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SIGNED STATES ON A LOGIC

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In the paper the notion of signed state on a logic will be studied. The decomposition of signed states as a difference of two positive signed states on a modular logic of finite rank will be shown. Finally, the important case of signed states on a logic $L(H)$ of all closed subspaces of a separable Hilbert space H of dimension at least 3 will be solved; it deals with a generalization of the known Gleason theorem. This paper is based on a dissertation [2].

1. Signed states on a logic

Let L be a σ — lattice with the first and the last elements 0 and 1, respectively, and an orthocomplementation $\perp$: $a \mapsto a^\perp$, a, $a^\perp \in L$, which satisfies

(i) $(a^\perp)^\perp = a$ for all $a \in L$;

(ii) if $a < b$, then $b^\perp < a^\perp$; (1)

(iii) $a \vee a^\perp = 1$ for all $a \in L$.

We further assume that if $a, b \in L$ and $a < b$, then $b = a \vee (a^\perp \wedge b)$. A σ — lattice L satisfying the above axioms will be called a logic ([5]).

Let L be a logic. We say that $a, b \in L$ are orthogonal and write $a \perp b$ if $a < b^\perp$. An observable is a map x from the Borel sets $B(R_1)$ of R_1 into a logic L, which satisfies (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$, $\{E_i\} \subset B(R_1)$.

A signed state is a map m from L into $R_1 \cup \{+\infty\} \cup \{-\infty\}$ such that

(i) $m(O) = O$;

(ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$, $a_i \perp a_j$, $i \neq j$, (2)

$\{a_i\} \subset L$;

and it may attain at most one of the values $\pm\infty$. A signed state m is positive (negative) if $m(a) \geqq 0$ ($m(a) \leqq 0$) for all $a \in L$. A state is a positive signed state m such that $m(1) = 1$.

Lemma 1. 1. *Let m be a signed state, $a<b$, $|m(b)|<\infty$. Then $|m(a)|<\infty$.*

Lemma 1. 2. *If $\{a_i\}$ is a sequence of mutually orthogonal elements of L such that $\left|m\left(\bigvee_{i=1}^{\infty} a_i\right)\right|<\infty$, then the series $\sum_{i=1}^{\infty}|m(a_i)|$ converges.*

Lemma 1.3. *If $a_1<a_2<\ldots$, $\bigvee_{i=1}^{\infty} a_i=a$, then $m(a)=\lim_i m(a_i)$. If $a_1>a_2>\ldots$, $\bigwedge_{i=1}^{\infty} a_i=a$, $|m(a_i)|<\infty$ at least for one i, then $m(a)=\lim_i m(a_i)$.*

The proofs of the last three lemmas are omitted; they are the same as in [3].

Let $O(L)$ be the set of all observables of L. A signed state function is a map P: $x\mapsto P_x$, $x\in O(L)$, which assigns to each observable $x\in O(L)$ a signed measure P_x on $B(R_1)$ such that for any real valued function f on R_1 and any observable x one has

$$P_{f\circ x}(E)=P_x(f^{-1}(E)),\quad E\in B(R_1), \tag{3}$$

where $f\circ x$ is an observable $f\circ x$: $E\mapsto x(f^{-1}(E))$. The notion of the signed state function closely connects with the concept of the signed state:

Theorem 1.4. *Let L be a logic, $O(L)$ the set of all observables and m be a signed state on a logic. If we define, for any observable $x\in O(L)$ and any Borel set $E\in B(R_1)$, a function*

$$P_x^m\colon E\mapsto m(x(E)),$$

then P_x^m is a signed state function. Coversely, if P is a signed state function, then there is a unique signed state m on L such that $P_x(E)=m(x(E))$ for all x and E.

Proof. The first part of our theorem is evident.

The second part. Let P be a signed state function. Let us put

$$m(a)=P_{q_a}(\{1\}) \tag{4}$$

for each $a\in L$, where q_a is a question, that is such a unique observable that $q_a(\{0\})=a^{\perp}$, $q_a(\{1\})=a$. Then m is a signed state on L. Indeed, $m(O)=P_{q_O}(\{1\})=0$. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of mutually orthogonal elements of L and $a=\bigvee_{i=1}^{\infty} a_i$. Let x be a unique observable such that $x(\{0\})=a^{\perp}$, $x(\{i\})=a_i$, $i=1, 2, \ldots$. If $f_i=X_{\{i\}}$, $i=1, 2, \ldots$, and $f=X_{\{1,2,\ldots\}}$, then $f_i\circ x$ is a question q_{a_i}, $f\circ x$ is a question q_a and, by (4) and (3), we have $m(a_i)=P_x(\{i\})$, $m(a)=P_x(\{1, 2, \ldots\})$. Since P_x is a signed measure it follows that $m(a)=P_{q_a}(\{1\})=P_{f\circ x}(\{1\})=P(\{1, 2, \ldots\})=\sum_{i=1}^{\infty}P_x(\{i\})=\sum_{i=1}^{\infty}m(a_i)$.

The uniqueness of m follows from the equation (4).

Q.E.D.

Theorem 1.5. *Let $M(L)$ be the set of all bounded signed states on L, then $M(L)$ is a real vector space with respect to the usual addition and the multiplication by real scalars. The number $\|m\| = \sup_{a \in L} |m(a)|$, $m \in M(L)$, defines the norm of m with respect to which $M(L)$ is a Banach space.*

Proof. Only the completeness of the norm. Let $\{m_n\}$ be a Cauchy sequence of elements from $M(L)$. There is a number $m(a) = \lim_n m_n(a)$ for any $a \in L$. We shall show that a function m: $a \mapsto m(a)$, $a \in L$, is a bounded signed state. We have $m(O) = 0$ and m is finitely additive function.

Now let $a = \bigvee_{i=1}^{\infty} a_i$, $a_i \perp a_j$, $i \neq j$, then

$$\left|m(a) - \sum_{i=1}^{k} m(a_i)\right| \leq |m(a) - m_n(a)| + \left|m_n(a) - m_n\left(\bigvee_{i=1}^{k} a_i\right)\right| + \\ + \left|m_n\left(\bigvee_{i=1}^{k} a_i\right) - m\left(\bigvee_{i=1}^{k} a_i\right)\right| .$$

If $\varepsilon > 0$ is given, then the first and the third member is smaller than $\frac{\varepsilon}{3}$ for some n, by the uniform convergence of $\{m_n\}$. The middle member is smaller than $\frac{\varepsilon}{3}$, by the σ — additivity of m_n. Hence m is a bounded signed state on L and the property $\|m_n - m\| \to 0$ follows from the uniform convergence.

Q.E.D.

2. Decomposition of signed states on a modular logic of finite rank

Let L be a logic. By a chain in L we mean a strictly increasing finite sequence of nonzero elements, that is, $\{a_1, a_2, \ldots, a_n\}$, $a_1 < a_2 < \ldots < a_n$, $a_1 \neq O$, $a_i \neq a_j$ if $i \neq j$. We shall say that L has a finite rank if there is an integer k such that every chain in L has at most k elements. A logic L is modular if for any three elements $a, b, c \in L$ for which $c < a$, one has

$$a \wedge (b \vee c) = (a \wedge b) \vee c . \tag{5}$$

A nonzero element x in a logic L is called an atom if for any element $y < x$ either $y = x$ or $y = O$.

By a valuation on a logic L we mean a real valued function v defined on L with the following properties

(i) $v(O) = 0$, $v(a) \geq 0$ for all $a \in L$;

(ii) if $a<b, a\neq b$, then $v(a)<v(b)$; (6)

(iii) $v(a\wedge b)+v(a\vee b)=v(a)+v(b)$ for all a, $b\in L$.

In [5] it is shown that if L is a modular logic of finite rank, then every nonzero element of L is a lattice sum of orthogonal atoms. Moreover, there exists a unique valuation v on L such that $v(x)=1$ for every atom $x\in L$, and a logic of finite rank is modular if and only if it possesses a valuation (Theorem 2.8 [5]).

From this proposition it follows that if $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ are sets of orthogonal atoms such that $a_1\vee a_2\vee \ldots\vee a_n = b_1\vee b_2\vee\ldots\vee b_m$, then $n=m$ and if v is a valuation from the above proposition, then $v(b)$ is the maximal number of orthogonal atoms $x<b$.

Now we are in a position to say the main theorem of this section.

Theorem 2.1. *Let L be a modular logic of finite rank. Then every bounded signed state m on L may be expressed as a difference of two positive signed states m_1, m_2, that is,*

$$m=m_1-m_2. \tag{7}$$

Proof. From our assumption it follows that there is a valuation v on L such that $v(x)=1$ for every atom $x\in L$. Let m be an arbitrary bounded signed state. We define a real valued function w_m on atoms of L by $w_m(a)=m(a)$, a — atom. w_m is a bounded function on atoms of L. Let $K=-\inf\{w_m(a)\colon a-\text{atom}\}$, then $K\geq 0$ and $w_1(\cdot)=w_m(\cdot)+K$ is a nonnegative function. The function $m_2(b)=Kv(b)$ is a positive signed state and so is also $m_1=m+m_2$, because if $b\in L$ and $\{a_1, a_2, \ldots, a_n\}$ is a set of orthogonal atoms such that $\bigvee_{i=1}^{n} a_i=b$, then $m_1(b)=m(b)+m_2(b)$ $=\sum_{i=1}^{n} m(a_i)+Kv(b) = \sum_{i=1}^{n}(m(a_i)+K) = \sum_{i=1}^{n} w_1(a_i)\geq 0$.

It is easily seen that $m=m_1-m_2$ is the searched decomposition.

Q.E.D.

In the third section of this paper the decomposition of bounded states on a logic of a separable Hilbert space will be investigated (Theorem 3.5).

3. Signed states on a logic od a Hilbert space

Let H be a separable Hilbert space over complex or real scalars with the inner product $(\cdot, \cdot)$. The complete lattice $L(H)$ of all closed subspaces M of a Hilbert space H is a logic. This logic is one of the most important examples of logics. Modern quantum theory works with the assumption that the logic of any atomic system is isomorphic to logic $L(H)$ for some separable Hilbert space H.

Since there is a one-to-one correspondence between the closed subspace M of H and its projector P^M, we shall write M for a subspace as well as its projector.

The excellent theorem of Gleason ([5]) asserts that every state m on $L(H)$ (that is, positive signed state m such that $m(H)=1$) is of the form $m(M)=\operatorname{tr}(TM)$, $M \in L(H)$, where T is a Hermitean positive operator of trace class, $\operatorname{tr}(T)=1$.

A Hermitean operator T on H is an operator of trace class if there is an orthonormal bas $\{f_i\}$ such that $\sum_i |(Tf_i, f_i)| < \infty$, then the sum $\operatorname{tr}(T)=\sum_i (Tf_i, f_i)$ is called a trace of T and it is independent of the used basis ([4]).

Let T be a Hermitean operator of trace class, then the function $m_T(M)=\operatorname{tr}(TM)$ is a bounded signed state ([4]). A signed state m is called regular if there is a Hermitean operator of trace class such that $m=m_T$. Then T is unique (use all one-dimensional subspaces of H), and $m_T+m_S=m_{T+S}$, $am_T=m_{\alpha T}$ holds, where T, S are Hermitean operators of trace class and α is a real scalar.

Theorem 3.1. *Let $R(H)$ be the set of all regular signed states on $L(H)$ and $M(H)$ be the set of all bounded signed states on $L(H)$. Then $R(H)$ is a Banach subspace of $M(H)$.*

Proof. Let $\{m_{T_n}\}$ be a Cauchy sequence of regular signed states. There is a bounded signed state $m \in M(H)$ such that $\|m - m_{T_n}\| \to 0$. We shall show that m is regular. Let us denote by $\hat{f}$ the projector operator generated by a unit vector f, that is, $\hat{f}x=(x,f)f$ for $x \in H$. Then $m(\hat{f})=\lim_n m_{T_n}(\hat{f}) = \lim_n \operatorname{tr}(T_n\hat{f})=\lim_n (T_nf, f)$. Hence there is a Hermitean operator $T=w-\lim_n T_n$.

If $\{f_i\}$ is an orthonormal base, then $(Tf_i, f_i)=\lim_n (T_nf_i, f_i) = m(\hat{f}_i)$, $i=1, 2, \ldots$.

The series $\sum_i (Tf_i, f_i)$ converges absolutely because $\sum_i (Tf_i, f_i) = \sum_i m(\hat{f}_i)=m(H)$ and the series $\sum_i m(\hat{f}_i)$ converges absolutely, by Lemma 1.2. Hence T is an operator of trace class.

It remains to show that $m=m_T$. Let $M \in L(H)$ and let $\{g_i\}$ be a base in M. Then

$$m(M) = \sum_j m(\hat{g}_j) = \sum_j (Tg_j, g_j) = \operatorname{tr}(TM).$$

Q.E.D.

There arises a natural question: is $R(H)$ equal to $M(H)$? For the definite answer we need the following notions and lemmas.

Let H be a Hilbert space, by $G(H)$ we denote the unit sphere in a Hilbert space. A real valued function w on $G(H)$ is called a weight function if

(i) $w(cf)=w(f)$, $|c|=1$, $f \in G(H)$;

(ii) there is a constant W such that $\sum_i w(f_i)=W$ for every orthonormal base $\{f_i\}$.

The number W is called the weight of the weight function w.

Suppose T to be a Hermitean operator of trace class, then $w_T(f)=(Tf,f)$, $f \in G(H)$, is a bounded weight function with the weight $\operatorname{tr}(T)$. A weight function w is regular if there is a Hermitean operator T of trace class such that $w=w_T$.

Let m be a bounded signed state, then a function $w_m(f)=m(\hat{f})$ is a bounded weight function, too. To prove the equality $R(H)=M(H)$ it is necessary and sufficient to show that every w_m is a regular weight function.

Lemma 3.2. *Let w be a bounded weight function on $G(H)$, $\dim H=3$. Then w is regular.*

Proof. Denote $k=\inf\limits_{\|f\|=1} w(f)$, then $w_1(f)=w(f)-k$ is a positive weight function. By Lemma 7.22 ([5]), w_1 is regular and hence there is a Hermitean operator T on H such that $w_1(f)=(Tf,f)$. Therefore $w(f)=w_1(f)+k = (Tf,f)+k(f,f) = ((T+kI)f,f)$, where I is the identic operator on H. We have proved to w is a regular weight function.

Q.E.D.

Lemma 3.3. *Let H be a separable Hilbert space of dimension at least 3. Then every bounded weight function w on $G(H)$ is regular.*

Proof. The restriction of a bounded weight function w on any subspace is a bounded weight function, too. Since every two-dimensional subspace N may be imbedded in a threedimensional subspace, it follows from Lemma 3.2 that the restriction of w to the unit sphere $G(N)$ of N is regular, and consequently there is a unique symmetric bilinear form Φ_N such that $\Phi_N(u,u)=w(u)$, $u \in G(N)$.

We shall now define a bilinear functional $\Phi(\cdot,\cdot)$ on H as follows: let f, g be two vectors of H and N be a two-dimensional subspace containing f and g, then $\Phi(f,g)=\Phi_N(f,g)$.

Φ is well defined if f and g are linearly independent, since then N is unique. If at most one of them is zero, then $\Phi_N(f,g)=0$ for every subspace N. If f, g are dependent and nonzero, then they span a one-dimensional subspace. Let now N_1, N_2 be two-dimensional subspaces containing f, g and let M be a tree-dimensional subspace containing N_1 and N_2. For a symmetric bilinear form Φ_M we have $\Phi_M(a,a)=w(a)$, $a \in G(M)$. The restrictions of Φ_M to N_1 and N_2 are symmetric bilinear forms whose quadratic forms coincide with w on $G(N_1)$ and $G(N_2)$, respectively. Therefore $\Phi_{N_1}(f,g)=\Phi_M(f,g)$, $\Phi_{N_2}(f,g)=\Phi_M(f,g)$, which proves that Φ is well defined.

Φ is symmetric and homogeneous. We claim to show that $\Phi(f,g+h) = \Phi(f,g)+\Phi(f,h)$ for any three vectors f, g, h. Let N_1, N_2, N_3 be two dimensional subspaces containing f,g; f,h and $f,g+h$, respectively, and let M be a three-dimensional subspace containing f,g,h. For a bilinear form Φ_M on M we have $\Phi_M(f,g) = \Phi_{N_1}(f,g)$, $\Phi_M(f,h) = \Phi_{N_2}(f,h)$, $\Phi_M(f,g+h) = \Phi_{N_3}(f,g+h)$.

We see that $\Phi(f, g+h) = \Phi_{N_3}(f, g+h) = \Phi_M(f, g+h) = \Phi_M(f\ g) + \Phi_M(f, h)$ $= \Phi_{N_1}(f, g) + \Phi_{N_2}(f, h) = \Phi(f, g) + \Phi(f, h)$.

A symmetric bilinear form Φ is bounded because $|\Phi(f, f)| = |w(f)| \leq \sup\limits_{\|f\| 1} |w(f)| < \infty$. Therefore there is a unique Hermitean operator T on H such that $\Phi(f, g) = (Tf, g)$ ([4]). If $\{f_i\}$ is an orthonormal base, then $\sum\limits_i (Tf_i, f_i) = \sum\limits_i w(f_i) = W$ and the series $\sum\limits_i (Tf_i, f_i)$ converges absolutely because of absolute convergence of $\sum\limits_i w(f_i)$.

Q.E.D.

Theorem 3.4. *Let H be a separable Hilbert space over complex or real scalars of dimension at least 3 and $L(H)$ be its logic of all closed subspaces of H. If T is a Hermitean operator of trace class, then m_T: $m_T(M) = \mathrm{tr}(TM)$, $M \in L(H)$, is a bounded signed state on $L(H)$.*

Conversely, for every bounded signed state m there is a unique Hermitean operator T of trace class such that $m(M) = \mathrm{tr}(TM)$, $M \in L(H)$.

Proof. The first part of our theorem is evident.
Let now m be a signed state from $M(H)$. Then $w_m(f) = m(\hat{f})$, $f \in G(H)$, is a bounded weight function with the weight $m(H)$. Therefore, by Lemma 3.3 there is a unique Hermitean operator T of trace class such that $w_m(f) = (Tf, f), f \in G(H)$.

Let now $M \in L(H)$ and let $\{g_j\}$ be an orthonormal base in M, then $m(M) = \sum\limits_j m(\hat{g}_j) = \sum\limits_j (Tg_j, g_j) = \mathrm{tr}(TM)$.

Q.E.D.

Theorem 3.4 does not hold for a two-dimensional Hilbert space as can easily be seen from the following. Let f, g be an orthonormal base in a two-dimensional Hilbert space H. We define the state m by $m(O) = 0$, $m(H) = 1$, $m(\hat{g}) = 0$, $m(\hat{f}) = 1$ and $m(M) = \frac{1}{2}$ for other one-dimensional subspaces of H.

If the state m was regular, then the respective weight function $w_m(h) = m(\hat{h})$, $\|h\| = 1$, would be continuous, which contradicts our example.

The Gleason theorem follows from Theorem 3.4, but the proof of Lemma 3.2 is based on the important Lemma 7.22 [5] from Gleason's proof. In paper [1] there was given a proof of Gleason theorem for any separable complex Hilbert space (for a two-dimensional space, too) which is incorrect, of course.

Theorem 3.5. *Let H be a separable Hilbert space (real or complex) and $L(H)$ be a logic of all closed subspaces of H. Then every bounded signed state m on $L(H)$ may be expressed as a difference of two positive signed states m_1, m_2.*

Proof. If H is not a two-dimensional Hilbert space, then every bounded signed state is regular, by Theorem 3.4, that is, m is of the form $m(M) = \mathrm{tr}(TM)$, $M \in L(H)$. A Hermitean operator T of trace class may be written as $T = T^{+} - T$, where T^{+}, T^{-} are positive Hermitean operators of trace class ([4]). Therefore $m = m_T = m_{T^+} - m_{T^-} = m_1 - m_2$, where $m_1 = m_{T^+}$, $m_2 = m_{T^-}$ are positive signed states.

For a two-dimensional Hilbert space the proposition follows from Theorem 2.1.

Q.E.D.

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ОБОБЩЕННЫЕ СОСТОЯНИЯ НА ЛОГИКЕ

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Резюме

В работе исследуется понятие обобщенного состояния на логике. Показано здесь разложение обобщенного состояния в виде разности двух позитивных обобщенных состояний на модулярной логике конечного ранга и исследуются обобщенные состояния на логике $L(H)$, $\dim H \geq 3$.