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Mathematica Slovaca, Vol. 56 (2006), No. 1, 53--78

Persistent URL: http://dml.cz/dmlcz/128690

Math. Slovaca, **56** (2006), No. 1, 53–78

Mathematica
Slovaca

Dedicated to Professor Tibor Katriňák

CANONICAL EXTENSIONS OF STONE AND DOUBLE STONE ALGEBRAS: THE NATURAL WAY

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(*Communicated by Sylvia Pulmannová*)

ABSTRACT. A construction of canonical extensions of Stone algebras is presented that uses the natural duality based on the three-element generating algebra **3** rather than the Priestley duality based on **2** that is traditionally used to build the canonical extension. The new approach has the advantage that the canonical extension so constructed inherits its algebra structure pointwise from a power of the generator, so that the extension of the fundamental operations and closure of the variety under the formation of canonical extensions occur in a transparent way. An analogous construction is outlined for two further varieties, indicating that the method has the potential to be applied in a similar manner to other classes of bounded distributive lattice expansions.

1. Introduction

Canonical extensions were first studied, in the context of Boolean algebras with operators (BAOs), in the classic work of B. Jónsson and A. Tarski [14]. The development of a corresponding theory in a less restricted setting than that of BAOs lagged far behind. However major advances have been made in the past ten years by M. Gehrke in collaboration with B. Jónsson, J. Harding, Y. Venema and others; see in particular [8], [9] and [10]. Appropriate definitions have been found for canonical extensions of distributive lattice expansions (that is, bounded distributive lattices with additional operations, henceforth called DLEs) and more generally of lattice and even poset

2000 Mathematics Subject Classification: Primary 06D50, 06B23, 03G10.
Keywords: canonical extension, Stone algebra, dual equivalence of categories.

The research for this paper was carried out while the first author was visiting Oxford under the auspices of the Oxford Colleges Hospitality Scheme. This support, and that provided by Slovak VEGA grant 1/0267/03, are gratefully acknowledged.

expansions. Such structures arise naturally as semantic models of non-classical logics. A variety of algebras is *canonical* if it is closed under the formation of canonical extensions. When the members of a variety are the algebraic models of a logic, canonicity leads *inter alia* to completeness results for the associated logic (see [7] for selected examples and also [10]). The work of Gehrke et al. has brought many new insights and has revealed symbiotic relationships between duality and canonical extensions. This is resulting in a unification of the hitherto largely separate research tracks of duality, involving topologised relational structures, and the relational semantics which has been such an important tool in modal and intuitionistic logic. We emphasise in particular the virtues, as enunciated in [10], of the canonical extension of an algebra in a canonical variety of DLEs:

- the extension is a concrete structure of the same type as the original algebra but with stronger properties;
- encoded within the extension are both the lattice reduct of the original algebra and its topological dual;
- the construction is functorial.

An important canonicity result proved by Gehrke and Jónsson (see [10; §4]) asserts that any finitely generated variety of DLEs is canonical. Their route to this theorem is indirect. The result emerges as a consequence of a more general one, which can be viewed as an algebraic formulation of an extension of the Fine-van Benthem-Goldblatt theorem from modal logic.

Given the importance of canonical extensions for both logic and algebra, it is worthwhile fully to understand their structure, and how canonicity occurs. This paper seeks, through a study of three particular finitely generated varieties of DLEs, to contribute to such an understanding. The varieties we consider are Stone algebras, double Stone algebras and regular double Stone algebras, denoted respectively $\mathcal{S}$, $\mathcal{DS}$ and $\mathcal{R}$. The structure of Stone, double Stone and regular double Stone algebras is well understood, thanks to major contributions of T. Katriňák going back to the mid 1960s. In particular his triple representations provided an important tool for studying these algebras, and contributed in part to interest in canonical extensions for regular double Stone algebras. In [2], S. D. Comer constructs canonical extensions for expansions of algebras in $\mathcal{R}$. The motivation for his paper comes from the relevance of Stone and regular double Stone algebras to rough sets; see [2] for references to this area of application. Comer firstly adapts the original Jónsson-Tarski construction for BAOs and then exploits a representation due to Katriňák [19], which reveals the close relationship of the algebras in $\mathcal{R}$ to Boolean algebras. Our method is quite different from Comer's. We remark that the very special nature of regular double Stone algebras implies the existence of a Boolean product representation, so that Comer's result may be seen as stemming from the preservation of canonical extensions under the formation of Boolean products; see [10; Sec. 3]. However, the same cannot be said for Stone, or double Stone, algebras. Although sheaf representations for Stone algebras are available, it is not the case that these yield decompositions of Boolean product type which would enable the natural dual representation of an arbitrary Stone algebra to be viewed in such terms. See, for example, [13], in particular Theorem 3.6.

Traditionally, canonical extensions of distributive lattices have been obtained via Priestley duality and canonical extensions for DLEs have then been constructed by enriching the canonical extensions of their lattice reducts by bolting on extensions of the additional operations. More explicitly, let us consider a DLE $\big(L; \wedge, \vee, 0, 1, \{f_\lambda\}_{\lambda\in\Lambda}\big)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and Λ is a non-empty index set. By Priestley duality, $(L; \wedge, \vee, 0, 1)$ can be identified with the lattice of clopen up-sets of its dual space Y of prime filters, or ultrafilters in the case that the lattice is Boolean. In this paper we take the canonical extension L^σ of L to be the complete lattice of *all* up-sets of Y (this is equivalent to the definition in [10]; see Section 3), together with the natural embedding of L into L^σ.

One now seeks to lift each operation f_λ to an operation f_λ^σ on L^σ so as to obtain a complete extension of the expansion $\big(L; \wedge, \vee, 0, 1, \{f_\lambda\}_{\lambda\in\Lambda}\big)$. In [8] and [10], formulae for such extensions are proposed: the idea is to exploit the density property possessed by the extension; this asserts that every element of L^σ is a join of meets, and a meet of joins, of the copy in L^σ of the original lattice L; see Section 3. In [10] a detailed analysis of extensions of maps is undertaken, in which canonical extensions are endowed with various topologies and an interesting interplay between algebra and topology is explored. The analysis, which is quite subtle, leads to various conditions guaranteeing canonicity and in particular to the canonicity of finitely generated varieties of DLEs.

Furthermore, canonicity may be used to derive dualities via correspondence. A systematic study is undertaken by M. Gehrke, H. Nagahashi and Y. Venema in [9] in the context of distributive modal algebras (DMAs, for short); these are bounded distributive lattices whose additional operations are unary and either preserve $\vee$ and 0, or preserve $\wedge$ and 1, or convert $\vee$ and 0 to $\wedge$ and 1 or vice versa. This setting certainly encompasses the particular varieties considered in this paper. However, the heavy machinery of [9], which is inspired by and extends sophisticated techniques from modal logic, is not needed for these varieties. Indeed, it is our purpose to show how, in these special cases, a different approach is available.

Our new approach is based on natural dualities. The theory of dualities of this kind, of which Priestley duality is one example, is very well developed, especially for finitely generated varieties and quasi-varieties of DLEs. A comprehensive account can be found in the monograph [1] by D. M. Clark and B. A. Davey.

The varieties $\mathcal{S}$, $\mathcal{DS}$ and $\mathcal{R}$ on which we focus have particularly simple natural dualities, all covered by the framework in [4]. We are thereby able to present our constructions without having to delve too deeply into the machinery of natural duality theory, which can be daunting to those unfamiliar with it. (A further, more technical, paper will treat a wider class of varieties.)

To explain the relevance of natural dualities to canonical extensions of DLEs and to contrast the "natural" approach with the one hitherto adopted we first need to make a few remarks about the dualities based on Priestley duality for varieties of DLEs which abound in the literature. Under quite weak conditions on the additional operations, these expanded Priestley dualities are derived by enriching the Priestley duals of the lattice reducts with relations or, where possible, functions, so as to capture the operations dually. The first systematic study of this process was undertaken by R. Goldblatt in [12]. The dualities for varieties of distributive modal algebras presented in [9] follow the same pattern as those in [12]. Roughly speaking, DMAs provide the largest class of algebras involving additional operations which are at most unary in which varieties will possess dualities based on relational structures in the traditional way.

We stress that, in the standard approach, the canonical extension of a DLE is built by first forming the extension of its lattice reduct and then superimposing extensions of the additional operations. Furthermore, the extended operations lead to the expanded duality in a very direct way. Thus the traditional canonical extension construction and the derivation of an expanded Priestley duality for a variety of DLEs rely, and in an interconnected way, on first working with the underlying lattices, rather than by giving constructions which are intrinsic to the variety under consideration. By replacing expanded Priestley dualities by natural dualities we are able to construct canonical extensions intrinsically.

For a bounded distributive lattice L the canonical extension L^σ, as defined above, may be viewed as the lattice of order-preserving maps from its Priestley dual space into the two-element chain $\underset{\sim}{2}$. The arbitrary joins and meets in L^σ are derived *pointwise* from those of $\underset{\sim}{2}$, now viewed as a complete lattice. However, when we move to DLEs, the extensions of the additional operations to L^σ, unlike the extensions of $\wedge$ and $\vee$, are not given pointwise: see the formulae for extensions of maps given in [8; §4], and [10; §3]. By working instead with a natural duality for a variety of DLEs we are able, as with Priestley duality itself, to extend *all* the operations in a pointwise fashion. As a result, the structure and the features of the canonical extensions become highly transparent. There is an additional advantage: in a natural duality, as in Priestley duality for bounded distributive lattices, the duals of free algebras are always obtained by forming concrete products. This rarely happens for expanded Priestley dualities. In the context of logic, the free algebra on a countable set of generators is, of course, of relevance, namely as the Lindenbaum-Tarski algebra.

By moving to a natural duality we do generally sacrifice one good feature possessed by expanded Priestley dualities: the concrete representation of algebras by algebras of sets. Instead, we have a concrete representation by algebras of functions. In [4], Davey and Priestley showed that, for a wide class of DLEs, it is simple to translate backwards and forwards between a natural duality and an expanded Priestley duality. This allows one to reap the benefits of both approaches: most notably, the representation of free algebras from the first and the concrete representation of algebras in terms of sets from the second. We show that, for the varieties $\mathcal{S}$, $\mathcal{DS}$ and $\mathcal{R}$, a similar two-perspective view is available for canonical extensions.

The paper is organised as follows. In the next section we review the available concrete representations for Stone algebras: via an expanded Priestley duality and via a natural duality. In Section 3 we look at the canonical extension of (the reduct of) a Stone algebra as hitherto constructed and show how this construction can be recast in terms of the natural duality. Section 4 develops a categorical framework. In Section 5 we discuss the different ways in which one can arrive at canonical extensions of Stone algebras, including one based in a free-standing way on the natural duality. In the final section we outline, without proofs, parallel constructions and results for double Stone algebras and regular double Stone algebras.

2. Preliminaries: Stone algebras and their dual representations

The variety $\mathcal{S}$ of Stone algebras can be identified with the quasi-variety $\mathbb{ISP}(\mathbf{3})$ generated by the three-element Stone algebra $\mathbf{3} = (\{0, a, 1\}; \vee, \wedge, {}^*, 0, 1)$, where the underlying three-element bounded lattice $0 < a < 1$ is equipped with a unary operation * of pseudocomplementation so that $1^* = a^* = 0$ and $0^* = 1$. We draw attention explicitly to the fact that when a Stone algebra is identified with a subalgebra of a power of $\mathbf{3}$, its lattice structure and its pseudocomplementation are obtained by lifting pointwise the corresponding operations on $\mathbf{3}$.

The Stone identity $x^* \vee x^{**} = 1$ characterises members of $\mathbb{ISP}(\mathbf{3})$ among the distributive pseudocomplemented lattices. With a slight abuse of terminology we shall also use the term Stone algebra when we are regarding a member $(L; \vee, \wedge, {}^*, 0, 1)$ of $\mathcal{S}$ simply as a special kind of distributive lattice.

In this paper, we generally work with functions rather than with sets. Accordingly, as in [4], we use hom-functors to set up Priestley duality between the categories

$\mathcal{D}$: $\{0,1\}$-distributive lattices and homomorphisms preserving 0, 1,

$\mathcal{P}$: Priestley spaces and continuous order-preserving maps.

Denote the two-element bounded distributive lattice, with elements 0 and 1, by $\mathbf{2}$ and its alter ego, the two-element chain, with order $0 < 1$ and the discrete topology, by $\underset{\sim}{2}_t$. The two-element chain without topology is denoted by $\underset{\sim}{2}$. Now define functors $H\colon \mathcal{D} \to \mathcal{P}$ and $K\colon \mathcal{P} \to \mathcal{D}$ as follows: for $L \in \mathcal{D}$ and $Y \in \mathcal{P}$,

On objects:

$$H\colon L \mapsto \mathcal{D}(L, \mathbf{2}) \le \underset{\sim}{2}_t{}^{|L|},$$
$$K\colon Y \mapsto \mathcal{P}\big(Y, \underset{\sim}{2}_t\big) \le \mathbf{2}^{|Y|}.$$

On morphisms:

$$H\colon f \mapsto - \circ f,$$
$$K\colon \phi \mapsto - \circ \phi.$$

(Here $|\text{-}|$ denotes the forgetful functor to the category of sets and $\le$ means "is a substructure of".) Then H and K set up a dual equivalence. Specifically:

$$\big(\forall L \in \mathcal{D}\big)\big(k_L\colon L \to KH(L) \text{ is an isomorphism}\big),$$
$$\text{where} \quad k_L(a)(f) = f(a) \quad (a \in L,\ f \in H(L)),$$

$$\big(\forall Y \in \mathcal{P}\big)\big(\kappa_Y\colon Y \to HK(Y) \text{ is an isomorphism}\big),$$
$$\text{where} \quad \kappa_Y(x)(\phi) = \phi(x) \quad (x \in Y,\ \phi \in K(Y)).$$

The maps k_L and κ_Y given at each point by evaluation are, respectively, the unit and co-unit of the dual adjunction. We note that the version of the duality in terms of sets comes from identifying $H(L)$ with the prime filters of L and $K(Y)$ with the clopen up-sets of Y.

Stone algebras can be characterised amongst pseudocomplemented distributive lattices by the property that every prime filter is contained in a unique maximal prime filter. The dual spaces of (the lattice reducts of) Stone algebras are exactly those Priestley spaces in which there exists a continuous order-preserving retraction $\rho = \rho \circ \rho$ taking each point x to the unique maximal point $\rho(x)$ above it. This result was first obtained in [20] (where a dual space of prime ideals, rather than prime filters, was used, so that the order on the dual space is the reverse of that used here).

We denote by $\mathcal{Y}$ the non-full subcategory of $\mathcal{P}$ whose objects consist of the spaces $H(L)$ for $L \in \mathcal{S}$; besides the order $\preccurlyeq$ and topology $\mathcal{T}$ they carry by virtue of being Priestley spaces, these are also regarded as equipped with the retraction ρ. The morphisms of $\mathcal{Y}$ are the $\mathcal{P}$-morphisms which preserve the map ρ. We also denote by H and K the restrictions of these functors to, respectively, the non-full subcategories $\mathcal{S}$ and $\mathcal{Y}$ of $\mathcal{D}$ and $\mathcal{P}$. We remark in passing that, for an expanded Priestley duality in the true sense we must regard the retraction map ρ as part of the type of the objects in the dual category, and we elect to do this. Stone algebras have the special property that the additional operation * is determined by the operations of the underlying lattice and the dual structures can be treated simply as Priestley spaces having a special property (which is first-order definable in the language of posets). Therefore we could, had we so wished, have treated the duality for $\mathcal{S}$ as a restriction of Priestley duality rather than as an expansion of it.

In contrast to the duality presented above for $\mathcal{S}$, which is based on the 2-element generating algebra $\mathbf{2}$ for $\mathcal{D}$, the (full) natural duality for Stone algebras is based on $\mathbf{3}$. To describe this, we must introduce the alter ego of $\mathbf{3}$. This is the structure $\underset{\sim}{3}_t = \big(\{0, a, 1\}; \preccurlyeq, \rho, \tau\big)$, where the underlying three-element set is equipped with the order relation $\big\{(0,0), (a,a), (1,1), (a,1)\big\}$, denoted $\preccurlyeq$, a unary operation ρ (an endomorphism of $\mathbf{3}$) with graph

$$\operatorname{graph}(\rho) = \big\{(0,0), (a,1), (1,1)\big\},$$

and the discrete topology τ. The structure in $\underset{\sim}{3}_t$ is *algebraic over* $\mathbf{3}$ in the sense that $\preccurlyeq$ and $\operatorname{graph}(\rho)$ are subalgebras of $\mathbf{3}^2$; the theory of natural dualities relies in a critical way on the fact that the structure chosen on the relational side is algebraic. The dual topological quasi-variety $\mathcal{X} = \mathbb{IS}_{\mathrm{C}}\mathbb{P}(\underset{\sim}{3}_t)$ derived from $\underset{\sim}{3}_t$ consists of topological relational structures which are isomorphic to closed substructures of powers of $\underset{\sim}{3}_t$; powers carry the product topology and $\preccurlyeq$ and ρ are extended pointwise to subsets of powers. The morphisms of $\mathcal{X}$ are the continuous maps which preserve both $\preccurlyeq$ and ρ. The class $\mathcal{X}$ can be characterised as the class of structures $(X; \preccurlyeq, \rho, \mathcal{T})$, where $(X; \preccurlyeq, \mathcal{T})$ is a Priestley space and ρ is a continuous retraction ρ mapping each element up to the unique maximal element above it (cf. [1; Theorem 4.3.7] with the reversed order $\preccurlyeq$ and ρ mapping down rather than up). We set up functors as follows:

On objects:

$$D\colon L \mapsto \mathcal{S}(L, \mathbf{3}) \leq \underset{\sim}{3}_t{}^{|L|},$$
$$E\colon X \mapsto \mathcal{X}(X, \underset{\sim}{3}_t) \leq \mathbf{3}^{|X|}.$$

On morphisms:

$$D\colon f \mapsto \text{-} \circ f,$$
$$E\colon \phi \mapsto \text{-} \circ \phi.$$

The full natural duality theorem for $\mathcal{S}$ asserts that D and E define a dual equivalence between $\mathcal{S}$ and $\mathcal{X}$. Specifically:

$$\begin{aligned}
&(\forall L \in \mathcal{S})\ \big(e_L \colon L \to ED(L) \text{ is an isomorphism}\big)\,,\\
&\qquad\qquad\text{where}\quad e_L(a)(f) = f(a)\ \ (\,a \in L\,,\ \ f \in D(L)\,),\\
&(\forall X \in \mathcal{X})\big(\varepsilon_X \colon X \to DE(X) \text{ is an isomorphism}\big)\,,\\
&\qquad\qquad\text{where}\quad \varepsilon_X(x)(\phi) = \phi(x)\ \ (\,x \in X\,,\ \ \phi \in E(X)\,).
\end{aligned}$$

Here again the maps e_L and ε_X given at each point by evaluation are, respectively, the unit and co-unit of the dual adjunction.

This set-up exactly parallels that for the Priestley duality, with $\mathbf{3}$ and $\underset{\sim}{3}_t$ taking the place of $\mathbf{2}$ and $\underset{\sim}{2}_t$. However, as the descriptions of the dual categories $\mathcal{X}$ and $\mathcal{Y}$ suggest, there is a much closer connection. In [3], B. A. Davey revealed that the Priestley-type duality for Stone algebras led directly to the natural duality (and likewise for double Stone algebras). Davey's paper was a first step towards natural duality theory, and also towards the theory of piggyback dualities within this. The key to the link between the two dualities, and to the results in this paper, is the way in which we can move backwards and forwards between, on the one hand, maps into the underlying set $\{0, a, 1\}$ of $\mathbf{3}$ and into its alter ego and, on the other hand, maps into $\{0, 1\}$. Define $g \colon \{0, a, 1\} \to \{0, 1\}$ by $g(1) = 1$, $g(a) = g(0) = 0$. The following proposition originates in [3]. It is set in a wider context in [4] (see in particular Theorem 3.7); an account of the piggyback duality framework as it applies to Stone algebras (viewed as a variety of Ockham algebras) can also be found in [1; §7.5]. The proposition shows that in the case of Stone algebras the natural and expanded Priestley dualities are very closely related.

2.1. Proposition. *Let $L \in \mathcal{S}$. Then there is a structure-preserving bijective correspondence between $X := D(L) = \mathcal{S}(L, \mathbf{3})$ and $Y \colon = H(L) = \mathcal{D}(L, \mathbf{2})$ by the maps $\Phi \colon \mathcal{S}(L, \mathbf{3}) \to \mathcal{D}(L, \mathbf{2})$ and $\Theta \colon \mathcal{D}(L, \mathbf{2}) \to \mathcal{S}(L, \mathbf{3})$ given by*

$$\Phi(x) = g \circ x \qquad (\,x \in \mathcal{S}(L, \mathbf{3})\,)$$

and

$$\Theta(y)(b) = \begin{cases} 1 & \text{if } y(b) = 1\,,\\ a & \text{if } \rho(y)(b) = 1 \text{ and } y(b) = 0\,,\\ 0 & \text{if } \rho(y)(b) = 0 \end{cases}$$

for $y \in \mathcal{D}(L, \mathbf{2})$ and $b \in L$.

In saying that the correspondence set up by Φ and Θ is structure-preserving, we mean that these maps preserve $\preccurlyeq$ and ρ and are continuous.

3. Canonical extensions of Stone algebras

In introducing canonical extensions of lattices we follow [8; §2]. A *completion* of a (bounded) lattice L is defined to be a pair (e, C) where C is a complete lattice and $e\colon L \to C$ is an embedding. Elements of C representable as meets (joins) of elements from $e(L)$ are called *closed* (*open*). The sets of such elements are denoted $K(C)$ ($O(C)$), respectively. A completion (e, C) is said to be *dense* if every element of C is both a join of meets and a meet of joins of elements from $e(L)$; it is said to be *compact* if, for any sets $A \subseteq O(C)$ and $B \subseteq K(C)$ with $\bigwedge A \leq \bigvee B$, there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leq \bigvee B'$. A *canonical extension* of L is by definition a dense and compact completion. It is shown in [8] that any bounded lattice has a canonical extension, and that this is unique up to isomorphism.

Now let $L \in \mathcal{D}$. As in the presentation in [10], we take the canonical extension to be the pair (k_L, L^σ), where L^σ denotes the set of all order-preserving maps from $Y = H(L)$ into $\{0,1\}$, with the pointwise (lattice) order, and $k_L\colon L \to KH(L) \leq L^\sigma$ is the embedding described in Section 2. Viewed another way, L^σ is isomorphic to the lattice of all up-sets of Y, *qua* poset. (Note that for any poset P we have lattice isomorphisms

$$\mathcal{U}(P) \cong \mathcal{U}(P^\partial)^\partial \cong \mathcal{O}(P^\partial),$$

where $\mathcal{O}(P)$ and $\mathcal{U}(P)$ denote, respectively, the lattices of all down-sets and all up-sets of P ordered by $\subseteq$, and $^\partial$ denotes the order dual.)

An element x in a complete lattice C is called *completely join-irreducible* if $x > \bigvee\{y \in C : y < x\}$. A *completely meet-irreducible* element is defined dually. Let $J^\infty(C)$ and $M^\infty(C)$ denote the sets of all completely join-irreducible and completely meet-irreducible elements, respectively. By our definition, 0 (the least element of C) is not completely join-irreducible, because it is the supremum of the empty set. Similarly, $1 \notin M^\infty(C)$. Observe that in a lattice $\mathcal{U}(P)$ the completely join- and meet-irreducible elements are, respectively, the elements ${\uparrow}x$ and $P \setminus {\downarrow}y$ for $x, y \in P$. As we have defined L^σ for $L \in \mathcal{D}$, we have $H(L) \cong J^\infty(L^\sigma)^\partial$ under the map $y \mapsto {\uparrow}y$.

We collect together in Theorem 3.1 facts about lattices arising as L^σ for $L \in \mathcal{D}$. This portmanteau statement is part of the foundational folklore of the theory of complete lattices. For fuller accounts, see for example [5; Theorem 10.29] and in particular [11; Proposition VII-2.10], of which we need the algebraic counterpart, and the references cited there. We remark that it is elementary to prove that any order-isomorphism between complete lattices automatically preserves arbitrary joins and meets (see [5; Lemma 2.27]). Below, "isomorphism" can therefore be interpreted as "isomorphism of complete lattices".

3.1. THEOREM. *Let L be a distributive lattice. Then the following are equivalent:*

(i) *L is isomorphic to $\mathcal{U}(P)$ for some ordered set P;*
(ii) *L is isomorphic to a complete sublattice of $\mathbf{2}^P$, for some set P;*
(iii) *L is completely distributive and is algebraic;*
(iv) *L is doubly algebraic (that is, both L and its order dual L^∂ are algebraic);*
(v) *L is complete, L satisfies the join-infinite distributive law and $J^\infty(L)$ is join-dense in L;*
(vi) *the map $\eta\colon c \mapsto L \setminus \{x \in J^\infty(L) : x \leqslant c\}$ is an isomorphism from L onto $\mathcal{U}\big(J^\infty(L)^\partial\big)$.*

Following the notation of [10], we denote the class of distributive lattices satisfying the equivalent conditions of Theorem 3.1 by DL^+. The following proposition is in the same spirit as results given in [7], where objects in DL^+ are viewed as Heyting algebras. Note also the discussion of Stone negation in [9; Sec. 6].

3.2. PROPOSITION. *Let the lattice L, with underlying order $\leqslant$, belong to DL^+, and represent L as $\mathcal{U}(Y)$ where $Y = J^\infty(L)^\partial$ and $J^\infty(L)$ carries the order induced by $\leqslant$. Then L is pseudocomplemented, with $U^* = Y \setminus {\downarrow}U$. Furthermore, the following are equivalent:*

(i) *each element of $J^\infty(L)$ majorises a unique $\leqslant$-minimal element of $J^\infty(L)$;*
(ii) *L is a Stone algebra.*

Proof. It is well known that any complete lattice satisfying the join-infinite distributive law is pseudocomplemented, and it is easily seen that the pseudocomplement of $U \in \mathcal{U}(Y)$ is given by $Y \setminus {\downarrow}U$; compare [5; 11.23]. In just the same way as in the topologised version, we obtain that $U^* \cup U^{**} = Y$ for all $U \in \mathcal{U}(Y)$ if and only if (i) holds; this is simply an elementary order-theoretic calculation. □

Given a Stone algebra L, the order-theoretic dual $H(L)^\partial$ of its Priestley dual satisfies condition (i) of the above proposition. Since $H(L)$ is shown above to be isomorphic to $Y = J^\infty(L^\sigma)^\partial$, it follows that $J^\infty(L^\sigma)$ also satisfies condition (i) of the proposition. Thus the proposition shows that the canonical extension of a Stone algebra is again a Stone algebra, so that the variety $\mathcal{S}$ is canonical. We denote by $\mathcal{S}^+$ the class $\mathrm{DL}^+ \cap \mathcal{S}$ of doubly algebraic Stone algebras and make $\mathcal{S}^+$ into a category by taking as morphisms the complete Stone algebra homomorphisms, that is, maps preserving arbitrary joins and meets and the pseudocomplement. Note that such maps are automatically $0,1$-preserving, because they preserve the join and the meet of the empty set.

The structures we wish to use on the dual side to represent canonical extensions of Stone algebras are of the form $(X; \preccurlyeq, \rho)$, where $\preccurlyeq$ is a partial order and ρ is a retraction on X and is such that $\rho(x)$ is the unique maximal point in X majorising x. We denote the class of all such structures by $\mathcal{X}^+$ and make this into a category by taking as morphisms the maps preserving the order and the map ρ. We obtain a structure $\underset{\sim}{3} := (\{0, a, 1\}; \preccurlyeq, \rho) \in \mathcal{X}^+$ simply by removing the discrete topology from $\underset{\sim}{3}_t$, the alter ego of $\mathbf{3}$ for the natural duality for $\mathcal{S}$. In addition, Proposition 3.2 shows that, for any $L \in \mathcal{S}^+$, the set $J^\infty(L)^\partial$ carries the structure of an object in $\mathcal{X}^+$.

Observe that in both the categories $\mathcal{S}^+$ and $\mathcal{X}^+$, structures are isomorphic if and only if there is an order-isomorphism between them: an order-isomorphism between the underlying ordered sets of pseudocomplemented complete lattices automatically preserves the pseudocomplement as well as arbitrary joins and meets. Likewise, an order-isomorphism between $\mathcal{X}^+$-objects automatically commutes with the ρ-map.

3.3. Theorem. *The class $\mathcal{X}^+$ of structures is exactly the (purely relational) quasi-variety $\mathbb{ISP}(\underset{\sim}{3})$ consisting of (order-)isomorphic copies of substructures of powers of $\underset{\sim}{3} = (\{0, a, 1\}; \preccurlyeq, \rho)$, the relational structure being lifted pointwise to powers.*

Proof. This is essentially just a topology-free version of results about topological quasi-varieties stated in [1; 1.4.3, 1.4.4, 1.4.7]. □

Before we can obtain an analogous result for $\mathcal{S}^+$ and set up a dual equivalence between $\mathcal{S}^+$ and $\mathcal{X}^+$, which we do in the next section, we need to examine the objects of these categories more closely. In particular, for $X \in \mathcal{X}^+$, we investigate $\underset{\sim}{3}^X$, the set of maps from X into $\underset{\sim}{3}$ which preserve $\preccurlyeq$ and ρ. We order $\underset{\sim}{3}^X$ "schizophrenically"; that is, we define $u \leqslant v$ in $\underset{\sim}{3}^X$ if and only if, for all $x \in X$, we have $u(x) \leqslant v(x)$ with respect to the lattice ordering $0 < a < 1$ of $\mathbf{3}$. Since $\mathbf{3}$ is in fact a complete lattice and a Stone algebra, the power $\mathbf{3}^{|X|}$ is, too, under pointwise-defined $\bigvee$, $\bigwedge$ and *. We shall see shortly that $\underset{\sim}{3}^X$, which sits inside $\mathbf{3}^{|X|}$, is closed under the inherited operations.

3.4. Lemma. *Let $X \in \mathcal{X}^+$ and let $u, v \in \underset{\sim}{3}^X$. Then $u \leqslant v$ if and only if $u^{-1}(1) \subseteq v^{-1}(1)$.*

Proof. If $u \leqslant v$ in $\underset{\sim}{3}^X$, then clearly $u^{-1}(1) \subseteq v^{-1}(1)$. Conversely, let us assume that $u^{-1}(1) \subseteq v^{-1}(1)$. Then we have $\rho^{-1}(u^{-1}(1)) \subseteq \rho^{-1}(v^{-1}(1))$. Because u preserves ρ, for every $x \in X$, it follows that $u(x) = a$ implies $u(\rho(x)) = 1$, whence $u^{-1}(a) \subseteq \rho^{-1}(u^{-1}(1))$. So, for every $x \in u^{-1}(a)$, $\rho(v(x)) = v(\rho(x)) = 1$, that is, $v(x) \in \{a, 1\}$. This shows that $u \leqslant v$. □

For $X \in \mathcal{X}^+$, we have $\mathcal{U}(X) \in \mathcal{S}^+$, thanks to Proposition 3.2. We define $\Gamma\colon \underset{\sim}{3}^X \to \mathcal{U}(X)$ by $\Gamma(c) = c^{-1}(1)$. Since 1 is maximal in $\underset{\sim}{3}$, the set $c^{-1}(1)$ is certainly an up-set in X, so Γ is well defined. We now construct a map going in the opposite direction. For an up-set U in X we define a map $\Psi(U)\colon X \to \underset{\sim}{3}$, called the map determined by U, by

$$\Psi(U)(z) = \begin{cases} 1 & \text{if } z \in U\,, \\ a & \text{if } z \in \rho^{-1}(U) \setminus U\,, \\ 0 & \text{otherwise.} \end{cases}$$

3.5. Lemma. *Let $(X; \preccurlyeq, \rho) \in \mathcal{X}^+$. Then $\Psi(U) \in \underset{\sim}{3}^X$ for each $U \in \mathcal{U}(X)$.*

Proof. We first note that $\rho^{-1}(U)$ is an up-set in $(X; \preccurlyeq)$ and that $U \subseteq \rho^{-1}(U)$. Let $z \preccurlyeq w$ in X. To see that $\Psi(U)(z) \preccurlyeq \Psi(U)(w)$, we only need to show that it cannot happen that $w \in \rho^{-1}(U)$ and $z \notin \rho^{-1}(U)$. But this is easy since $z \preccurlyeq w$ leads to $\rho(w) = \rho(z)$. To see that $\Psi(U)$ preserves ρ, consider $z \in \rho^{-1}(U)$. Then $\Psi(U)(z) \in \{a, 1\}$ and $\rho(z) \in U$, whence $\Psi(U)\big(\rho(z)\big) = 1 = \rho\big(\Psi(U)(z)\big)$. The remaining case is easy as $z \notin \rho^{-1}(U)$ yields $\rho(z) \notin \rho^{-1}(U)$. □

The next proposition enables us to use the maps Ψ and Γ to transfer backwards and forwards between the setting of up-sets on the one hand and of maps into $\{0, a, 1\}$ on the other.

3.6. Proposition. *Let $X \in \mathcal{X}^+$. Then the maps Ψ and Γ are mutually inverse order-isomorphisms between $\mathcal{U}(X)$ and $\underset{\sim}{3}^X$. As a consequence, $\underset{\sim}{3}^X$, with $\bigwedge$, $\bigvee$ and * given pointwise, belongs to $\mathcal{S}^+$. The pseudocomplement α^* of $\alpha \in \underset{\sim}{3}^X$ is determined by the set*

$$(\alpha^*)^{-1}(1) = \big\{x \in X :\ \rho\big(\alpha(x)\big) \neq 1\big\}.$$

Proof. Lemma 3.4 tells us that Γ is an order-embedding from $\underset{\sim}{3}^X$ into $\mathcal{U}(X)$. Also $\Gamma\big(\Psi(U)\big) = U$ for all $U \in \mathcal{U}(X)$, and it is easily checked that, because elements of $\underset{\sim}{3}^X$ preserve ρ, we have $\Psi\big(\Gamma(c)\big) = c$ for all $c \in \underset{\sim}{3}^X$.

The penultimate statement follows from the remarks above concerning isomorphisms. For the explicit description of the pseudocomplement we note that

$$\begin{aligned} x \in \Gamma(\alpha^*) &\iff x \in \Gamma(\alpha)^* && \text{(since } \Gamma \text{ is an order-isomorphism)} \\ &\iff x \in X \setminus {\downarrow}\big(\Gamma(\alpha)\big) && \text{(from Proposition 3.2)} \\ &\iff x \in X \setminus \rho^{-1}\big(\Gamma(\alpha)\big) && \text{(since } \Gamma(\alpha) \text{ is an up-set)} \\ &\iff \rho\big(\alpha(x)\big) \neq 1\,. \end{aligned}$$

(Compare Clark and Davey [1; 4.3.8].) □

For arbitrary x, y in $X \in \mathcal{X}^+$, let $J_x := \Psi(\uparrow x)$ and $M_y := \Psi(X \backslash \downarrow y)$ denote the maps in $\underset{\sim}{3}^X$ determined by the up-sets $\uparrow x$ and $X \setminus \downarrow y$, respectively. The next lemma follows from the description of completely join- and meet-irreducible elements in an up-set lattice and the preservation of such elements under order-isomorphism. Alternatively it can be verified directly.

3.7. Lemma. *Let $X \in \mathcal{X}^+$ and let $C = \underset{\sim}{3}^X$. Then:*

(1) $J^\infty(C) = \{J_x : x \in X\}$ *and* $M^\infty(C) = \{M_y : y \in X\}$.

(2) *Let $c \in C$. Then*

$$c = \bigvee_{x \in c^{-1}(1)} J_x \qquad \textit{and} \qquad c = \bigwedge_{y \in (X \backslash c^{-1}(1))} M_y .$$

3.8. Theorem.

(1) *Let $X \in \mathcal{X}^+$. Then X is isomorphic to $J^\infty\big(\underset{\sim}{3}^X\big)^\partial$.*

(2) *Let $L \in \mathcal{S}^+$. Then L is isomorphic to $\underset{\sim}{3}^{J^\infty(L)^\partial}$.*

Proof.

(1) By Proposition 3.6 we have that $C := \underset{\sim}{3}^X$ belongs to $\mathcal{S}^+$, and from Lemma 3.7 that $J^\infty(C) = \{J_x : x \in X\}$, where J_x is as defined above. The map $x \mapsto J_x$ is an order-isomorphism of X onto $J^\infty\big(\underset{\sim}{3}^X\big)^\partial$.

(2) The required isomorphism is obtained by composing the order-isomorphism from L onto $\mathcal{U}\big(J^\infty(L)^\partial\big)$ provided by Theorem 3.1 with the order-isomorphism Ψ from Proposition 3.6. (As already noted, an order-isomorphism automatically preserves * and arbitrary joins and meets.) □

We let $\mathbf{3}_c$ denote the three-element algebra $\mathbf{3}$ regarded as a complete lattice and a Stone algebra. Thus $\mathbf{3}_c \in \mathcal{S}^+$. Partnering Theorem 3.3 we now have the following characterisation theorem for $\mathcal{S}^+$.

3.9. Theorem. *For a distributive lattice L the following are equivalent:*

(i) $L \in \mathcal{S}^+$;

(ii) $L \in \mathbb{IS}_c\mathbb{P}(\mathbf{3}_c)$, *that is, the class of those algebras which are isomorphic to a complete subalgebra of a power of $\mathbf{3}_c$, the structure being given pointwise on powers.*

Proof.

(i) implies (ii): this is a consequence of Theorem 3.8.

Assume L satisfies (ii). Then L is a Stone algebra, since $\mathcal{S} = \mathbb{ISP}(\mathbf{3})$. Also, L is distributive and doubly algebraic, since any complete subalgebra of a power of $\mathbf{3}_c$ has these properties (see, for example, [11; I-4.12, I-4.14]). Hence the distributive lattice reduct of L satisfies condition (iv) of Theorem 3.1. We conclude that $L \in \mathcal{S}^+$. □

4. Categorical duality

In this section we extend our representations from the previous section into a categorical duality between the categories $\mathcal{S}^+$ and $\mathcal{X}^+$ introduced there. The result we prove is in the same spirit as that of S. K. Thomason [24] for modal logic. His result extends to the more general class of logics and the associated algebras considered by Gehrke, Nagahashi and Venema [9], though it is not explicitly stated in full. Stone algebras are a very special instance of the algebras of the type studied in [9]. What is new here is not that an equivalence exists between the category $\mathcal{S}^+$ of complex algebras and an associated category of frames, but rather the presentation of an equivalence set up by natural hom-functors resembling those used in the full duality theorem for Stone algebras.

We set up the required functors as follows:

On objects:

$$D^+ \colon L \mapsto \mathcal{S}^+(L, \mathbf{3}_c) \le \underset{\sim}{3}^{|L|},$$
$$E^+ \colon X \mapsto \mathcal{X}^+(X, \underset{\sim}{3}) \le \mathbf{3}^{|X|}.$$

On morphisms:

$$D^+ \colon f \mapsto \text{-} \circ f,$$
$$E^+ \colon \phi \mapsto \text{-} \circ \phi.$$

For $X \in \mathcal{X}^+$, what we are now calling $E^+(X)$ is of course just the structure $\underset{\sim}{3}^X$ considered in Section 3. We already know that this belongs to $\mathcal{S}^+$. Clearly, $D^+(L) = \big(\mathcal{S}^+(L, \mathbf{3}_c); \preccurlyeq, \rho\big)$ belongs to $\mathcal{X}^+$ since the order $\preccurlyeq$ of $\mathcal{S}^+(L, \mathbf{3}_c)$ is the extension to $\underset{\sim}{3}^{|L|}$ of the order $\preccurlyeq$ of $\underset{\sim}{3}$ and ρ also acts pointwise in $\mathcal{S}^+(L, \mathbf{3}_c)$.

Observe that, since we are now considering general morphisms and not, as in the preceding section, isomorphisms, there is work to be done to make sure that D^+ and E^+ are well defined on maps. Certainly they then satisfy the conditions to be functors.

4.1. Lemma.

(1) *Let $\phi \colon X \to Y$ be an $\mathcal{X}^+$-morphism. Then $E^+\phi \colon E^+(Y) \to E^+(X)$ is an $\mathcal{S}^+$-morphism.*
(2) *Let $f \colon L \to M$ be an $\mathcal{S}^+$-morphism. Then $D^+f \colon D^+(M) \to D^+(L)$ is an $\mathcal{X}^+$-morphism.*

Proof. By way of illustration we demonstrate preservation of arbitrary joins in (1). It is likewise shown by bracket-pushing that $E^+\phi$ preserves meets and *.

Let $\{\alpha_i\}_{i\in I}$ be a subset of $E^+(Y)$ and define $\alpha := \bigvee_{i\in I} \alpha_i$. We have to show that $E^+\phi(\alpha) = \bigvee_{i\in I} E^+\phi(\alpha_i)$. For all $x \in X$,

$$\begin{aligned}
&\big((E^+\phi)(\alpha)\big)(x)\\
&\quad= \Big(\Big(\bigvee_{i\in I}\alpha_i\Big)\circ\phi\Big)(x) &&\text{(by definition of } E^+\phi \text{ and of } \alpha)\\
&\quad= \Big(\bigvee_{i\in I}\alpha_i\Big)(\phi(x))\\
&\quad= \bigvee_{i\in I}\alpha_i(\phi(x)) &&\text{(joins in } E^+(Y) \text{ being formed pointwise)}\\
&\quad= \bigvee_{i\in I}(\alpha_i\circ\phi)(x)\\
&\quad= \bigvee_{i\in I}(E^+\phi(\alpha_i))(x)\\
&\quad= \Big(\bigvee_{i\in I}E^+\phi(\alpha_i)\Big)(x) &&\text{(joins in } E^+(X) \text{ being formed pointwise).}
\end{aligned}$$

In exactly the same way, in (2), D^+f preserves $\preccurlyeq$ and ρ, since these, too, are given pointwise. □

4.2. Proposition. *For every $L \in \mathcal{S}^+$, the structure $D^+(L)$ is $\mathcal{X}^+$-isomorphic to $J^\infty(L)^\partial$.*

Proof. Let $\alpha\colon L \to \mathbf{3}_c$ be an $\mathcal{S}^+$-morphism. Then the filter $\alpha^{-1}(1)$ of L has a least element, necessarily non-zero, and it is easy to see that this element must be completely join-irreducible. Hence, we can define $j_L\colon D^+(L) \to J^\infty(L)$ by $j_L(\alpha) = \min \alpha^{-1}(1)$. To show that j_L is onto, let $c \in J^\infty(L)$. Let $\alpha\colon L \to \underset{\sim}{3}_c$ be defined as follows:

$$\alpha(x) = \begin{cases} 1 & \text{if } x \geqslant c,\\ a & \text{if } x \geqslant \rho(c) \text{ and } x \ngeqslant c,\\ 0 & \text{otherwise.}\end{cases}$$

(Note that the order here is that induced from L, with respect to which $\rho(c)$ is a minimal element of $J^\infty(L)$.)

We must verify that α is an $\mathcal{S}^+$-morphism. If so, j_L maps $D^+(L)$ onto $J^\infty(L)^\partial$.

We first check that α preserves the lattice order. Let $x \leqslant y$ in L. By transitivity of $\leqslant$ we see that $\alpha(x) = 1$ implies $\alpha(y) = 1$ and $\alpha(x) = a$ implies $\alpha(y) = a$

or 1. Certainly $\alpha(1) = 1$ and $\alpha(0) = 0$, the latter because $\rho(c)$, being completely join-irreducible, cannot be 0. Now consider a non-empty set $\{x_i\}_{i\in I}$ in L. Because α is order-preserving, $\alpha\Big(\bigvee_{i\in I} x_i\Big) \geqslant \bigvee_{i\in I} \alpha(x_i)$ and $\alpha\Big(\bigwedge_{i\in I} x_i\Big) \leqslant \bigwedge_{i\in I} \alpha(x_i)$. We must prove the reverse inequalities. Suppose $\alpha\Big(\bigvee_{i\in I} x_i\Big) = 1$. Each x_i is expressible as $\bigvee_{j\in I_i} x_{ij}$ where $x_{ij} \in J^\infty(L)$ for each $j \in I_i$. Then $\bigvee_{i\in I, j\in I_i} x_{ij} \geqslant c$, from which it follows that there exists $i \in I$ and $j \in I_i$ such that $x_{ij} \geqslant c$. But then $x_i \geqslant c$, too. Hence $\bigvee_{i\in I} \alpha(x_i) = 1$. In a similar way, $\alpha\Big(\bigvee_{i\in I} x_i\Big) = a$ implies that there exists k such that $x_k \geqslant \rho(c)$, so that $\bigvee_{i\in I} \alpha(x_i) \geqslant a$. We deduce that $\bigvee_{i\in I} \alpha(x_i) = \alpha\Big(\bigvee_{i\in I} x_i\Big)$. An easy calculation shows that α preserves meets.

It remains to prove that α preserves *. To do this, it is convenient to identify L with the lattice $\mathcal{O}(Z)$ of down-sets of $Z := J^\infty(L)$, with the completely join-irreducible elements as the down-sets ${\downarrow}p$ for $p \in Z$. For $x \in \mathcal{O}(Z)$, we have $x^* = Z \setminus {\uparrow}x$. Because L is a Stone algebra, $x^* \cup x^{**} = Z$, with the union disjoint; cf. [20; Proposition 2]. Consequently x^* is both an up-set and a down-set and each minimal point of Z belongs to either x^* or to x^{**}, but not to both. It follows that $x \not\geqslant \rho(c)$ if and only if $x^* \geqslant \rho(c)$ (note that x and $x^{**} = Z \setminus x^*$ contain exactly the same set, M say, of minimal points and that a minimal point is in M if and only if it is not in x^*). Accordingly $\alpha(x^*) = a$ never occurs. Since * on $\mathbf{3}_c$ maps $\{0, a, 1\}$, it now suffices to show that $\alpha(x^*) = 0$ if and only if $\alpha(x)^* = 0$ $(x \in L)$. But

$$\alpha(x^*) = 0 \iff x^* \not\geqslant \rho(c) \iff x \geqslant \rho(c) \iff \alpha(x) \in \{a, 1\} \iff \alpha(x)^* = 0.$$

To complete the proof we must show that j_L is an order-isomorphism. Let $\beta, \gamma \in D^+(L)$. By definition, $\beta \preccurlyeq \gamma$ if and only if $\beta(x) \preccurlyeq \gamma(x)$ in $\underset{\sim}{3}$ for all $x \in L$. So $\beta \preccurlyeq \gamma$ implies $\beta^{-1}(1) \subseteq \gamma^{-1}(1)$, whence $\min \beta^{-1}(1) \geqslant_L \min \gamma^{-1}(1)$. Therefore $j_L(\beta) \leqslant j_L(\gamma)$ in $J^\infty(L)^\partial$. Conversely, assume that the last condition holds. It follows that $\beta^{-1}(1) \subseteq \gamma^{-1}(1)$. Let $x \in L$ and suppose that $\beta(x) = 0$. Then $\beta(x^*) = \beta(x)^* = 1$, so $\gamma(x)^* = \gamma(x^*) = 1$. Therefore $\gamma(x) = 0$. In a similar way, we can prove by considering x^{**} that $\beta(x) = a$ implies $\gamma(x) \in \{a, 1\}$. We conclude that $\beta \preccurlyeq \gamma$. $\square$

For every $X \in \mathcal{X}^+$, $L \in \mathcal{S}^+$ we define the maps given at each point by evaluation $\varepsilon_X \colon X \to D^+E^+(X)$ and $e_L \colon L \to E^+D^+(L)$ in the usual way: for $x \in X$, $\alpha \in E^+(X)$, $c \in L$, $\beta \in D^+(L)$ we set

$$\varepsilon_X(x)(\alpha) = \alpha(x), \qquad e_L(c)(\beta) = \beta(c).$$

4.3. PROPOSITION. *For every $X \in \mathcal{X}^+$, $L \in \mathcal{S}^+$, the maps ε_X and e_L are isomorphisms.*

P r o o f . In Theorem 3.8 we constructed isomorphisms $F\colon X \to J^\infty(E^+(X))^\partial$ and $G\colon L \to E^+(J^\infty(L)^\partial)$. By Proposition 4.2 we have an isomorphism

$$i_{D^+(L)}\colon D^+(L) \to J^\infty(L)^\partial .$$

One can verify that $\varepsilon_X = i^{-1}_{D^+(E^+(X))} \circ F$ and $e_L = E^+(i_{D^+(L)}) \circ G$, which shows that ε_X and e_L are isomorphisms. $\square$

4.4. THEOREM. *The functors D^+ and E^+ establish a dual equivalence between the categories $\mathcal{S}^+$ and $\mathcal{X}^+$.*

P r o o f . It is routine to show that $e\colon 1_{\mathcal{S}^+} \to D^+E^+$ and $\varepsilon\colon 1_{\mathcal{X}^+} \to E^+D^+$ are natural transformations, that is, for any $\mathcal{X}^+$-morphism $\phi\colon X \to Y$ and any $\mathcal{S}^+$-morphism $f\colon L \to M$, the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ D^+E^+(X) & \xrightarrow{D^+E^+\phi} & D^+E^+(Y) \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle e_L}\downarrow & & \downarrow{\scriptstyle e_M} \\ E^+D^+(L) & \xrightarrow{E^+D^+f} & E^+D^+(M) \end{array}$$

(We note that the calculations here very much resemble those in B. A. D a v e y and H. W e r n e r [6; 1.5].) $\square$

5. The canonical extension revisited

Fix $L \in \mathcal{S}$ and let Y and X denote respectively its Priestley dual $H(L)$ and its natural dual $D(L)$, regarded as structures in $\mathcal{X}^+$. In Section 3, we investigated the canonical extension of L, defined to be the lattice of up-sets of Y. We showed that this is isomorphic in $\mathcal{S}^+$ to the lattice of $\mathcal{X}^+$-morphisms from Y into $\underset{\sim}{3}$, with pointwise-defined operations from $\mathbf{3}$. This gives some insights into the canonical extension, but the viewpoint is a hybrid one. We moved to the natural duality setting at the second stage (the formation of the completion as a function lattice), but not at the first (the construction of the dual space Y). We now build in the natural correspondence between the Priestley dual and the natural dual of L given in Proposition 2.1, so transferring fully to the natural duality setting.

We can see that this is possible by noting that the order-isomorphism between X and Y lifts to an order-isomorphism between $\mathcal{U}(X)$ and $\mathcal{U}(Y)$, which is necessarily an $\mathcal{S}^+$-isomorphism. Thus the canonical extension can be manifested as $\underset{\sim}{3}^X$, instead of as $\underset{\sim}{3}^Y$.

We now show explicitly how, working wholly within the setting of maps into $\mathbf{3}$ and $\mathbf{3}_c$ and their respective alter egos $\underset{\sim}{3}_t$ and $\underset{\sim}{3}$, we can verify that $(e_L, E^+D(L))$ does indeed yield the canonical extension of $L \in \mathcal{S}$. To do this, it suffices to verify the properties of density and compactness that characterise a canonical extension of L. We first derive two results revealing exactly how L sits inside $E^+D(L)$.

5.1. Lemma. *Let L be a Stone algebra and let $L^\nu := E^+D(L)$. Then*

(1) *every completely join-irreducible element of L^ν can be expressed as*

$$J_x = \bigwedge\bigl(\alpha \in ED(L) \mid \alpha(x) = 1\bigr);$$

(2) *every completely meet-irreducible element of L^ν can be expressed as*

$$M_y = \bigvee\bigl(\alpha \in ED(L) \mid \alpha(y) \neq 1\bigr).$$

Proof.

(1) Let $X := D(L)$ and $x \in D(L)$. Let $A_x := \{\alpha \in ED(L) : \alpha(x) = 1\}$ and let us denote $\alpha_x := \bigwedge(\alpha \mid \alpha \in A_x) \in L^\nu$. (We note that $A_x \neq \emptyset$ since $e_A(1) \in A_x$.) As each $\alpha \in A_x$ is continuous and order-preserving, $\alpha^{-1}(1)$ is a clopen up-set of $D(L)$ containing x and it is easy to see that $\alpha_x^{-1}(1) = \bigcap_{\alpha \in A_x} \alpha^{-1}(1)$. We claim that $\alpha_x^{-1}(1) = {\uparrow}x$. If $z \in {\uparrow}x$, then clearly $z \in \alpha_x^{-1}(1)$. For the converse, assume that $z \notin {\uparrow}x$. We wish to find $\alpha \in A_x$ such that $\alpha(z) \neq 1$. Since the order on $D(L)$ is defined pointwise, there exists $b \in L$ such that $e_L(b)(z) = z(b) \notin {\uparrow}x(b) = {\uparrow}e_L(b)(x)$ in $\underset{\sim}{3}$. We now separate cases. First, if $x(b) = 1$, then we have $e_L(b) \in A_x$, but $e_L(b)(z) \neq 1$, so $\alpha = e_L(b)$ serves. Next, if $x(b) = a$, then $z(b) = 0$. We have $e_L(b^{**})(x) = x(b^{**}) = x(b)^{**} = 1$ while $e_L(b^{**})(z) = z(b)^{**} = 0$. In this case we take $\alpha = e_L(b^{**})$. Finally, if $x(b) = 0$, then $z(b) = a$ or 1. We have $e_L(b^*)(x) = x(b^*) = x(b)^* = 1$ while, similarly, $e_L(b^*)(z) = 0$. So in this case take $\alpha = e_L(b^*)$. We have proved, as claimed, that $\alpha_x^{-1}(1) = {\uparrow}x$. As J_x is the map in $\underset{\sim}{3}^X$ determined by ${\uparrow}x$, this yields $\alpha_x = J_x$ according to Lemma 3.4, which completes the proof of (1).

To prove (2), for any $y \in D(L)$ we let $B_y = \{\alpha \in ED(L) : \alpha(y) \neq 1\}$ and define $\beta_y := \bigvee(\beta \mid \beta \in B_y) \in L^\nu$. (We note that B_y contains $e_L(0)$, and so is non-empty.) Then $z \in \beta_y^{-1}(1)$ if and only if there exists $\beta \in ED(L)$ with $\beta(z) = 1$ and $\beta(y) \neq 1$. We are going to show that β_y is the map in $\underset{\sim}{3}^X$ determined by $D(L) \setminus {\downarrow}y$. Suppose $z \notin {\downarrow}y$. Then there exists $b \in L$ such that $z(b) \notin {\downarrow}y(b)$ in $\underset{\sim}{3}$. In the case that $z(b) = 1$ and $y(b) \neq 1$ we see that $e_L(b)(z) = 1$, but $e_L(b)(y) \neq 1$. Hence $z \in \beta_y^{-1}(1)$. If $z(b) = a$ and $y(b) = 0$, then $e_L(b^{**})(z) = 1$ and $e_L(b^{**})(y) = 0$ yielding $z \in \beta_y^{-1}(1)$. Finally, if $z(b) = 0$ and $y(b) \neq 0$, then $e_L(b^*)(z) = 1$, but $e_L(b^*)(y) \neq 1$. Again we have $z \in \beta_y^{-1}(1)$. So $D(L)\backslash{\downarrow}y \subseteq \beta_y^{-1}(1)$. Conversely, suppose for a contradiction that $z \in {\downarrow}y$. Then for all $\beta \in ED(L)$ we have $\beta(z) \preccurlyeq \beta(y)$ so that there does not exist β with $\beta(z) = 1$, but $\beta(y) \neq 1$. We deduce that $\beta_y^{-1}(1) = M_y^{-1}(1)$, whence $\beta_y = M_y$, as required. □

5.2. Proposition. *Let L be a Stone algebra and let $L^\nu := E^+D(L)$. Then*

(1) *every element $c \in L^\nu$ can be expressed as*

$$c = \bigvee_{x \in c^{-1}(1)} \Big(\bigwedge \{ \alpha \in ED(L) : \ \alpha(x) = 1 \} \Big);$$

(2) *every element $c \in L^\nu$ can be expressed as*

$$c = \bigwedge_{y \in (X \backslash c^{-1}(1))} \Big(\bigvee \{ \alpha \in ED(L) : \ \alpha(y) \neq 1 \} \Big).$$

Proof. According to Lemma 3.7, every element $c \in L^\nu$ can be expressed as

$$c = \bigvee_{x \in c^{-1}(1)} J_x = \bigwedge_{y \in (X \backslash c^{-1}(1))} M_y .$$

Now (1) and (2) follow from Lemma 5.1 above. □

As a consequence of Proposition 5.2, $(e_L, E^+D(L))$ is a dense completion of L. We remark that it is not difficult to show that density is equivalent to the following statement for L and L^ν as above: every interval $[c,d]$ in L^ν with $c \in J^\infty(L^\nu)$ and $d \in M^\infty(L^\nu)$ contains an element of $ED(L)$.

Compactness of L^ν is equally easy to establish, since it follows from the (known) compactness of $D(L)$ as a topological space. This is immediate from [8; Lemma 2.4], which asserts that compactness of L^ν is equivalent to the requirement that for S, $T \subseteq L$,

$$\bigwedge e(S) \leqslant \bigvee e(T) \iff \bigwedge e(S') \leqslant \bigvee e(T') \qquad \text{for some finite } \ S' \subseteq S, \ T' \subseteq T .$$

We remark that, because $X = D(L)$ has a subbasis of clopen sets each of which is either an up-set or a down-set, it can easily be shown that compactness of the space X follows from the compactness condition on L^ν. However, in this direction it is perhaps easier still to draw on the well-known facts about Priestley duality and the fact that $D(L)$ and $H(L)$ are homeomorphic and order-isomorphic.

We arrive at the following theorem.

5.4. THEOREM. *Let L be a Stone algebra. Then $\big(e_L, E^+D(L)\big)$ is a dense and compact completion of L and so is, up to isomorphism, the canonical extension of L.*

As the notation in the preceding discussion indicates, we can view the formation of the canonical extension on objects as a transition from $\mathcal{S}$ to $\mathcal{S}^+$ via the composition of D and E^+ on objects. Since each of D and E^+ is a functor, we also have a corresponding extension of homomorphisms, whereby an $\mathcal{S}$-morphism $f\colon L \to M$ extends to an $\mathcal{S}^+$-morphism $f^\nu\colon L^\nu \to M^\nu$, given by $f^\nu = E^+Df$. A definition-chase shows that

$$\big(f^\nu(\alpha)\big)(x) = \alpha(x \circ f) \qquad (\alpha \in L^\nu,\ \ x \in \mathcal{S}(M, \mathbf{3})\,).$$

This shows the way in which our construction is functorial.

There is a little more that can instructively be said about the alternative perspectives on canonical extensions for Stone algebras. First we show how the canonical extension fits into the piggyback duality framework. The idea behind the piggyback method as it applies to varieties of DLEs is to use a known expanded Priestley duality to validate a hoped-for natural duality. Specifically, let us consider $\mathcal{S}$. It can be shown by very general arguments that the map $e_L\colon L \mapsto ED(L)$ given at each point by evaluation is an embedding for each $L \in \mathcal{S}$. The piggyback method seeks to show that e_L is surjective by exploiting the known surjectivity of the map $k_L\colon L \mapsto KH(L)$ given at each point by evaluation for each $L \in \mathcal{S}$. It is an elementary set theoretic exercise to show that e_L is surjective if we can construct a well-defined injection $\Delta\colon ED(L) \to KH(L)$ such that $\Delta \circ e_L = k_L$. The construction is carried out in the following way. Let $\phi \in ED(L)$. Then ϕ maps $\mathcal{S}(L, \mathbf{3})$ into $3 = \{0, a, 1\}$. We now try to define $\Delta(\phi)$ on $Y = H(L)$ so that the right-hand diagram in Fig. 1 commutes. (Recall that $g\colon \{0, a, 1\} \to \{0, 1\}$ is given by $g(1) = 1$, $g(a) = g(0) = 0$.) The situation is simpler here than in the piggyback method in general because Φ is a bijection, so that $\Delta(\phi)$ is definable directly as the composite $g \circ \phi \circ \Gamma$.

$$\begin{tikzcd} & KH(L) \\ L \arrow[ur, "k_L"] \arrow[dr, "e_L"'] & \\ & ED(L) \arrow[uu, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} X = \mathcal{S}(L, \mathbf{3}) \arrow[r, "\phi"] \arrow[d, shift left, "\Phi"] & \underset{\sim}{3}_t \arrow[d, "g"] \\ \mathcal{D}(L, \mathbf{2}) \arrow[u, shift left, "\Gamma"] \arrow[r, "\Delta(\phi)"] & \underset{\sim}{2}_t \end{tikzcd}$$

FIGURE 1.

If we drop the continuity restriction on ϕ, so that we consider ϕ as an element of $E^+D^+(L)$ rather than an element of $ED(L)$, then we extend the injection $\Delta\colon ED(L) \to KH(L)$ to an injection $\Delta^+\colon E^+D^+(L) \to L^\nu$. What we have shown is that the piggyback construction lifts up to the level of the canonical extensions. Thus the canonical extension translates into the natural duality setting in a piggyback way, as we would expect.

6. Double Stone algebras and regular double Stone algebras

A double Stone algebra is of the form $(L; \vee, \wedge, {}^*, {}^\circ, 0, 1)$ for which $(L; \vee, \wedge, {}^*, 0, 1)$ and $(L; \wedge, \vee, {}^\circ, 1, 0)$ are Stone algebras. The variety $\mathcal{DS}$ coincides with the quasi-variety generated by the algebra $\mathbf{4} = (\{0, a, b, 1\}; \vee, \wedge, {}^*, {}^\circ, 0, 1)$, where the underlying bounded distributive lattice is the four-element chain with $0 < a < b < 1$ and

$$1^* = b^* = a^* = 0\,, \quad 0^* = 1\,, \qquad 0^\circ = a^\circ = b^\circ = 1\,, \quad 1^\circ = 0\,.$$

We let $\mathcal{DS}^+$ denote $\mathrm{DL}^+ \cap \mathcal{DS}$ and $\mathcal{X}^+_{\mathcal{DS}}$ the class of structures $(X; \preccurlyeq, \rho, m)$ where $\preccurlyeq$ is a partial order and ρ and m are retractions on X for which $\rho(x)$ and $m(x)$ are, respectively, the unique maximal point above x and the unique minimal point below it.

A natural duality for $\mathcal{DS}$ is set up in the customary way by taking as alter ego $\underset{\sim}{4}_t = (\{0, a, b, 1\}; \preccurlyeq, \rho, m, \tau)$, where, as usual, τ is the discrete topology, $\preccurlyeq$ is the partial order shown in Fig. 2, and $(\{0, a, b, 1\}; \preccurlyeq, \rho, m) \in \mathcal{X}^+_{\mathcal{DS}}$.

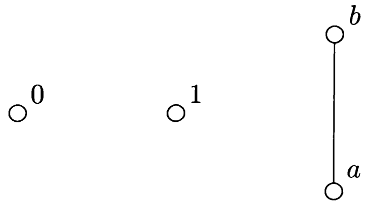

FIGURE 2.

Note that, as for Stone algebras, the alter ego structure we use does not exactly match that given in [1]. Our choice allows a smooth translation to Priestley duality (in terms of up-sets, or homomorphisms into $\mathbf{2}$) and fits into the wider framework of [22].

Just as for Stone algebras, this natural duality for $\mathcal{DS}$ is "essentially the same as" the expanded Priestley duality. This good behaviour stems from there being a $\mathcal{DS}$ analogue of Proposition 2.1.

The development of a hom-functor based theory of canonical extensions for double Stone algebras now proceeds in exactly the same way as that for Stone algebras.

Let $X \in \mathcal{X}^+_{\mathcal{DS}}$. We now define maps $\Psi_{\mathcal{DS}}\colon \mathcal{U}(X) \to \underset{\sim}{4}^X$ and $\Gamma_{\mathcal{DS}}\colon \underset{\sim}{4}^X \to \mathcal{U}(X)$, which play the same role for $\mathcal{DS}$ as the maps Ψ and Γ from Section 3 do for $\mathcal{S}$. These maps are given by $\Gamma_{\mathcal{DS}}(c) = c^{-1}(\{1,b\})$ for every $c \in \underset{\sim}{4}^X$ and

$$\Psi_{\mathcal{DS}}(U)(z) = \begin{cases} 1 & \text{if } z \in m^{-1}(U)\,, \\ b & \text{if } z \in U \setminus m^{-1}(U)\,, \\ a & \text{if } z \in \rho^{-1}(U) \setminus U\,, \\ 0 & \text{if } z \notin \rho^{-1}(U)\,. \end{cases}$$

The entire framework of Sections 3 and 4 can now be transferred, *mutatis mutandis*, to yield corresponding results for $\mathcal{DS}$. We omit the details.

Finally we consider the variety $\mathcal{R}$ of regular double Stone algebras. More advanced duality theory is involved here, and our purpose in including a discussion of this variety is to show that, notwithstanding, a "fully pointwise" construction of canonical extensions is still available.

Regular double Stone algebras have several different incarnations. For our purposes it is most appropriate to treat $\mathcal{R}$ as the subvariety generated by the three-element chain $\mathbf{3} = \{0, a, 1\}$ *qua* double Stone algebra. It is characterised implicationally within $\mathcal{DS}$ as follows: ($x^* = y^*$ and $x^\circ = y^\circ$) implies $x = y$. Alternatively, $\mathcal{R}$ can be viewed as the variety of three-valued Łukasiewicz algebras. For further details see [25].

It is easy to show that in fact $\mathcal{R} = \mathbb{ISP}(\mathbf{3})$. From this it follows, with the aid of [1; 1.3.1] and the expanded Priestley duality for double Stone algebras, that the Priestley dual spaces of algebras in $\mathcal{R}$ have the characterising property of having each point either maximal or minimal, or both. Every such space arises in the following manner. Take two disjoint copies Z^1 and Z^2 of a Boolean space Z, and denote by z^1 and z^2 the points of these sets coming from $z \in Z$. Give $X := Z^1 \cup Z^2$ the disjoint union topology; order it by putting $z^1 < z^2$ for all $z \in Z$ and imposing no other non-trivial comparabilities. Fix a closed subset W of Z and form an ordered quotient Y of X by identifying z^1 and z^2 for $z \in W$. Pictorially, we have two homeomorphic "layers", with certain corresponding points pinched together. Let A be the regular double Stone algebra whose dual is Y. Then Z is the dual of the centre of A:

$$\begin{aligned} C(A) &= \{a \in A : \ a = a^{**}\} = \{a^* : \ a \in A\} = \{a \in A : \ a = a^{++}\} \\ &= \{a^+ : \ a \in A\}\,. \end{aligned}$$

Under the duality for Boolean algebras, there is a bijective correspondence between filters of the Boolean lattice $C(A)$ and closed subsets of its dual space Z.

The algebra A is completely determined by the pair (Z, W) or, algebraically, by the pair $\big(C(A), F(A)\big)$, where $F(A)$ is the filter dual to W. Katriňák's representation of regular double Stone algebras is obtained by explicitly building an algebra A out of a pair (B, F), where B is a Boolean algebra and F is a filter of B, either in a sheaf-theoretic way or in a purely algebraic manner, as described in [2].

Comer's construction of the canonical (*alias* perfect) extension A^σ of A in $\mathcal{R}$ is obtained as the algebra associated with the pair (C_1, F_1) where C_1 is the canonical extension of $C(A)$ and F_1 is the principal filter generated by $\bigwedge_{C_1} F(A)$. From the way that it is obtained, A^σ is a regular double Stone algebra. But some work is needed to show that it has the properties (now known as density and compactness) characteristic of a canonical extension and it is not immediately visible how the additional operations $*$ and $+$, on A^σ, are derived from the corresponding operations on A.

We now consider how the canonical extension can be viewed from the perspective of natural duality. When compared with $\mathcal{S}$ and $\mathcal{DS}$, the variety $\mathcal{R}$ raises some new issues as regards natural duality theory. The reason for this is that we cannot, for arbitrary $A \in \mathcal{R}$, find a surjective map from $\mathcal{R}(A, \mathbf{3})$ onto $\mathcal{D}(A, \mathbf{2})$ of the form $\alpha \circ$ -, where α is a fixed element of $\mathcal{D}(\mathbf{3}, \mathbf{2})$ (cf. Proposition 2.1). However, trivially, $\mathcal{D}(A, \mathbf{2})$ is the union of the images of $\mathcal{R}(A, \mathbf{3})$ under the maps $\alpha_1 \circ$ - and $\alpha_2 \circ$ -, where the carrier maps α_1 and α_2 are, respectively, the elements of $\mathcal{D}(\mathbf{3}, \mathbf{2})$ which map a to 1 and a to 0. An analogous observation, in the context of Kleene algebras, initiated the theory of multi-sorted dualities [4] (see also [1; §7.1]).

To obtain the multi-sorted duality which we require for $\mathcal{R}$, we may draw on the unified study of varieties of double MS-algebras in [23; §4], in which $\mathcal{R}$ is the subvariety labelled $\mathbf{2}$ (and $\mathcal{DS}$ is labelled as $\mathbf{4}$). Alternatively, we may appeal to the theory as presented for Łukasiewicz algebras in [21]. Either way, the dual space of an algebra $A \in \mathcal{R}$ consists of two disjoint copies X^1 and X^2 of $\mathcal{D}(A, \mathbf{3})$, one for each carrier map. The schizophrenic object $\underset{\sim}{M}_t$ may be viewed as being based on disjoint copies $M^i := \{0^i, a^i, 1^i\}$ $(i = 1, 2)$ of $\{0, a, 1\}$. For the algebraic persona we regard each copy as an algebra isomorphic to $\mathbf{3}$. On the relational side, we have relations which are subsets of $M^i \times M^j$ $(i, j \in \{1, 2\})$. Specifically, we include

$$r = \big\{(0^1, 0^2), (a^1, a^2), (1^1, 1^2)\big\} \subseteq M^1 \times M^2$$

suppressing (because these relations will be preserved automatically by the maps we need to consider) the discrete orders on M^1 and M^2. Together, these three relations can be viewed as yielding a partial order on $M^1 \cup M^2$. Finally we consider the two relations

$$r_{12} = \{(0^1,0^2),(1^1,1^2)\} \subseteq M^1 \times M^2\,,$$
$$r_{21} = \{(0^2,0^1),(1^2,1^1)\} \subseteq M^2 \times M^1\,.$$

We only need to include one of these two relations in our relational type, and shall take r_{12}. The dual space $D(A) = X = X^1 \cup X^2$ of an algebra A acquires its relational structure pointwise. Thus, for $y \in X^1$ and $z \in X^2$, we have $(y,z) \in r^X$ if and only if $\big(y(b),z(b)\big) \in r$ for all $b \in A$. This says precisely that $y = x^1$ and $z = x^2$, where x^1 and x^2 are set-theoretically the same map from A into $\{0,a,1\}$, regarded respectively as having codomain M^1 and M^2. Also, $(y,z) \in r_{12}^X$ if and only if $\big(y(b),z(b)\big) \in r_{12}$ for all $b \in A$. This happens if and only if y and z both come from the same map $x\colon A \to \{0,1\}$. A disjoint union topology, lifted from the discrete topology on $M^1 \cup M^2$, is then imposed on X.

To recapture the algebra A from its dual structure X, we consider the set $E(X)$ of all continuous maps $\phi\colon X^1 \cup X^2$ to $M^1 \cup M^2$ which map X^i into M^i $(i = 1,2)$ and which preserve the relations r and r_{12} in the obvious sense; this set acquires the structure of an algebra in $\mathcal{R}$ pointwise from $\mathbf{3}$. All this can be made functorial, so that D and E become functors between $\mathcal{R}$ and a suitable topologico-relational category $\mathcal{X}_{\mathcal{R}}$ generated, in an appropriate sense, from $\underset{\sim}{M}_t$. Indeed, we have a dual equivalence and in particular an isomorphism $A \cong ED(A)$ for each $A \in R$.

Now (at last!) our definition of the canonical extension A^σ for $A \in \mathcal{R}$ is obvious: as a set, A^σ consists of all maps $\phi\colon X^1 \cup X^2$ to $M^1 \cup M^2$ which map X^i into M^i $(i = 1,2)$ and which preserve the relations r and r_{12}. Because ϕ is required to preserve r and r_{12}, $\phi(x^1) = \phi(x^2)$ for all $(x^1,x^2) \in r^X$ and $\phi(x) \in \{0,1\}$ whenever $\operatorname{im} x = \{0,1\}$. We make A^σ into an algebra by imposing * and $^+$, as well as the lattice operations, pointwise from $\mathbf{3}$. In particular, ϕ^* and ϕ^+ are obtained as follows from ϕ: for all $x \in X$,

$$\phi^*(x) = \big(\phi(x)\big)^* = \begin{cases} 1 & \text{if } \phi(x) = 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi^+(x) = \big(\phi(x)\big)^+ = \begin{cases} 0 & \text{if } \phi(x) = 1\,, \\ 1 & \text{otherwise.} \end{cases}$$

In the same way that we obtained discrete (that is, topology-free) dual categories for $\mathcal{S}^+$ and $\mathcal{DS}^+$, we can define a category $\mathcal{X}_{\mathcal{R}}^+$, generated by $\underset{\sim}{M}$, the same structure as $\underset{\sim}{M}_t$, but with the topology omitted, which will be dually equivalent to $\mathcal{R}^+ = \mathrm{DL}^+ \cap \mathcal{R}$.

Finally, we note that the natural duality for $\mathcal{R}$ and its expanded Priestley duality are related in exactly the way one would expect. The requisite translation process is described in [23; Theorem 3.8], or alternatively in [21; Theorem 3.9].

For the very special case of $\mathcal{R}$ this is extremely simple. The two disjoint pieces X^1 and X^2 of the natural dual $X = D(A)$ are the spaces we previously called Z^1 and Z^2. The relations r and r_{12}, interpreted on X, encode respectively the ordering of X as two antichain layers and the subset W determining the quotienting map which identifies points of $Z^1 \cup Z^2$ to yield the Priestley dual space $H(A)$.

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Received August 16, 2004
Revised November 30, 2005

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