x \geqq d$. Hence by Lemma 2.1'

$$\liminf_{x \to +\infty} \frac{u(x)}{xu'(x)} \geqq \frac{1}{2}.$$

Thus, since $m < 1/2$, there would exist a number $d_1 \geqq d$, such that $u(x)/u'(x) > mx$ for $x \geqq d_1$. By writing (R) in the form of a system

$$u' = w \tag{6}$$

$$w'' + p(x)w' + 2A(x)w + (A'(x) + b(x))u = 0,$$

we could write the second equation in the form

$$(\exp(\int_a^x p(\eta)\,\mathrm{d}\eta)w')' + \exp(\int_a^x p(\eta)\,\mathrm{d}\eta)\left[2A(x) + (A'(x) + b(x))\frac{u}{w}\right]w = 0.$$

Since by the above

$$2A(x) + (A'(x) + b(x))\frac{u(x)}{w(x)} = 2A(x) + (A'(x) + b(x))\frac{u(x)}{u'(x)} >$$

$$> 2A(x) + (A'(x) + b(x))mx$$

for $x \geqq d_1$, hence also

$$\exp(\int_a^x p(\eta)\,\mathrm{d}\eta)\left[2A(x) + (A'(x) + b(x))\frac{u(x)}{u'(x)}\right] >$$

$$> \exp(\int_a^x p(\eta)\,\mathrm{d}\eta)\,[2A(x) + (A'(x) + b(x))mx].$$

It would follow from the Sturm comparison theorem that since the differential equation

$$(\exp(\int_a^x p(\eta)\,\mathrm{d}\eta)w')' + \exp(\int_a^x p(\eta)\mathrm{d}\eta)[2A(x) + (A'(x) + b(x))mx]w = 0$$

is oscillatory, all nonzero solutions of

$$(\exp(\int_a^x p(\eta)\,\mathrm{d}\eta)y')' + \exp(\int_a^x p(\eta)\,\mathrm{d}\eta) \times \tag{7}$$

$$\times\left[2A(x) + (A'(x) + b(x))\frac{u(x)}{w(x)}\right]y = 0,$$

defined for $x \geqq d_1$, would oscillate. But this contradicts the fact that the particular solution $w(x) = u'(x)$ is the solution of (7). Thus, the assumption that $u(x)$ is nonoscillatory leads to a contradiction.

3. Lemma 3.1. *If* $p(x) \geqq 0$ *and* $b(x) - A(x)p(x) \geqq 0$ *and not identically zero in any interval, and* (R) *has one oscillatory solution* $v(x)$, *then any nontrivial solution* $u(x)$ *of* (R) *such that*

$$u(c) = v(c) = 0$$

(c arbitrary) is oscillatory.

Proof. We will apply the identity (I). We consider the solution

$$z(x) = c_1u(x) + c_2v(x),\ z(c) = 0.$$

Then

$$F[z(c), c] = z'^2(c) \geqq 0 \tag{8}$$

and $F[z(x), c] > 0$ for $x > c$. We consider the Wronskian

$$W(u(x), v(x)) = u(x)v'(x) - u'(x)v(x).$$

If $W(u(x), v(x))$ vanished at a point $d > c$, then there would exist constants c_1 and c_2 such that

$$c_1u(d) + c_2v(d) = 0$$

$$c_1u'(d) + c_2v'(d) = 0$$

$$c_1^2 + c_2^2 \neq 0.$$

Then $F[z(d), c] = 0$ and by (8) $F[z(c), c] \geqq 0$. But

$$F[z(d), c] > F[z(c), c] \geqq 0.$$

This contradiction shows that $W(u(x), v(x)) \neq 0$ for $x > c$. Hence since $v(x)$ is oscillatory, $u(x)$ is oscillatory.

Lemma 3.2. *If $p(x) \geqq 0$ and $b(x) - A(x)p(x) \geqq 0$ and not identically zero in any interval, and* (R) *has one oscillatory solution, then any solution which vanishes is oscillatory.*

Proof. Let $v(x)$ be an oscillatory solution of (R) which vanishes at x_1 and let $u(x)$ be a nontrivial solution of (R) such that $u(x_0) = 0$. Construct a solution $z(x)$ of (R) such that $z(x_0) = z(x_1) = 0$, $z(x) \not\equiv 0$. Applying Lemma 3.1 first to the solution $v(x)$ and $z(x)$ at the point x_1, we see that $z(x)$ is oscillatory. Next applying Lemma 3.1 to the solutions $z(x)$ and $u(x)$ at the point x_0, we see that $u(x)$ is oscillatory and the proof is complete.

Theorem 3.2. *If $p(x) \geqq 0$ and $b(x) - A(x)p(x) \geqq 0$ and not identically zero in any interval, and* (R) *has one oscillatory solution, then a necessary and sufficient condition for a solution $u(x) \not\equiv 0$ to be nonoscillatory is that $F[u(x), a] < 0$ for all $x \in \langle a, \infty)$.*

Proof. The sufficiency is trivial. Indeed, if $F[u(x), a]$ is negative for all $x \in \langle a, \infty)$, it is clear that $u(x) \neq 0$ for all $x \in \langle a, \infty)$. To prove the necessity we will show that if (R) has one oscillatory solution and $u(x) \not\equiv 0$, $F[u(c), a] \geqq$ $\geqq 0$, $c \in \langle a, \infty)$ arbitrary, then $u(x)$ is oscillatory. If $u(c) = 0$, the assertion follows from Lemma 3.2. If $u(c) \neq 0$, we consider a second solution defined by the initial conditions

$$v(c) = 0, \ v'(c) = u(c), \ v''(c) = u'(c).$$

Since $v(x)$ is not identically zero and vanishes at c, we see from Lemma 3.2 that $v(x)$ is oscillatory. Furthermore, for any constants c_1 and c_2 both not zero

$$(9) \quad F[(c_1u(c) + c_2v(c)), a] = \{c_1^2F[u(c), a] + c_2^2F[v(c), a]\} \exp\left(\int_a^c p(\eta)\, d\eta\right) =$$

$$= \{c_1^2F[u(c), a] + c_2^2u^2(c) \exp\left(\int_a^c p(\eta)\, d\eta\right)\} \exp\left(\int_a^c p(\eta)\, d\eta\right) \geqq 0.$$

Consider the Wronskian $W(u(x), v(x)) = u(x)v'(x) - u'(x)v(x)$. If $W(u(x), v(x))$ vanished at a point $d > c$, then there would exist constants c_1 and c_2 such that

$$c_1u(d) + c_2v(d) = 0$$

$$c_1u'(d) + c_2v'(d) = 0$$

and

$$c_1^2 + c_2^2 \neq 0.$$

If $z(x)$ were the solution $c_1u(x) + c_2v(x)$, then $F[z(d), a] = 0$ and by (9) $F[z(c), a] \geqq 0$.

But

$$F[z(d), a] > F[z(c), a] \geqq 0.$$

This contradiction shows that $W(u(x), v(x)) \neq 0$ for $x > c$. Hence, since $v(x)$ is oscillatory, $u(x)$ is oscillatory too.

The next theorem shows that solutions satisfying the conditions of Theorem 3.2 actually exist. Since the method of construction has already been given by M. Greguš [4], we will only outline the proof.

Theorem 3.3. *If $p(x) \geqq 0$ and $b(x) - A(x)p(x) \geqq 0$, and not identically zero in any interval, then* (R) *has a solution $y(x)$ for which $F[y(x), a]$ is always negative. Consequently $y(x)$ is nonoscillatory.*

Proof. For each integer $n > a$, we consider the solution $y_n(x)$ defined by the initial conditions

$$y_n(n) = y_n'(n) = 0, \; y_n''(n) \neq 0,$$

and the normalization

$$y_n(x) = c_{1n}z_1(x) + c_{2n}z_2(x) + c_{3n}z_3(x),$$

with

$$(10) \qquad c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1,$$

where $z_1(x)$, $z_2(x)$, $z_3(x)$ is the fundamental system of solutions of (R). Since the three sequences $\{c_{in}\}$, $i = 1, 2, 3$ are bounded, there exists a sequence of integers $\{n_j\}$ such that the subsequences $\{c_{in_j}\}$ converge to numbers c_i, $i = 1, 2, 3$. From (10) we see that

$$c_1^2 + c_2^2 - c_3^2 = 1.$$

The sequences $\{y_{n_j}(x)\}$, $\{y'_{n_j}(x)\}$, $\{y''_{n_j}(x)\}$ converge uniformly on any finite subinterval of $\langle a, \infty)$ to the functions $y(x)$, $y'(x)$ and $y''(x)$, where $y(x)$ is a nontrivial solution of (R).

Let c be an arbitrary point in the interval $\langle a, \infty)$. Since $F[y_{n_j}(n_j), a] = 0$ and $F[y_{n_j}(x), a]$ is strictly increasing, $F[y_{n_j}(c), a] < 0$ for $c < n_j$. Since $F[y(c), a] = \lim_{n_j \to \infty} F[y_{n_j}(c), a]$, therefore $F[y(c), a] \leqq 0$. If c is arbitrary, $F[y(x), a] \leqq 0$ for all $x \in \langle a, \infty)$. But the equality cannot hold at any point d, since this would imply that $F[y(x), a] > 0$ for $x > c$, as $F[y(x), a]$ is strictly increasing.

REFERENCES

[1] LAZER, A. C.: The behavior of solutions of the differential equation $y''' + p(x)y' + q(x)y = 0$, Pacif. J. Math., 17, 1966, 435—465.

[2] LAZER, A. C. and AHMAD SHAIR: On the oscillatory behavior of a class of linear third order differential equations, J. Math. Anal. Appl., 28, 1969, 681—689.

[3] GREGUŠ, M.: Oszillatorische Eigenschaften der Lösungen der linear Differentialgleichungen dritter Ordnung $y''' + 2Ay' + (A' + b)y = 0$, wo $A = A(x) \leqq 0$ ist, Czech. Math. J. (84), 9, 1959, 416—428.

[4] GREGUŠ, M.: Über einige Eigenschaften der Lösungen der Differentialgleichungen $y''' + 2Ay' + (A' + b)y = 0$, $A \leqq 0$, Czech. Math. J., 11 (86), 1961, 106—114.

[5] HANAN, M.: Oscillation criteria for third-order linear differential equation, Pacif. J. Math., 11, 1961, 914—944.

[6] MORAVSKÝ, L.: Einige oszillatorische und asymptotische Eigenschaften der Lösungen der Differentialgleichung $y''' + p(x)y'' + 2A(x)y' + (A'(x) + b(x))y = 0$, Acta F. R. N. Univ. Comen. — Mathematica, XVI, 1967.

Received May 23, 1972

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