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ON PSEUDOCOMPLEMENTED SEMILATTICES WITH STONE CONGRUENCE LATTICES

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§ 1. Introduction

It is known [9] that the congruence lattices of pseudocomplemented semilattices are pseudocomplemented. In fact, such lattices were the subject of detailed analysis in [9], [10] and [11] (also see [8]). After characterizing in [11] the pseudocomplemented semilattices whose congruence lattices are Heyting lattices we were led to study the following question: Under what conditions on a pseudocomplemented semilattice its congruence lattice is a Stone lattice.

In this paper[1]) we give an answer to this question and derive from it a result due to Katriňák [7] characterizing the Boolean algebras whose congruence lattices (or lattices of filters) are Stone lattices. We also describe the pseudocomplemented semilattices with Boolean congruence lattices.

§ 2. Basic concepts and results

We follow the notation and terminology of [9], [10] and [11] (whose familiarity is helpful but is not necessary). Recall that a pseudocomplemented semilattice is an algebra $\langle S; \wedge, {}^*, 0\rangle$, where $\langle S; \wedge, 0\rangle$ is a $\wedge$-semilattice with 0 and $*$ is the operation of pseudocomplementation (i.e. $x \wedge a = 0$ iff $x \leq a^*$ in S). 0^* is the largest element in S and is denoted by 1. Let S denote an arbitrary pseudocomplemented semilattice (PCS) and let $B(S)$ denote the set of closed (i.e. $a = a^{**}$) elements of S.

[1]) The results of this paper are taken from the author's Ph. D. thesis. The author is extremely grateful to his supervisor Professor Stanley Burris for his guidance and to Dr. Bulman—Fleming for his keen interest in this work as well as to the Government of Ontario for financial support through the Ontario Graduate Fellowship Programme. The final draft of this paper was prepared in February, 1977 when the author was visiting the Department of Pure Mathematics, University of Waterloo and was supported by NRC Grant A7256.

It is well known [2] that $B(S)$ is both a subalgebra of S and a Boolean algebra whose meet and complementation are the same as those in S and whose Boolean join is defined by $a \dot{\vee} b = (a^* \wedge b^*)^*$. Thus a Boolean algebra can be regarded as a PCS in which every element is closed. For b in $B(S)$ let $D_b(S)$ denote the set of elements a such that $a^{**} = b$, and we write $D_1(S)$ as $D(S)$ and call its elements as dense. $D_b(S)$ is a subsemilattice of S. We let Con S denote the congruence lattice of S whose least and greatest elements are denoted by Δ_S and ∇_S (or simply Δ and ∇); the congruence Φ is defined by

$$x \varphi y \quad \text{iff} \quad x^* = y^*.$$

For a filter F in S the filter congruence $\hat{F}$ is defined by

$$x \hat{F} y \quad \text{iff} \quad x \wedge f = y \wedge f \quad \text{for some} \quad f \text{ in } F;$$

similarly for a in S the congruence $\hat{a}$ is defined by

$$x \hat{a} y \quad \text{iff} \quad x \wedge a = y \wedge a.$$

We need the following facts, the first of which was proved in [9] and the last two in [11]:

2.1. *If* $\psi \in \operatorname{Con} S$, *then* $\psi = ([1]_\psi)^\wedge \vee (\psi \wedge \varphi)$.

2.2. *The following statesments are equivalent:*

i) Con S *is distributive,*

ii) S *satisfies:*

$$\text{(D)} \quad \forall x \forall y (x < y^{**} \to x \leq y \ \ or \ \ y \leq x),$$

iii) S *satisfies:*

$$(\mathrm{D_W}) \quad \forall x \forall y (x^* = y^* \to x \leq y \ \ or \ \ y \leq x)$$

and

$$(\mathrm{U}') \quad \forall x \forall y ((x = x^{**} \,\&\, x < y^{**}) \to x < y),$$

iv) Con S *is modular.*

2.3. *The following statesments are equivalent:*

i) *The interval* $[\Delta, \varphi]$ *in* Con S *is distributive,*

ii) S *satisfies* $(\mathrm{D_W})$.

2.4. *The congruence* φ *is the intersection of all maximal elements in* Con S.

§ 3. Stone congruence lattices

In this section we give a characterization of the PCS's whose congruence lattices are Stone lattices.

We shall begin with a lemma.

Lemma 3.1. *Let* $\theta_1, \theta_2 \in \mathrm{Con}\, S$. *Then* $\theta_1 \vee \theta_2 = \nabla$ *iff, for some* $b \in B(S)$, $b \in [1]\theta_1$ *and* $b^* \in [1]\theta_2$.

Proof. $\theta_1 \vee \theta_2 = \nabla$ iff $\langle 1, 0\rangle \in \theta_1 \vee \theta_2$ iff $\langle 1, 0\rangle \in (\theta_1 \vee \theta_2)_B$[2]) iff $\langle 1, 0\rangle \in [\theta_1]_B \vee [\theta_2]_B$ iff for some $b \in B(S)$, $\langle 1, b\rangle \in \theta_1$ and $\langle b, 0\rangle \in \theta_2$. Then it follows immediately that b $[1]\theta_1$ and $b^* \in [1]\theta_2$.

Now we give a necessary condition on S (which is a first-order sentence) that Con S be a Stone lattice.

Theorem 3.2. *Suppose* Con S *is a Stone lattice. Then* S *satisfies*

$$\text{(S1)} \qquad \forall x \forall y((x \neq x^{**} \,\&\, y \neq y^{**} \,\&\, x \neq y) \rightarrow x \wedge y = 0).$$

Proof. First note that if Con S is Stone, then S satisfies (D), by 2.2.

Suppose $x, y \in S$ are such that $x < y$ and $x^* = y^*$ and let $\theta = \theta(x, y)$. Then we have $\theta = ([x, y] \times [x, y]) \cup \Delta$. Also $\theta \wedge \hat{y} = \Delta$, and so $[1]\theta^* \supseteq [y, 1]$. On the other hand if $z < y$, then by (D), $x \leqslant z$ or $z \leqslant x$, and so $z \notin [1]\theta^*$, and $[1]\theta^* = [y, 1]$.

Now suppose $x, y \in S$ with $x < y < x^{**}$. Then it is routine to verify that $\theta(x, y)^* \supseteq \theta(y, x^{**})$, hence $\theta(x, y)^{**} \subseteq \theta(y, x^{**})^*$. This leads to $[1]\theta(x, y)^{**} \subseteq [x^{**}, 1]$, and since $[1]\theta(x, y)^* = [y, 1]$, it follows from Lemma 3.1 that $\theta(x, y)^* \vee \theta(x, y)^{**} \neq \nabla$. This is contrary to the hypothesis.

Now suppose $x, y \in S$ with $x < x^{**}$, $y < y^{**}$ and $x^{**} \neq y^{**}$. It is simple to verify that $\theta(x, x^{**}) \wedge \theta(y, y^{**}) = \Delta$, and hence that $[1]\theta(x, x^{**})^{**} \subseteq [1]\theta(y, y^{**})^* = [y^{**}, 1]$. Since $[1]\theta(x, x^{**})^* = [x^{**}, 1]$ and since $\theta(x, x^{**})^* \vee \theta(x, x^{**})^{**} = \nabla$, it follows by Lemma 3.1 that $x^{**} \wedge y^{**} = 0$, hence $x \wedge y = 0$. Thus the theorem is proved.

At this point we introduce a few notations.

$$\begin{aligned} N(S) &= \{n \in S : n \text{ is non-closed}\}, \quad \text{i.e., } N(S) = \{n \in S : n \neq n^{**}\}, \\ C(S) &= \{c \in S : c \wedge n = 0,\ n \in N(S)\}, \\ N^{**}(S) &= \{n^{**} : n \in N(S)\} \text{ and } C^*(S) = \{c^* : c \in C(S)\}. \end{aligned}$$

It is clear that $C(S) \subseteq B(S)$ (in fact $C(S)$ is an ideal in $B(S)$) and $0 \in C(S)$. Also note that if $c \in C(S)$ and $n \in N(S)$, then $c \wedge n^{**} = (c \wedge n)^{**} = 0$ and hence it follows immediately that $C(S)$ can also be defined as

$$C(S) = \{c \in S : c \wedge n^{**} = 0 \text{ for all } n \in N(S)\}.$$

Since there cannot be any confusion, we simply write N, C, N^{**}, C^* for $N(S)$, $C(S)$, $N^{**}(S)$ and $C^*(S)$ respectively.

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[2]) $(\theta)_B$ denotes the restriction to $B(S)$ of a congruence θ.

Definition 3.3. *Let* $\psi \in \mathrm{Con}\, S$. *Then define*

$$N_\psi = \{n \in N \colon \langle n, n^{**}\rangle \in \psi\}$$

and

$$C_\psi = \{c \in C \colon \langle 0, c\rangle \in \psi\}.$$

It should be noted that C_ψ is an ideal in $B(S)$, $N_\psi = N_{\varphi \wedge \psi}$ and $N_\varphi = N$.

Definition 3.4. *Let S be a PCS. We say S is a neo-Boolean algebra (in short, NBA) iff S satisfies* (S1).

We remark that every Boolean algebra is an NBA.

Lemma 3.5. *Every NBA satisfies condition* (D).

Proof. Suppose (D) fails in S. Then there exist $x, y \in S$ such that $x < y^{**}$ and x are incomparable. The latter of these implies $x \wedge y < x$ and $x \wedge y < y$, and it is also clear that $y < y^{**}$. We know that $(x \wedge y)^{**} = x^{**}$, and since $x \wedge y < x \leqslant x^{**}$, $x \wedge y$ is a non-closed element and hence $x \wedge y \neq 0$. Thus y and $x \wedge y$ are two distinct nonclosed elements such that $y \wedge (x \wedge y) = x \wedge y \neq 0$, which implies that (S1) fails in S and the lemma is proved.

We shall now examine the congruences of NBA's in some detail. The following lemma is obvious.

Lemma 3.6. *Let S be an NBA. Then*

$$\varphi = \{\langle n, n^{**}\rangle : n \in N\} \cup \Delta_S.$$

If $A \subseteq S$, then A^u denotes the set of upper bounds of A.

Lemma 3.7. *Let S be an NBA. Then* $(N^{**})^u = C^*$.

Proof. Let us assume that N is non-empty, since the case N is empty is easily handled. It is obvious that $C^* \subseteq (N^{**})^u$, so let $y \in (N^{**})^u$. Then $y \geqslant n^{**}$ for $n \in N$, hence $n^{**} \wedge y^* = 0$ for $n \in N$, which implies $y^* \in C$ and hence $y \in C^*$ as y is clearly closed.

Lemma 3.8. *If S is an NBA and* $X \subseteq N(S) = N$, *then* $N_{((X^{**})^u)^\wedge} = N - X$.

Proof. Suppose y belongs to the left side. Then y is nonclosed, and $y \wedge f = y^{**} \wedge f$ for some $f \in (X^{**})^u$. If $y \in X$, then $y^{**} \leqslant f$ and so $y = y^{**}$, contrary to y being nonclosed. Thus $N_{((X^{**})^u)^\wedge} \subseteq N - X$.

Conversely, if $y \in N - X$, then $y \wedge x = 0$ for all $x \in X$, and so $y^* \geqslant x^{**}$ for all $x \in X$; i.e. $y^* \in (X^{**})^u$. Now clearly $y \wedge y^* = 0 = x^{**} \wedge y^*$, and so $x \in N_{((X^{**})^u)^\wedge}$.

Lemma 3.9. *Let S be an NBA and let* $\beta \in \mathrm{Con}\, S$ *with* $\beta \subseteq \varphi$. *Then* $\beta^* = ((N_\beta^{**})^u)^\wedge$.

Proof. Let $\langle x, y\rangle \in \beta \wedge ((N_\beta^{**})^u)^\wedge$ and $x \neq y$. Since $\beta \subseteq \varphi$, in view of Lemma 3.6 we may suppose $x \in N$ and $y = x^{**}$, thus $\langle x, x^{**}\rangle \in \beta$ and $\langle x, x^{**}\rangle \in ((N_\beta^{**})^u)^\wedge$.

Then $x \wedge f = x^{**} \wedge f$ for some $f \in (N_\beta^{**})^u$. However, $x^{**} \in N_\beta^{**}$, and so $f \geqslant x^{**} > x$, implying $x = x^{**}$, which is a contradiction, hence $\beta \wedge ((N_v^{**})^u)^\wedge = \Delta$.

Next, let $\alpha \in \mathrm{Con}\, S$ be such that $\alpha \wedge \beta = \Delta$. If $n \in N$ and $\langle n, n^{**} \rangle \in \alpha$, then $\langle n, n^{**} \rangle \notin \beta$, and so $n \in N - N_\beta$. Then by Lemma 3.8 we get $\langle n, n^{**} \rangle \in ((N_\beta^{**})^u)^\wedge$ and hence $\alpha \wedge \varphi \subseteq ((N_\beta^{**})^u)^\wedge$. Thus by 2.1, in order to show $\alpha \subseteq ((N_\beta^{**})^u)^\wedge$, it suffices to show that $[1]_\alpha \subseteq (N_\beta^{**})^u$. Suppose $x \in [1]_\alpha$ and $x \notin (N_\beta^{**})^u$. Then $x \ngeqslant n^{**}$ for some $n \in N_\beta$ and so $x \wedge n^{**} < n^{**}$. It is clear that $x \wedge n^{**}$ is closed (being in fact 0 if $x \neq x^{**}$) and so we obtain from Lemma 3.5 that $x \wedge n^{**} < n < n^{**}$ and therefore we get $\langle n, n^{**} \rangle \in \alpha$. This is impossible since $\langle n, n^{**} \rangle \in \beta$, showing that $[1]_\alpha \subseteq (N_\beta^{**})^u$. The lemma is thus proved.

Corollary 3.10. *Let S be an NBA. Then $\varphi^* = ((N^{**})^u)^\wedge = (c^*)^\wedge$.*

Proof. Immediate from Lemma 3.9 and 3.7.

Corollary 3.11. *Let S be an NBA and let $N = \{n\}$. Then $\varphi^* = (n^{**})^\wedge$.*

Proof. From the hypothesis we see that $(N^{**})^u = [n^{**}, 1]$ and hence from Corollary 3.10 the result is immediate.

Lemma 3.12. *Let S be an NBA and let $\psi \in \mathrm{Con}\, S$ be such that $[1]_\psi \cap N$ is non-empty. Then ψ is dense (i.e. $\psi^* = \Delta$) in* Con S.

Proof. Let $n_0 \in [1]_\psi \cap N$. Since $\langle n_0, 1 \rangle \in \psi$, $\langle n_0, n_0^{**} \rangle \in \psi$. If $|N| = 1$, then $N = \{n_0\}$ and hence $\varphi^* = (n_0^{**})^\wedge$ by Corollary 3.11, which implies $\varphi^* \subseteq \psi$, and clearly $\varphi \subseteq \psi$, giving $\varphi \vee \varphi^* \subseteq \psi$. By Lemma 3.5, Con S is distributive. It can easily be shown that in this case $(\theta \vee \psi)^* = \theta^* \wedge \psi^*$ for any $\theta, \psi \in \mathrm{Con}\, S$. Thus $(\varphi \vee \varphi^*)^* = \varphi^* \wedge \varphi^{**} = \Delta$, i.e. $\varphi \vee \varphi^*$ is dense and so ψ is dense. Thus we assume $|N| \geqslant 2$ and let $n \in N$ be such that $n \neq n_0$. Then $n \wedge n_0 = 0$ by (S1). Therefore from $\langle n_0, 1 \rangle \in \psi$ we get $\langle 0, n \rangle \in \psi$, hence $\langle 0, n^{**} \rangle \in \psi$, yielding $\langle n, n^{**} \rangle \in \psi$, thus we have $\varphi \subseteq \psi$. Next we wish to show $\varphi^* \subseteq \psi$. We know, by Corollary 3.10, $\varphi^* = ((N^{**})^u)^\wedge$. Now if $f \in (N^{**})^u$, then $f \geqslant n_0^{**} \geqslant n_0$ and so from $n_0 \in [1]_\psi$ we get $f \in [1]_\psi$, thus $(N^{**})^u \subseteq [1]_\psi$, which implies $\varphi^* \subseteq \psi$. We thus have $\varphi \vee \varphi^* \subseteq \psi$, proving as above that ψ is dense.

Definition 3.13. *Let F be a filter in S. F is Boolean iff $F \cap N$ is empty, or equivalently $F \subseteq B(S)$.*

Remark 3.14. Lemma 3.12 implies that if $\psi \in \mathrm{Con}\, S$ is not dense, then $[1]_\psi$ is a Boolean filter in S.

Definition 3.15. *Let $A \subseteq C$. Then we define*

$$d_C(A) = \{c \in C : c \wedge a = 0,\ a \in A\}.$$

Lemma 3.16. *Let S be an NBA and let $I \subseteq C$ be an ideal in $B(S)$. Then $(N^{**} \cup I \cup d_C(I))^u = \{1\}$.*

Proof. Let $f \in (N^{**} \cup I \cup d_C(I))^u$. Then f is an upper bound for $N^{**} \cup I \cup d_C(I)$. Hence $f^* \wedge x = 0$ for $x \in N^{**} \cup I \cup d_C(I)$, thus $f^* \wedge n^{**} = 0$ for $n \in N$, $f^* \wedge i = 0$ for

$i \in I$ and $f^* \wedge j = 0$ for $j \in d_C(I)$. From $f^* \wedge n^{**} = 0$, $n \in N$ we get $f^* \in C$. Thus $f^* \in C$ and $f^* \wedge i = 0$, $i \in I$, hence $f^* \in d_C(I)$. Since $f^* \wedge j = 0$, $j \in d_C(I)$, and $f^* \in d_C(I)$, it follows, in particular, that $f^* \wedge f^* = 0$ and hence $f^* = 0$, i.e. f is dense. Since $f \in (N^{**})^u$, we see that f is closed and so we conclude that $f = 1$, proving the lemma.

By taking $I = \{0\}$ we immediately have:

Corollary 3.17. *Let S be an NBA. Then*

$$(N^{**} \cup C)^u = \{1\}.$$

Lemma 3.18. *Let S be an NBA and let F be a Boolean filter in S. Then*

$$F \subseteq ((N^{**} - N_{\hat{F}}^{**}) \cup d_C(C_{\hat{F}}))^u.$$

Proof. Suppose $f \in F$ is such that $f \not\geqslant n^{**}$ for some $n^{**} \in N^{**} - N_{\hat{F}}^{**}$. Then $f \wedge n^{**} < n^{**}$, giving $f \wedge n^{**} < n < n^{**}$ using Lemma 3.5. Now $\langle f \wedge n^{**}, n^{**} \rangle \in \hat{f} \subseteq \hat{F}$, which implies $n^{**} \in N_{\hat{F}}^{**}$ and this is a contradiction, thus $F \subseteq (N^{**} - N_{\hat{F}}^{**})^u$. Next let $f \in F$ and $y \in d_C(C_{\hat{F}})$. Since $\langle f^*, 0 \rangle \in \hat{F}$, we have $\langle f^* \wedge y, 0 \rangle \in \hat{F}$ and so $f^* \wedge y \in C_{\hat{F}}$. From this we get $f^* \wedge y = 0$, since $y \in d_C(C_{\hat{F}})$ whence $f \geqslant y$, showing $F \subseteq (d_C(C_{\hat{F}}))^u$ and the lemma now follows.

Lemma 3.19. *Let S be an NBA and let F be a non-trivial (i.e. $F \neq \{1\}$) Boolean filter in S. Then $(N_{\hat{F}}^{**} \cup C_{\hat{F}})^u \subseteq B(S)$.*

Proof. Let $g \in (N_{\hat{F}}^{**} \cup C_{\hat{F}})^u$. First suppose $N_{\hat{F}}$ is non-empty. Then let $n \in N_{\hat{F}}$ and hence $g \geqslant n^{**}$. From this it follows by (S1) that g is closed. Next suppose $N_{\hat{F}}$ is empty. Then $F \subseteq (N^{**})^u$ by Lemma 3.18. Since F is non-trivial, choose $f \in F$ such that $f \neq 1$. Then $f^* \in C$, since $(N^{**})^u = C^*$ by Lemma 3.7. From the hypothesis we have that f is closed, and $f \neq 1$ and hence $f^* \neq 0$. Also $f^* \in C_{\hat{F}}$, as we know $\langle f^*, 0 \rangle \in \hat{F}$. From this we see that $g \geqslant f^* \neq 0$, from which it immediately follows, because $f^* \in C$, that g is closed; thus the lemma is proved.

The following theorem gives a nice description of pseudocomplements for congruences associated with filters.

Theorem 3.20. *Let F be a Boolean filter in an NBA S. Then*

$$(\hat{F})^* = ((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^\wedge.$$

Proof. If $F = \{1\}$, then $\hat{F} = \Delta$ and hence $N_{\hat{F}}^{**}$ is empty and $C_{\hat{F}} = \{0\}$, so the theorem clearly follows. We now suppose $F \neq \{1\}$. Let $\langle x, y \rangle \in \hat{F} \wedge ((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^\wedge$ with $x \neq y$. We wish to show that x and y are both closed. Without loss of generality we can assume $x \leqslant y$, whence $x = x^{**}$ or $y = y^{**}$. Therefore

$$\langle x, x^{**} \rangle, \quad \langle y, y^{**} \rangle \in \hat{F} \wedge ((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^\wedge.$$

But $\langle z, z^{**} \rangle \in \hat{F} \wedge ((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^\wedge$ implies $z = z^{**}$, since $g \geqslant z^{**}$ for all $g \in N_{\hat{F}}^{**}$. Thus x and y are closed.

We have $x\wedge f=y\wedge f$ and $x\wedge h=y\wedge h$ for some $f\in F$ and $h\in(N_{\hat F}^{**}\cup C_{\hat F})^u$. We know by hypothesis that f is closed, and since F is non-trivial, we get by Lemma 3.19 that h is closed. Then $(x\wedge f)\ \dot\vee\ (x\wedge h)\ =\ (y\wedge f)\ \dot\vee\ (y\wedge h)$ and so $x\wedge(f\dot\vee h)\ =\ y\wedge(f\dot\vee h)$. Now $f\in F\subseteq((N^{**}-N_{\hat F}^{**})\cup d_C((C_{\hat F}))^u$ by Lemma 3.18 and hence $f\dot\vee h\in(N^{**}\cup C_{\hat F}\cup d_C(C_{\hat F}))^u$, since $h\in(N_{\hat F}^{**}\cup C_{\hat F})^u$. By Lemma 3.16, $f\dot\vee h=1$. Thus we see that $x=y$, contrary to the assumption $x\neq y$. This shows that $\hat F\wedge((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge=\Delta$.

Next suppose $\alpha\in\operatorname{Con}S$ to be such that $\hat F\wedge\alpha=\Delta$. We show that $\alpha\subseteq$ $\subseteq((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge$. Let $F_1=[1]_\alpha$ and we claim that $F_1\subseteq(N_{\hat F}^{**}\cup C_{\hat F})^u$. For, suppose $f\in F_1$ to be such that $f\not\geq n^{**}$ for some $n\in N_{\hat F}$, then $f\wedge n^{**}\neq n^{**}$, i.e. $f\wedge n^{**}<n^{**}$. Since $f\wedge n^{**}$ is closed, $f\wedge n^{**}<n<n^{**}$ using Lemma 3.5. From this it follows that $\langle n,n^{**}\rangle\in\alpha$, since $\langle f\wedge n^{**},n^{**}\rangle\in(F_1)^\wedge\subseteq\alpha$. But $\langle n,n^{**}\rangle\in\hat F$, since $n\in N_{\hat F}$ and so we have a contradiction, showing that $F_1\subseteq(N_{\hat F}^{**})^u$. Next let $f\in F_1$ be such that $f\not\geq c$ for some $c\in C_{\hat F}$. Then $f\wedge c\neq c$, and also $\langle f\wedge c,\ c\rangle\in\hat F_1\subseteq\alpha$. But since $\langle 0,c\rangle\in\hat F$, $\langle f\wedge c,c\rangle\in\hat F$ and so $\langle f\wedge c,c\rangle\in\hat F\wedge\alpha$, which is a contradiction, proving $F_1\subseteq(C_{\hat F})^u$. Thus $F_1\subseteq(N_{\hat F}^{**}\cup C_{\hat F})^u$, proving the claim and thus we get $(F_1)^\wedge\subseteq$ $\subseteq((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge$. To complete the proof, by 2.1 we need to show $\alpha\wedge\varphi\subseteq$ $\subseteq((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge$, and so let $\langle x,x^{**}\rangle\in\alpha\wedge\varphi$. Then $\langle x,x^{**}\rangle\notin\hat F$, since $\hat F\wedge\alpha=$ $=\Delta$ and so $x^{**}\notin N_{\hat F}^{**}$, and therefore we conclude $x^*\in(N_{\hat F}^{**})^u$. As $x^{**}\in N^{**}$, we $x^*\in C^u$ and hence $x^*\in(C_{\hat F})^u$. Thus $x^*\in(N_{\hat F}^{**}\cup C_{\hat F})^u$, so that $\langle 0,x^{**}\rangle\in$ $((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge$, from which we get $\langle x,x^{**}\rangle\in((N_{\hat F}^{**}\cup C_{\hat F})^u)^\wedge$, implying $\alpha\wedge\varphi\subseteq$ $((N_{\hat F}^{**}\cup C_{\hat F})^u$. This completes the proof.

Corollary 3.21. *Let S be an NBA and let $A\subseteq N^{**}$. Then*

$$((A^u)^\wedge)^*=(((N^{**}-A)\cup C)^u)^\wedge;$$

in particular

$$(((N^{**})^u)^\wedge)^*=(C^u)^\wedge.$$

Proof. Let $F=A^u$. Then $N_{\hat F}^{**}=N^{**}-A$ by Lemma 3.8 and one easily checks that $C_{\hat F}=C$. The corollary now follows from the above theorem.

Corollary 3.22. *Let S be an NBA and let $K\subseteq C$. Then*

$$((K^u)^\wedge)^*=((N^{**}\cup d_C(K))^u)^\wedge;$$

in particular

$$((C^u)^\wedge)^*=((N^{**})^u)^\wedge.$$

Proof. Observe first that if $K\subseteq\{0\}$, then the first equality holds trivially. If K contains a non-zero element, then K^u is a Boolean filter, and so we may apply Theorem 3.20 with $F=K^u$ to establish the result; for, in this case $N_{\hat F}^{**}=N^{**}$ and $C_{\hat F}=d_C(K)$.

Lemma 3.23. *Let S be an NBA, let $A\subseteq N^{**}$ and let $K\subseteq C$. Then $(A^u)^\wedge\wedge(K^u)^\wedge$ $=\ ((A\cup K)^u)^\wedge$.*

Proof. Since $(A \cup K)^u \subseteq A^u$, we get $((A \cup K^u)^{\wedge} \subseteq (A^u)^{\wedge}$; $((A \cup K)^u)^{\wedge} \subseteq (K^u)^{\wedge}$ be symmetry and so $((A \cup K))^u)^{\wedge} \subseteq (A^u)^{\wedge} \wedge (K^u)^{\wedge}$. Let $\langle x, y \rangle \in (A^u)^{\wedge} \wedge (K^u)^{\wedge}$ with $x \neq y$. Then there exist $f \in A^u$ and $g \in K^u$ such that $x \wedge f = y \wedge f$ and $x \wedge g = y \wedge g$. If A is empty or K contains no non-zero elements, then the lemma holds trivially, hence we assume that A is non-empty and K contains a non-zero element. This implies that f and g are closed. Now without loss of generality we can assume that $x \leqslant y$ and $y = y^{**}$. If x is closed, then $x \wedge (f \dot{\vee} g) = y \wedge (f \dot{\vee} g)$ and $f \dot{\vee} g \in (A \cup K)^u$, implying that $\langle x, y \rangle \in ((A \cup K)^u)^{\wedge}$. If $x < x^{**}$, then we have $\langle x, x^{**} \rangle \in (A^u)^{\wedge}$ and $\langle x, x^{**} \rangle \in (K^u)^{\wedge}$, which implies by Lemma 3.8 that $x^{**} \in N^{**} - A$, and so $x^* \in A^u$. Since $x^{**} \in N^{**}$ and $K \subseteq C$, we get $x^* \in K^u$ also, hence $x^* \in (A \cup K)^u$. Since $x \wedge x^* = 0 = x^{**} \wedge x^*$, we have that $\langle x, x^{**} \rangle \in ((A \cup K)^u)^{\wedge}$. Thus $\langle x, y \rangle \in ((A \cup K)^u)^{\wedge}$ and the lemma is proved.

Theorem 3.24. *Let S be an NBA and let $\psi \in \operatorname{Con} S$ be non-dense. Then*

$$\psi^* = ((N_{\psi}^{**} \cup C_{\psi})^u)^{\wedge}.$$

Proof. We know that $\psi = \hat{F} \vee (\psi \wedge \Phi)$, where $F = [1]_{\psi}$ and hence, since Con S is distributive, $\psi^* = (\hat{F})^* \wedge (\psi \wedge \varphi)^*$. Since ψ is non-dense, F is a Boolean filter by Lemma 3.12 (or by Remark 3.14). By Theorem 3.20 we have $(\hat{F})^* = ((N_F^{**} \cup C_F)^u)^{\wedge}$, and by Lemma 3.9 $(\psi \wedge \varphi)^* = (((N_{\psi \wedge \varphi})^{**})^u)^{\wedge}$. Thus we have

$$\begin{aligned} \psi^* &= ((N_F^{**} \cup C_F)^u)^{\wedge} \wedge ((N_{\psi}^{**})^u)^{\wedge} \quad \text{since} \quad N_{\psi \wedge \Phi} = N_{\psi} \\ &= ((N_F^{**})^u)^{\wedge} \wedge ((C_F)^u)^{\wedge} \wedge ((N_{\psi}^{**})^u)^{\wedge} \quad \text{by Lemma 3.23.} \end{aligned}$$

Now clearly $N_F^{**} \subseteq N_{\psi}^{**}$, and so $(N_{\psi}^{**})^u \subseteq (N_F^{**})^u$, which yields $((N_{\psi}^{**})^u)^{\wedge} \subseteq ((N_F^{**})^u)^{\wedge}$. Hence we have $\psi^* = ((N_{\psi}^{**})^u)^{\wedge} \wedge ((C_F)^u)^{\wedge}$, which yields by Lemma 3.23 again that $\psi^* = ((N_{\psi}^{**} \cup C_F)^u)^{\wedge}$, proving the theorem.

We now return to the problem of characterizing the PCS's whose congruences form a Stone lattice. We have already seen that (S1) is a necessary condition. The following example shows that it is not sufficient.

Example 3.25. Let B be the Boolean algebra of finite and cofinite subsets of ω. For each atom a in B we choose a new symbol n_a and let $S = B \cup \{n_a : a$ is an atom in $B\}$. Define $\wedge$ on S as follows:

(i) if $x, y \in B$, then $x \wedge y = x \wedge^B y$,

(ii) if $x \in B$, then $x \wedge n_a = \begin{cases} n_a & \text{if } x \geqslant a \\ 0 & \text{otherwise,} \end{cases}$

(iii) $n_a \wedge n_b = \begin{cases} n_a & \text{if } a = b \\ 0 & \text{if } a \neq b, \end{cases}$

and define $*$ on S as follows:

(i) if $x \in B$, then $x^* = x'$, x' being the complement of x in B,

(ii) $(n_a)^* = a^*$ for every atom a in B.

Then it is clear that $\mathscr{S} = \langle S, \wedge, {}^*, 0 \rangle$ is a PCS with $B(S) = B$ and $N(S) = \{n_a : a$ is an atom in $B\}$; moreover, $(n_a)^{**} = a$ and so $N^{**}(S)$ is the set of atoms in B.

Clearly $\mathscr{S}$ satisfies (S1), but we claim that Con $\mathscr{S}$ is not a Stone lattice. For, suppose Con $\mathscr{S}$ is Stone. Let A be an infinite subset of N^{**} whose complement is also infinite, e.g. $A = \{\{1\}, \{3\}, \{5\}, \ldots\}$. Then A^{u} and $(N^{**} - A)^{u}$ are Boolean filters in S. Since $C(S) = \{0\}$, we have by Corollary 3.21 that $((A^{u})^{\wedge})^{*} = ((N^{**} - A)^{u})^{\wedge}$ and $((A^{u})^{\wedge})^{**} = (A^{u})^{\wedge}$. Since Con $\mathscr{S}$ is Stone, we have $(A^{u})^{\wedge} \vee ((N^{**} - A)^{u})^{\wedge} = \nabla_{S}$. By restricting this equation to $B(S)$, and then translating to the lattice of filters $F(B(S))$, we get $A^{u} \vee (N^{**} - A)^{u} = B(S)$ in $F(B(S))$. But it is straightforward to verify that $A^{u} \vee (N^{**} - A)^{u}$ is the filter of cofinite sets in ω and so $A^{u} \vee (N^{**} - A)^{u} \neq B(S)$, which is a contradiction, proving the claim.

We shall now give a second necessary condition on S — suggested by the above example — in order that Con S be a Stone lattice.

Theorem 3.26. *Let* Con S *be a Stone lattice. Then S satisfies:*

(S2) (i) *if $A \subseteq N^{**}$, then $\dot{\vee} A$ exists,*

(ii) *if $K \subseteq C$, ther $\dot{\vee} K$ exists.*[3])

Proof. If A is empty of $K \subseteq \{0\}$, then (S2) holds trivially, and so in the rest of the proof we assume that neither of these instances occurs. From the hypothesis and Theorem 3.2 we obtain that S is an NBA.

We first show that $\dot{\vee} N^{**}$ and $\dot{\vee} C$ exist. By Corollaries 3.21 and 3.22 we have $(((N^{**})^{u})^{\wedge})^{*} = (C^{u})^{\wedge}$ and $((C^{u})^{\wedge})^{*} = ((N^{**})^{u})^{\wedge}$ and hence using the hypothesis again we get $((N^{**})^{u})^{\wedge} \vee (C^{u})^{\wedge} = \nabla_{S}$. By restricting this equation to $B(S)$ we get $(N^{**})^{u} \vee C^{u} = B(S)$ in $F(B(S))$. Hence there exist elements m and t in $B(S)$ such that $m \in (N^{**})^{u}$, $t \in (C)^{u}$ and $m \wedge t = 0$. Since $m \dot{\vee} t \in (N^{**} \cup C)^{u} = \{1\}$, we have $m \dot{\vee} t = 1$ and hence m and t are complements of each other in $B(S)$, i.e. $t = m^{*}$ and $m = t^{*}$. We claim that m is the least element in $(N^{**})^{u}$. For, if not, there exists $m_1 \in (N^{**})^{u}$ such that $m > m_1$. Then $m_1 \dot{\vee} t = 1$, and since $m \wedge t = 0$, $m_1 \wedge t = 0$ and so m_1 is a complement of t. However, complements are unique in $B(S)$ and we have a contradiction, proving that m is the least element in $(N^{**})^{u}$. Similarly one argues that m^{*} is the least element of C^{u}. Thus $\dot{\vee} N^{**} = m$ and $\dot{\vee} C = m^{*}$. It is clear that $m \in C$ and hence $C = [0, m^{*}]$, and also we note that C is, of course, a Boolean lattice.

We have by hypothesis that $((A^{u})^{\wedge})^{*} \vee ((A^{u})^{\wedge})^{**} = \nabla_{S}$. We also have by Corollary 3.21 that $((A^{u})^{\wedge})^{*} = (((N^{**} - A) \cup C)^{u})^{\wedge}$ and by Theorem 3.20 that $((((N^{**} - A) \cup C)^{u})^{\wedge})^{*} = (A^{u})^{\wedge}$. Thus we get $(A^{u})^{\wedge} \vee (((N^{**} - A) \cup C)^{u})^{\wedge} = \nabla_{S}$. From this, as before, one has $A^{u} \vee ((N^{**} - A) \cup C)^{u} = B(S)$ in $F(B(S))$ and arguing as in the previous paragraph we can find $a \in B(S)$ such that a is the least element in A^{u} and a^{*} is the least element in $((N^{**} - A) \cup C)^{u}$, thus $\dot{\vee} A$ exists and $\dot{\vee} A = a$.

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[3]) For $A \subseteq B(S)$, if $\sup A$ exists in the Boolean algebra $B(S)$, then we write $\dot{\vee} A$ for $\sup A$.

Again by hypothesis, $((K^{u})^{\wedge})^{*} \vee ((K^{u})^{\wedge})^{**} = \nabla_S$. From Corollary 3.22 we have $((K^{u})^{\wedge})^{*} = ((N^{**} \cup d_C(K))^{u})^{\wedge}$ and by Theorem 3.20, $(((N^{**} \cup d_C(K))^{u})^{\wedge})^{*} = ((d_C d_C(K))^{u})^{\wedge}$. Hence it follows that $((N^{**} \cup d_C(K))^{u})^{\wedge} \vee ((d_C d_C(K))^{u})^{\wedge} = \nabla_S$. Then there exist $f \in (d_C(K))^{u}$ (and $f \in (N^{**})^{u}$ also) and $g \in (d_C d_C(K))^{u}$ such that $f \dot{\vee} g = 1$ and $f \wedge g = 0$. From this it follows that $f \wedge m^*$ and $g \wedge m^*$ are complements of each other in the Boolean lattice $[0, m^*]$. Then arguing as before and noting that $(d_C d_C(K))^{u} = K^{u}$, we conclude that $\dot{\vee} K$ exists in $[0, m^*]$ and hence $\dot{\vee} K$ exists in $B(S)$. This completes the proof.

We shall now prove that (S1) and (S2) are indeed sufficient conditions on S in order that Con S be a Stone lattice.

Therefore 3.27. *Let S satisfy (S1) and (S2). Then* Con S *is a Stone lattice.*

Proof. First we note from the hypothesis and Lemma 3.25 that S satisfies (D) — hence Con S is distributive by 2.2.

Let ψ be any congruence on S. If ψ is dense, then the theorem is immediate. Thus we assume that ψ is not dense. Then by Theorem 3.24 we have $\psi^* = ((N_\psi^{**} \cup C_\psi)^{u})^{\wedge}$, and from Theorem 3.20 it follows that $\psi^{**} = ((N^{**} - N_\psi^{**}) \cup d_C(C_\psi))^{u})^{\wedge}$, since S satisfies (S1). Also we have from (S2) that $\dot{\vee} N^{**}$, $\dot{\vee} C$, $\dot{\vee} N_\psi^{**}$ and $\dot{\vee} C_\psi$ exist, and so let $\dot{\vee} N^{**} = m$, $\dot{\vee} N_\psi^{**} = u$ and $\dot{\vee} C_\psi = r$. Then $\dot{\vee} C = m^*$, $\dot{\vee}(N^{**} - N_\psi^{**}) = m \wedge u^*$ and $\dot{\vee} d_C(C_\psi) = m^* \wedge r^*$. From this we get $\dot{\vee}(N_\psi^{**} \cup C_\psi) = u \dot{\vee} r$ and $\dot{\vee}((N^{**} - N_\psi^{**}) \cup d_C(C_\psi)) = (m \wedge u^*) \dot{\vee} (m^* \wedge r^*) = u^* \wedge r^* = (u \dot{\vee} r)^*$. Hence we get $\psi^* = (u \dot{\vee} r)^{\wedge}$ and $\psi^{**} = ((u \dot{\vee} r)^*)^{\wedge}$. Now $\langle 0, (u \dot{\vee} r)^* \rangle \in \psi^*$ and $\langle (u \dot{\vee} r)^*, 1 \rangle \in \psi^{**}$ and so $\langle 0, 1 \rangle \in \psi^* \vee \psi^{**}$, implying that $\psi^* \vee \psi^{**} = \nabla_S$. Thus we see that Con S is Stone, proving the theorem.

We thus have proved the following Theorem.

Theorem 3.28. Con *S is a Stone lattice iff S satisfies (S1) and (S2).*

Corollary 3.29. (Katriňák [7]). *Let B be a BA. Then* Con *B is a Stone lattice iff B is complete.*

Proof. Treating B as a PCS, (S1) holds trivially in B and $N^{**}(B)$ is empty. Hence the corollary follows immediately from Theorem 3.28.

§ 4. Congruences forming Boolean lattices.

The purpose of this section is to describe those PCS's whose congruence lattices are Boolean lattices. It turns out that the only finite Boolean algebras treated as PCS's have this property. We also consider some related questions. We shall begin with the following lemma.

Lemma 4.1. *If* Con *S is complemented, then S is a Boolean algebra.*

Proof. Suppose S is not a Boolean algebra. Then there exists at least one non-closed element in S which implies that $\varphi \neq \Delta$. From the hypothesis we see that φ is complemented. Let φ' denote a complement of φ. Since $\varphi \wedge \varphi' = \Delta$, then $\varphi' \leqslant \varphi^*$. (Recall that Con S is pseudocomplemented.) Hence $\varphi \vee \varphi' \leqslant \varphi \vee \varphi^*$. We claim that $\varphi \vee \varphi^* \neq \nabla$. For, suppose $\varphi \vee \varphi^* = \nabla$. Then $\langle 0, 1 \rangle \in \varphi \vee \varphi^*$, and so $\langle 0, 1 \rangle \in (\varphi \vee \varphi^*)_B = (\varphi)_B \vee (\varphi^*)_B = (\varphi^*)_B$, in short $\langle 0, 1 \rangle \in (\varphi^*)_B$, which implies that $\langle 0, 1 \rangle \in \varphi^*$. Then $\nabla = \theta(0, 1) \leqslant \varphi^* \leqslant \nabla$, i.e. $\varphi^* = \nabla$, from which it follows that $\varphi \leqslant \varphi^*$, which is impossible and so the claim is proved. Then $\varphi \vee \varphi' \neq \nabla$, which is clearly a contradiction since φ' is a complement of φ. Thus we conclude that S is a Boolean algebra and so the lemma is proved.

Corollary 4.2. *If* Con S *is complemented, then* S *is a finite Boolean algebra.*

Proof. By Lemma 4.1 we see that S is a Boolean algebra. Then Con S is isomorphic with the ideal lattice of the Boolean algebra S. Now the corollary is an immediate consequence of the wellknown facts that there exists non-principal ideals in an infinite Boolean algebra and that non-principal ideals do not have complements in the ideal lattice of that algebra.

Corollary 4.3. *If* Con S *is complemented, then* Con S *is distributive.*

Proof. Immediate from Lemma 4.1 and the fact that the congruence lattice of a Boolean algebra is distributive.

The following theorem contains a characterization of PCS's whose congruence lattices are Boolean.

Theorem 4.4. The following statesments are equivalent:

(1) Con S *is complemented,*

(2) S *is a finite Boolean algebra,*

(3) Con S *is a finite Boolean lattice,*

(4) Con S *is a Boolean lattice.*

In this case, Con $S \cong S$.

Proof. (1) implies (2) by Corollary 4.2, that (2) implies (3) is well known, (3) implies that (4) is trivial and (4) implies that (1) is trivial as well.

The following Corollary is implicit in Jones [5].

Corollary 4.5. *The variety PCS has a unique simple algebra in it, namely the 2-element algebra.*

Proof. If S is a simple algebra in PCS, then Con S is the Boolean lattice, with two elements. The corollary now follows immediately from Theorem 4.4.

Corollary 4.6. *The class of all congruence lattices of PCS's cannot be characterized by a set of first order axioms.*

Proof. We only need to observe that an elementary class with arbitrarily large finite models has infinite models, and then apply Theorem 4.4.

Our next theorem gives a necessary and sufficient condition on S in order that the interval $[\Delta, \varphi]$ of Con S be a Boolean lattice. In this connection we need the following Theorem which was proved in Varlet [12].

Theorem 4.7. *Let L be a $\wedge$-semilattice and let* Con L *be the lattice of congruences on L. Then* Con L *is a Boolean lattice iff every closed interval in L is totally ordered and is of finite length.*

Theorem 4.8. *The interval $[\Delta, \varphi]$ of* Con S *is a Boolean lattice iff*

(i) (D) *holds in S,*

(ii) *for $c \in B(S)$ every interval $[a, b]$ in D_c is of a finite length.*

Proof. Suppose $[\Delta, \varphi]$ is a Boolean lattice. Then $[\Delta, \varphi]$ is distributive and hence by 2.3 we see that (D_w) holds in S. We shall show that (U') also holds in S. Suppose (U') is false in S. Then there exist two closed elements c_1 and c_2 and a non-closed element a in S such that $a^{**} = c_1$, $c_2 < c_1$ but $c_2 \nless a$. Then $(c_2 \wedge a)^* = c_2^*$. It is clear that $\theta(c_2 \wedge a, c_2) \subseteq \theta(a, c_1) \subseteq \varphi$. Since every interval of a Boolean lattice is a Boolean lattice and $[\Delta, \varphi]$ is a Boolean lattice, we have that the interval $[\Delta, \theta(a, c_1)]$ is a Boolean lattice. However, we claim that the congruence $\theta = \theta(c_2 \wedge a, c_2)$ is not complemented in $[\Delta, \theta(a, c_1)]$. For if θ were complemented, then let α be its complement in $[\Delta, \theta(a, c_1)]$. We know from [9] that every interval of Con S is pseudocomplemented; hence $[\Delta, \theta(a, c_1)]$ is pseudocomplemented. So let β denote the pseudocomplement of θ in $[\Delta, \theta(a, c_1)]$. Since $\alpha \wedge \theta = \Delta$, we have $\alpha \subseteq \beta$ and hence $\alpha \vee \theta \subseteq \beta \vee \theta$. It is clear that $\langle a, c_1 \rangle \notin \theta$ and also $\langle a, c_1 \rangle \notin \beta$ (because $\langle a, c_1 \rangle \in \beta$ would imply $\langle c_2 \wedge a, c_2 \rangle \in \beta$ contradicting the fact that β is a pseudocomplement of θ). We claim that in fact $\langle a, c_1 \rangle \notin \beta \vee \theta$. For in the opposite case there exists $x_1, \ldots, x_n$ in S such that $\langle a, x_1 \rangle \in \theta$, $\langle x_1, x_2 \rangle \in \beta$, $\langle x_2, x_3 \rangle \in \theta$, $\ldots$, $\langle x_n, c_1 \rangle \in \beta$. Since both θ and β are contained in $\theta(a, c_1)$, we have $a = x_1 \wedge a = x_2 \wedge a = \ldots = x_n \wedge a = c_1 \wedge a = a$; so $x_i \geqslant a$ for $i = 1, \ldots, n$. Note that $a \nless c_2$, hence it follows that $x_i \nless c_2$ for $i = 1, \ldots, n$. From this it follows that $a = x_1$, $x_2 = x_3$, etc. since $\langle u, v \rangle \in \theta(c_2 \wedge c_2)$ implies $u \leqslant c_2$ and $v \leqslant c_2$. Thus we see that $\langle a, c_1 \rangle \in \beta$, a possibility that has already been excluded. Hence the claim is proved and we $(a, c_1) \notin \beta \vee \theta$. This implies that $\beta \vee \theta \subset \theta\ (a, c_1)$, whence $\alpha \vee \theta \subset \theta\ (a, c_1)$. But we know $\alpha \vee \theta = \theta(a, c_1)$, since α is the complement of θ in $[\Delta, \theta(a, c_1)]$. Thus we have a contradiction. This shows that $\theta(c_2 \wedge a, c_2)$ is not complemented in $[\Delta, \theta(a, c_1)]$, which contradicts the fact that the interval $[\Delta, \theta(a, c_1)]$ is a Boolean lattice. Thus we see that (U') holds in S and so (D) holds in S, proving the theorem. To prove (ii): let $c \in B(S)$ be arbitrarily fixed. By (i) (D) holds in S, which implies that D_c is a chain. Let $\psi \leqslant \theta(D_c \times D_c)$. We can regard D_c as a $\wedge$-semilattice. Then we let $\psi_c = \psi \cap (D_c \times D_c)$. Treating ψ as just a semilattice congruence on (semilattice) S, we see from (D_w) and (U') that $\psi = \psi_c \cup \Delta_S$. From this it follows that the interval $[\Delta, \theta(D_c \times D_c)]$ is isomorphic with L_c, the lattice of semilattice congruences on the semilattice D_c. The hypothesis implies that

$[\Delta, \theta(D_c \times D_c)]$ is a Boolean lattice, hence L_c is a Boolean lattice. Then by Theorem 4.7 we get that every interval in D_c is of finite length. Thus (ii) is proved.

Conversely, suppose (i) and (ii) hold in S. Then by (i) we have, as before, that each D_c is a chain and $[\Delta, \theta(D_c \times D_c)]$ is isomorphic with L_c (as above). Since D_c is a chain and by (ii) every interval of D_c is of finite length, then again by Theorem 4.7 we see that L_c is a Boolean lattice and hence $[\Delta, \theta(D_c \times D_c)]$ is a Boolean lattice for every $c \in B(S)$. Moreover it is easy to see that $\varphi =$ $= \bigvee_{c \in B(S)} \theta(D_c \times D_c)$, and $\theta(D_{c_1} \times D_{c_1}) \wedge \theta(D_{c_2} \times D_{c_2}) = \Delta$ if $c_1 \neq c_2$ because of (D). Thus $[\Delta, \theta]$ can be regarded as a direct product of the Boolean lattices $[\Delta, \theta(D_c \times D_c)]$ for $c \in B(S)$, hence $[\Delta, \theta]$ is a Boolean lattice. This proves the theorem.

Corollary 4.9. *Suppose that S has the property:*

For $c \in B(S)$ every interval $[a, b]$ in $D_c(S)$ is of a finite length.

Then the following statesments are equivalent:

(1) *The interval $[\Delta, \theta]$ is a Boolean lattice,*

(2) *S satisfies* (D),

(3) Con *S is modular,*

(4) Con *S is distributive,*

(5) Con *S is a Heyting lattice.*

In particular, if every D_c in S is finite or if S itself is finite, then (1)—(5) are equivalent.

Proof. (1) and (2) are equivalent by Theorem 4.8, and the fact that (2)—(5) are equivalent to each other was shown in [11].

We shall make the following definition.

Definition 4.10. *A property P of a PCS S is (finite) Boolean iff the following is true:*

P holds in S iff S is a (finite) Boolean algebra.

We have already encountered one such property in Lemma 4.1, namely that of Con S being complemented. We shall now consider another such property.

Theorem 4.11. *The following statements are equivalent:*

(1) *S is a Boolean algebra,*

(2) *Every element of* Con *S is a meet of maximal elements.*

(3) *Δ is the intersection of all maximal elements in* Con *S.*

Proof. That (1) implies (2) is well known. On the other hand, (2) implies that the intersection of all maximal elements of Con S is Δ. From 2.4 one has immediately $\Delta = \varphi$, from which (1) follows.

Theorem 4.12.

(1) *Every two congruences on S permute with each other.*

(2) *Either S is a Boolean algebra, or else S is of the form $B \cup \{1\}$ where B is a Boolean algebra and 1 is the new largest element of S.*

Proof. (1) implies (2): First we note that (1) implies the modularity of Con S, as it is well known, which in turn implies by 2.2 that (D) holds in S. Now we claim that, for any closed element c, D_c contains at most one non-closed element. For suppose that for some c, D_c has at least two non-closed elements, say x and y. Without loss of generality we may assume that $x<y<c$. Let $\psi=\theta(x, y)$ and $\eta=\theta(y, c)$. Then $\langle x, c\rangle \in \psi\circ\eta$. Observe that the only non-trivial congruence class of ψ is $[x, y]$ and the only non-trivial congruence class of η is $[y, c]$. Hence it is obvious that $\langle x, c\rangle \notin \eta\circ\psi$, giving $\psi\circ\eta\neq\eta\circ\psi$. This contradicts (1), hence our claim is true; in particular S can have at most one dense element not equal to 1. Next we claim that if $c\neq 1$ is a closed element, then $D_c=\{c\}$. For, if D_c contains a non-closed element, say x, consider $\psi=\theta(x, c)$ and $\eta=\theta(c, 1)$. Then $\langle x, 1\rangle \in \psi\circ\eta$, and it is easy to see that $\langle x, 1\rangle \notin \eta\circ\psi$, contrary to (1) again. Thus we have shown that S cannot have any non-closed element except possibly one dense element which is different from 1, i.e. S can only be either a Boolean algebra, or else S is of the form $B\cup\{1\}$, where B is a Boolean algebra. On the other hand, if S is a Boolean algebra it is well known that (1) holds in S, and also from this well-known fact it easily follows that if S is of the form $B\cup\{1\}$, (1) holds in S. Thus the theorem is proved.

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О ПОЛУСТРУКТУРАХ С ПСЕВДОДОПОЛНЕНИЯМИ СО СТРУКТУРАМИ КОНГРУЭНЦИЙ СТОУНА

Г. П. Цанкаппанавар

Резюме

В работе дана характеризация полуструктур с псевдодополнениями со структурой конгруэнций Стоуна. Эта характеризация включает результат Катриняка, который характеризует булевые алгебры, структура конгруэнций которых является структурой Стоуна.

Рассматриваются также полуструктуры с псевдодополнениями, у которых структуры конгруэнций булевы.