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ON PARTIALLY ORDERED GROUPS OF LOCALLY FINITE LENGTH

BOŽENA ČERNÁKOVÁ

All partially ordered groups considered in this note are assumed to be abelian; the group operation is written additively. A partially ordered set is said to be of locally finite length if all its bounded chains are finite.

Each partially ordered set of locally finite length is a multilattice (Benado [2]).

Let α be a cardinal, $\alpha \geqq 2$. We denote by $\mathscr{G}_\alpha$ the class of all partially ordered groups G such that

(i) the partially ordered set $(G, \leqq)$ is directed, of locally finite length and all saturated chains from a to b are of the same length for every $a, b \in G, a < b$;

(ii) the set X of all elements of G covering the zero element of G has the cardinality α.

The structure of partially ordered groups G belonging to $\mathscr{G}_\alpha$ will be investigated in this note. All partially ordered groups of the class $\mathscr{G}_\alpha$ will be constructively described (cf. Thm. 3.5). It will be proved that for each cardinal $\alpha \geqq 2$ there exists an infinite set of non-isomorphic partially ordered groups belonging to $\mathscr{G}_\alpha$.

1. Preliminaries

We recall some basic notions which will be applied in the sequel.

Let P be a partially ordered set and let x, y be elements of P. Each nonempty subset of P is partially ordered by the induced partial order. We denote by $U(x, y)$ the set of all upper bounds of the set $\{x, y\}$ in P. Further let $x \vee y$ be the set of all minimal elements of the partially ordered set $U(x, y)$. The set $x \wedge y$ is defined dually.

The partially ordered set P is said to be a multilattice if it satisfies the following condition $(M1)$ and the condition $(M1')$ dual to $(M1)$:

$(M1)$ Whenever $x \in P$, $y \in P$ and $z \in U(x, y)$, then there is $z_1 \in x \vee y$ such that $z_1 \leqq z$.

If G is a partially ordered group such that the corresponding partially ordered set $(G, \leqq)$ is a directed multilattice, then G is said to be a multilattice group (m-group). m-groups (introduced by Benado [1]) were thoroughly investigated by McAlister ([4], [5]).

For the basic definitions concerning partially ordered groups cf. Fuchs [3].

Let $(G, +, \leqq)$ be a partially ordered group. If no misunderstanding is likely to arise, we write G instead of $(G, +, \leqq)$. However, if we wish to emphasize that the relation $\leqq$ is not taken into account, then we sometimes denote the group $(G, +)$ by $G^{(1)}$.

G is said to be trivially ordered if no distinct elements of G are comparable.

The condition mentioned in (i) above concerning saturated chains from a to b is related to modularity and distributivity; let us recall these notions.

Benado [2] has defined the notion of modular and distributive multilattices as follows:

A multilattice M is called modular if for every $a, b, c, u, v \in M$ satisfying the conditions $u \leqq b \leqq c \leqq v$, $v \in a \vee b$, $u \in a \wedge c$ we have $b = c$.

A multilattice M is called distributive if for every $a, b, c, u, v \in M$ satisfying the conditions $u \in a \wedge b$, $u \in a \wedge c$, $v \in a \vee b$, $v \in a \vee c$ we have $b = c$.

The following definition of a distributive multilattice group is due to McAlister [4].

A multilattice group G is said to be distributive if for any $a, b, c \in G$, the relations $(a \vee b) \cap (a \vee v) \neq \emptyset$, $(a \wedge b) \cap (a \wedge c) \neq \emptyset$ together imply $b = c$.

It is evident that both definitions of distributivity are equivalent in multilattices and that distributivity implies modularity.

For elements a, b of a multilattice M we write $a \prec b$ (b covers a) if $a > b$ and if there does not exist any element $c \in M$ such that $a < c < b$. The meaning of a $a \succ b$ is defined dually.

Let M be a multilattice of locally finite length $a, b \in M$, $a \leqq b$. Let C be a chain in M such that b is the greatest element of C and a is the least element of C; then C is said to be a chain from a to b. If card $C = n$, then we say that the length of the chain C is n. Let $C = \{a_0, a_1, \ldots, a_n\}$, $a_0 \prec a_1 \prec a_2 \prec \ldots \prec a_n$; then the chain C is said to be saturated.

1.1. Lemma. *Let $G \neq \{0\}$ be a directed multilattice group of locally finite length and let X be the set of all elements of G covering O. Then the set X generates the group $G^{(1)}$.*

Proof. Let $g \in G$, $g \neq 0$. Then there exists $h \in G$ with $h > g$, $h > 0$. Further there are elements $a_0, a_1, \ldots, a_n \in G$ and $b_0, b_1, b_2, \ldots, b_n \in G$ such that

$$0 = a_0 \prec a_1 \prec a_2 \prec \ldots \prec a_n = h;$$

$$g = b_0 \prec b_1 \prec b_2 \prec \ldots \prec b_m = h.$$

Put $a_i - a_{i-1} = x_i$ $(i = 1, 2, \ldots, n)$, $b_j - b_{j-1} = y_j$ $(j = 1, 2, \ldots, m)$. Then all x_i and all y_j belong to X and we have

$$g = h - (h - g) = (x_1 + x_2 + \ldots + x_n) - (y_1 + y_2 + \ldots + y_m).$$

Hence X generates $G^{(1)}$.

In [2] (Theoreme 4.5) the following assertion is proved:

Let M be a modular multilattice of locally finite length, $a, b \in M$. Then all saturated chains from a to b are of the same length.

Hence if G is a directed group such that the partially ordered set $(G, \leqq)$ is of locally finite length and is distributive, G fulfils (i). The converse fails to hold (cf. the example given in § 4 below).

Let G be a partially ordered group and let $\{G_i\}_{i \in I}$ be a system of convex subgroups of G (all G_i are partially ordered by the induced partial order). Assume that

(a) the group $G^{(1)}$ is a (discrete) direct product of groups $G_i^{(1)}$ $(i \in I)$;

(b) if $i_1, \ldots, i_n$ are distinct elements of I, $0 \neq g_k \in G_{i_k}$ $(k = 1, \ldots, n)$, then from $g_1 + \ldots + g_n \geqq 0$ it follows that $g_1 \geqq 0, \ldots, g_n \geqq 0$.

Under these assumptions the partially ordered group G is said to be a direct sum of partially ordered groups G_i $(i \in I)$ and we express this fact by writing $G = \Sigma_{i \in I} G_i$.

If $\{H_j\}_{j \in J}$ is any system of partially ordered groups, then there is a partially ordered group $H = \Sigma_{j \in J} H'_j$ such that H'_j is isomorphic to H_j for each $j \in J$.

The following assertion is easy to verify.

1.2. Lemma. *Let G and G_i $(i \in I)$ be partially ordered groups, $G = \sum_{i \in I} G_i$. Then G is a multilattice group if and only if all G_i are multilattice groups.*

Let H be a convex subgroup of a partially ordered group G. The partially ordered factor group G/H is the grou $G^{(1)}/H^{(1)}$ which is partially ordered as follows: for $x + H \in G/H$ we put $x + H > H$ if there is $x_1 \in x + H$ with $x_1 > 0$.

2. Homomorphisms s and s' with $s(X) = s'(X) = 1$

Let α be a cardinal, $\alpha \geqq 2$ and let $\mathscr{G}_\alpha$ be as in the introduction. Let $G \in \mathscr{G}_\alpha$. The set of all elements of G which cover 0 is denoted by X.

For every $x \in X$ and every $a \in G$, $0 < x$ implies $a < a + x$. Hence we obtain immediately

2.1. Lemma. *Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ and let $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m$ be positive integers. Assume that $\beta_1 x_1 + \ldots + \beta_n x_n \leqq \gamma_1 y_1 + \ldots + \gamma_m y_m$. Then $\beta_1 + \ldots + \beta_n \leqq \gamma_1 + \ldots + \gamma_m$.*

2.2. Lemma. *Let $x_1, \ldots, x_n \in X$ and let $\beta_1, \ldots, \beta_n$ be nonzero integers. Assume that $\beta_1 x_1 + \ldots + \beta_n x_n \geqq 0$. Then $\beta_1 + \ldots + \beta_n \geqq 0$. If $\beta_1 x_1 + \ldots + \beta_n x_n > 0$, then $\beta_1 + \ldots + \beta_n > 0$.*

Proof. Without loss of generality we can assume that there is a positive integer k with $1 \leqq k \leqq n$ such that $\beta_i > 0$ for $i = 1, 2, \ldots, k$ and $\beta_i < 0$ for $i = k + 1, \ldots, n$. If $k = n$, then the assertion of the lemma obviously holds. Let $k < n$. We have

$$\beta_1 x_1 + \ldots + \beta_k x_k \geqq -\beta_{k+1} x_{k+1} - \ldots - \beta_n x_n > 0,$$

and hence in view of 2.1

$$\beta_1 + \ldots + \beta_k \geqq -\beta_{k+1} - \ldots - \beta_n .$$

If $\beta_1 x_1 + \ldots + \beta_n x_n > 0$, then in the above part of the proof the relation $\geqq$ can be replaced by $>$.

From 2.2 we infer that the following corollaries 2.3 and 2.4 are valid:

2.3. Corollary. *Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ be integers. If $\alpha_1 x_1 + \ldots + \alpha_n x_n < \beta_1 y_1 + \ldots + \beta_m y_m$, then $\alpha_1 + \ldots + \alpha_m < \beta_1 + \ldots + \beta_m$.*

2.4. Corollary. *Let $x_i, \alpha_i, y_j, \beta_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ be as in 2.3. If $\alpha_1 x_1 + \ldots + \alpha_n x_n = \beta_1 y_1 + \ldots + \beta_m y_m$, then $\alpha_1 + \ldots + \alpha_n = \beta_1 + \ldots + \beta_m$.*

Let $0 \neq g \in G$. From 1.1 it follows that there are elements $x_1, \ldots, x_n \in X$ and integers $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 x_1 + \ldots + \alpha_n x_n = g$. Define the mapping $s\colon G \to Z$ by the rule $s(g) = \alpha_1 + \ldots + \alpha_n$. In view of 2.4 the integer $s(g)$ is uniquely determined by g.

Then 2.3 can be expressed by

$$g_1, g_2 \in G,\ g_1 < g_2 \Rightarrow s(g_1) < s(g_2). \tag{2.3.1}$$

Therefore s is a homomorphism of the partially ordered group G onto Z.

2.5. Lemma. *If $0 < g \in G$, then there are $x_1, \ldots, x_n \in X$ such that $g = x_1 + \ldots + x_n$.*

Proof. Let $0 < g \in G$. There are elements $a_0, \ldots, a_n \in G$ such that $0 = a_0 < a_1 < a_2 < \ldots < a_n = g$. For each positive integer i with $1 \leqq i \leqq n$ we have $a_i - a_{i-1} \in X$. Put $a_i - a_{i-1} = x_i$ $(i = 1, \ldots, n)$. Then $g = x_1 + \ldots + x_n$.

Let Z be the additive group of all integers with the natural linear order and let D_α be the direct sum of α copies of Z. Then $D_\alpha \in \mathcal{G}_\alpha$.

Let us denote by F_α the free abelian group with the set X of free generators. If f is a nonzero element of F_α, then there are (uniquely determined) distinct elements $x_1, \ldots, x_n \in X$ and uniquely determined nonzero integers $\alpha_1, \ldots, \alpha_n$ such that $f = \alpha_1 x_1 + \ldots + \alpha_n x_n$. Hence F_α is isomorphic to $(D_\alpha)^{(1)}$. If we put $f > 0$ whenever $\alpha_i > 0$ for $i = 1, \ldots, n$, then we obtain a partially ordered group F'_α isomorphic to D_α. Hence $F'_\alpha \in \mathcal{G}_\alpha$.

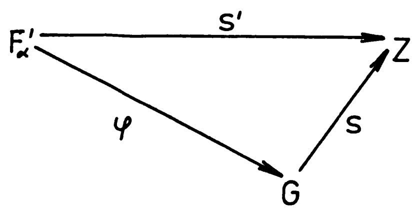
Let f be as above. Consider the mapping $\varphi\colon F_\alpha \to G$ defined by $\varphi(0) = 0$ and $\varphi(f) = g$, where the relation

$$g = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

holds in G.

Therefore by 2.5 we have $f > 0$ iff $\varphi(f) > 0$ and φ is an epimorphism of F'_α onto G.

Consider the mapping $s': F'_\alpha \to Z$ defined by the rule $s'(f) = \alpha_1 + \ldots + \alpha_n$ where $f \in F'_\alpha$, $f = \alpha_1 x_1 + \ldots + \alpha_n x_n$. Then s' is a homomorphism of F'_α onto Z and the following diagram is commutative.



We remark that $s'(x_i) = s(x_i) = 1$ for each $x_i \in X$. Denote $H_G = \text{Ker } \varphi$, $H = \text{Ker } s'$. Let $0 \neq f \in H$, then $s'(f) = 0$, hence the elements f and 0 are incomparable. Therefore the partial order on H is trivial. By using 2.2 for F'_α we infer that H_G is also trivially ordered in F'_α.

We obtain

2.6. Proposition. *Let $G \in \mathscr{G}_\alpha$. Then*

(a) *G is isomorphic to F'_α/H_G.*
(b) *If x_1, x_2 are distinct elements of X, then $x_1 - x_2 \notin H_G$.*
(c) *H_G is a subgroup of H.*

3. Subgroups of Ker s'

Let s' and H be as in section 2. In the present section we describe all groups which belong to $\mathscr{G}_\alpha$. In view of 2.6 it suffices to investigate partially ordered groups having the form F'_α/K, where K is a convex subgroup of F'_α which satisfies the following conditions:

(b') If x_1, x_2 are distinct elements of X, then $x_1 - x_2 \notin K$.
(c') K is a subgroup of H.

We shall show that under the mentioned conditions F'_α/K belongs to $\mathscr{G}_\alpha$.

Let $f \in F'_\alpha$, $f_1 \in F + K$. Then $f_1 = f + k$ for some $k \in K$, thus $s'(f_1) = s'(f)$. Define the mapping s'': $F'_\alpha/K \to Z$ by putting $s''(f + K) = s'(f)$.

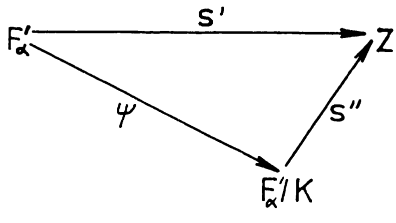
The partially ordered group F'_α belongs to $\mathscr{G}_\alpha$, hence the results from §2 can be applied to F'_α (if s is replaced by s').

3.1. Lemma. *Let A_1, $A_2 \in F'_\alpha/K$, $A_1 < A_2$. Then $s''(A_1) < s''(A_2)$.*

Proof. There are $f_i \in A_i$ $(i = 1,2)$ such that $f_1 < f_2$. In view of 2.3.1, $s'(f_1) < s'(f_2)$. Hence $s''(A_1) < s''(A_2)$.

3.2. Corollary. *F'_α/K is a partially ordered group of locally finite length.*

Let ψ be the natural homomorphism of F'_α onto F'_α/K (i.e., $\psi(f)=f+K$ for each $f\in F'_\alpha$). The following diagram is commutative.



Let $f_1, f_2\in F'_\alpha$. It is easy to verify that the following assertions are true:

(i) *If $f_1<f_2$, then $f_1\prec f_2$ iff $s'(f_1)+1=s'(f_2)$.*
(ii) *If $f_1<f_2$, then $f_1+K\prec f_2+K$ in F'_α/K iff $s''(f_1+K)+1=s''(f_2+K)$*
(iii) *$f_1\prec f_2$ iff $f_1+K\prec f_2+K$.*

From (i)—(iii) we obtain immediately:

(iv) *If $f_1<f_2$, then $f_1\prec f_2$ iff $f_1+A\prec f_2+A$.*

3.3. Lemma. *Let X' be the set of all elements of F'_α/K covering K. Then card $X'=\alpha$.*

Proof. Let $A\in X'$. In view of (iv) there is $a\in A$ such that $0\prec a$. Hence $a\in X$, $A=x+K$ for some $x\in X$. Conversely, let $x\in X$. Then we have $s''(x+K)=1$. From 3.1 we infer that there does not exist any $C\in F'_\alpha/K$ with $K<C<x+K$. Thus $x+K\in X'$. If $x_1, x_2\in X$, $x_1\neq x_2$, then by (b') we have $x_1-x_2\notin K$, $x_1+K\neq x_2+K$. Therefore card $X'=$ card $X=\alpha$.

Since all bounded saturated chains with the same endpoints in F'_α are finite and have the same length, from (iv) we infer that F'/K also satisfies this condition. Since F'_α is directed, F'_α/K is directed as well. Then from 3.2 and 3.3 we obtain

3.4. Theorem. *Let K be a subgroup of F'_α satisfying the conditions (b') and (c'). Then F'_α/K belongs to $\mathcal{G}_\alpha$.*

From 3.4 and 2.6 we infer:

3.5. Theorem. *Let G be a partially ordered group. Then the following conditions are equivalent:*

(i) $G\in\mathcal{G}_\alpha$.
(ii) G is isomorphic to F'_α/K where K is a subgroup of F'_α which satisfies the conditions (b') and (c').

4. Nonisomorphic types of partially ordered groups in $\mathcal{G}_\alpha$

Now we intend to show that for each cardinal $\alpha \geqq 2$ the class $\mathcal{G}_\alpha$ contains an infinite number of nonisomorphic partially ordered groups.

First let $\alpha = 2$ and let F'_2 be the free abelian group generated by elements $x_1, x_2 \in X$, $x_1 \neq x_2$. Let N be the set of all positive integers. For each $n \in N$, $n > 1$ let us form the set $K_n = \{f \in F'_2 : f = np\, x_1 - np\, x_2, p \in Z\}$. Then K_n is a subgroup of F'_α and satisfies the conditions (b') and (c'). In view of 3.4 we obtain $B_n = F'_{2/K_n} \in \mathcal{G}_\alpha$ for each $n \in N$, $n > 1$.

It is easy to varify that B_n is directly indecomposable for each $n > 1$.

We show that the partially ordered groups B_n and B_m are not isomorphic whenever $n, m \in N$, $n \neq m$.

The coset of B_n (B_m) containing an element $f \in F'_2$ will be denoted by $\bar{f}(f^*)$.

Assume that $n < m$ and that there exists an isomorphism φ of B_n onto B_m. Then $\{\bar{x}_1, \bar{x}_2\}$ is the set of all elements of B_n covering $\bar{0}$, and $\{x_1^*, x_2^*\}$ is the set of all elements of B_m covering 0^* (cf. the proof of 3.3). Hence either $\varphi(\overline{x_1}) = x_1^*$, $\varphi(\overline{x_2}) = x_2^*$ or $\varphi(\overline{x_1}) = x_2^*$, $\varphi(\overline{x_2}) = x_1^*$. Further we have $n(\overline{x_1} - \overline{x_2}) = n(\overline{x_1 - x_2}) = (\overline{nx_1 - nx_2}) = \bar{0} = K_n$. Hence $\varphi[n(\overline{x_1} - \overline{x_2})] = 0^* = K_m$. We shall prove that $\varphi[n(\overline{x_1} - \overline{x_2})] \neq 0^*$ holds true, a contradiction. In fact, in the case $\varphi(\overline{x^1}) = x_1^*$, $\varphi(\overline{x_2}) = x_2^*$ we get $\varphi[n(\overline{x_1} - \overline{x_2})] = n[\varphi(\overline{x_1}) - \varphi(\overline{x_2})] = n[x_1^* - x_2^*] = (nx_1 - nx_2)^* \neq$ $\neq 0^*$. The case $\varphi(\overline{x_1}) = x_2^*$, $\varphi(\overline{x_2}) = x_1^*$ is analogous.

We conclude that B_n and B_m are not isomorphic and so for $\alpha = 2$ there exists an infinite number of nonisomorphic partially ordered groups belonging to $\mathcal{G}_2$.

Let $\alpha > 2$ be a cardinal and let β be a cardinal such that $\alpha = \beta + 2$. Let M be a set, card $M = \beta$. For each $i \in M$ let Z_i be the additive group of all integers with the natural linear order. Let $A = \sum_{i \in M} Z_i$ be the direct sum of partially ordered groups Z_i.

An element $a \in A$ satisfies the relation $a > 0$ if and only if there is $i \in M$ such that $a(i) = 1$ and $a(j) = 0$ for each $j \in M$, $j \neq i$. Therefore the cardinality of the set of all elements from A covering the element 0 is equal to β.

Let us form the direct product $D'_n = A \times B_n$, $n \in N$, $n > 1$. Since $A \in G_\beta$, $B_n \in \mathcal{G}_2$, we obtain $D'_n \in \mathcal{G}_\alpha$.

Finally, we want to show that if m, $n \in N$, $m \neq n$, then the partially ordered groups D'_n and D'_m are not isomorphic. If D'_n and D'_m are isomorphic, then from $D'_n = A \times B_n$ and $D'_m = A \times B_m$ and from the fact that B_n, B_m, Z_i are directly indecomposable we would infer (by using the well-known theorem of Šimbireva

[6], cf. also [3], Chap. II, Thm 8) that the partially ordered groups B_n and B_m are isomorphic, which is a contradiction.

As an example, in Fig. 1 there is given the diagram of the partially ordered group $B_3 = F_2'/K_3$.

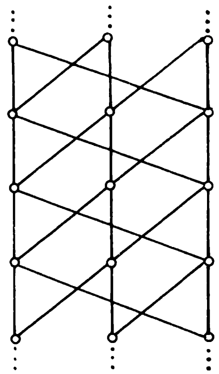


Fig. 1

We conclude by giving an example of a partially ordered group $G \in \mathcal{G}_3$ which fails to be distributive. Let G be the set of all pairs (x, y) with $x \in Z$, $y \in \{0, 1, 2\}$, with the operation $+$ defined coordinate-wise (for the second coordinate we apply the addition mod 3). We put $(x, y) \geqq 0$ iff either $x = y = 0$ or $x > 0$. Then $G \in \mathcal{G}_3$ and G is not distributive.

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О ЧАСТИЧНО УПОРЯДОЧЕННЫХ ГРУППАХ ЛОКАЛЬНО КОНЕЧНОЙ ДЛИНЫ

Božena Černáková

Резюме

В работе исследуется строение абелевой частично упорядоченной группы $(G, +, \leqq)$, для которой выполнены следующие условия: (i) $(G, \leqq)$ — направленное множество, (ii) если a, $b \in G$ и $a < b$, тогда все насыщенные цепы соединяющие элементы a и b конечны и одинаковой длины.