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GRAPH ISOMORPHISMS OF SEMIMODULAR LATTICES

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This note is a continuation of a former paper of the author [4], where it was proved that a condition concerning sublattices of type C (for denotations, cf. below) is sufficient for semimodular lattices $\mathscr{L}$ and $\mathscr{L}_1$ of locally finite length with isomorphic graphs to have direct product representations $f: \mathscr{L} \to \mathscr{A} \times \mathscr{B}$ and $g: \mathscr{L}_1 \to \mathscr{A} \times \mathscr{B}^\sim$ such that $h = g^{-1}f$ (where $\mathscr{B}^\sim$ is dual to $\mathscr{B}$ and h is the given graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$).

In the present paper it will be shown that the condition concerning sublattices of type C is also necessary for the existence of such direct product representations. A further result on graph isomorphisms of semimodular lattices (dealing with sublattices of type C_1) is established.

Graph isomorphisms of distributive lattices were studied in [7]; for the case of modular lattices cf. Birkhoff [1] and the author [3], [5].

We recall some notions of graphs of lattices. Let $\mathscr{L} = (L; \leqq)$ be a lattice. $\mathscr{L}$ is said to be of locally finite length if each bounded chain in $\mathscr{L}$ is finite. In what follows all lattices are assumed to be of locally finite length. If $a, b \in L$ and a is covered by b (i.e., $a < b$ and the interval $[a, b]$ is prime), then we write $a \prec b$ or $b \succ a$. The lattice $\mathscr{L}$ is called semimodular if and only if its elements satisfy

(ξ') *If x and y cover a, and $x \neq y$, then $x \vee y$ covers x and y.* (Cf. [2a], p. 100; in [2b], p. 15, the term 'semimodularity' has a different meaning.)

By the graph $G(\mathscr{L})$ we mean the undirected graph whose set of vertices is L and whose edges are those pairs $\{a, b\}$ which satisfy either $a \prec b$ or $b \prec a$. If $\mathscr{G}_1$ and $\mathscr{G}_2$ are graphs with sets of vertices G_1 and G_2 and if $h: G_1 \to G_2$ is a bijection such that, for any x and y from G_1 the pair $\{x, y\}$ is an edge in $\mathscr{G}_1$ if and only if $\{h(x), h(y)\}$ is an edge in $\mathscr{G}_2$, then h is said to be an isomorphism of $\mathscr{G}_1$ onto $\mathscr{G}_2$.

If $\mathscr{L}_1 = (L_1; \leqq_1)$ is a lattice and h is an isomorphism of $G(\mathscr{L})$ onto $G(\mathscr{L}_1)$, then h is called a graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$. The covering relation in $\mathscr{L}_1$ is denoted by $\prec_1$.

Now let $h: L \to L_1$ be any bijection and let $T \subseteq L$. The subset T is said to be preserved (reversed) under h if, whenever $t_1, t_2 \in T$, $x_1, x_2 \in L$ and $t_1 \leqq x_1 \prec x_2 \leqq t_2$, then $h(x_1) \prec_1 h(x_2)$ (or $h(x_1) \succ_1 h(x_2)$, respectively).

Let C be the lattice in Fig. 1. A lattice is said to be of type C if it is isomorphic to C. Consider the following conditions for the lattices $\mathcal{L}$ and $\mathcal{L}_1$ and for the mapping h:

(α_1) All sublattices of type C of $\mathcal{L}$ are preserved under h and all sublattices of type C of $\mathcal{L}_1$ are preserved under h^{-1}.

(α_2) There are lattices $\mathcal{A}$ and $\mathcal{B}$ and direct product representations $f\colon \mathcal{L}\to \mathcal{A}\times\mathcal{B}$, $g\colon \mathcal{L}_1\to\mathcal{A}\times\mathcal{B}$ such that $h=g^{-1}f$.

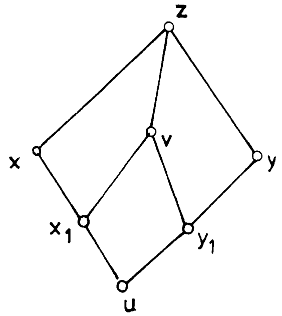

Fig 1

The following result was proved in [4]:

(A) ([4], Theorem 2.) *Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices and let h be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then* $(\alpha_1)\Rightarrow(\alpha_2)$.

(In [4] it was assumed that $\mathcal{L}$ and $\mathcal{L}_1$ are finite, but the proof established in [4] remains valid in the case when $\mathcal{L}$ and $\mathcal{L}_1$ are of locally finite length. Also, in Thm. 2 of [4] it was asserted only that there are lattices $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{L}\cong\mathcal{A}\times\mathcal{B}$ and $\mathcal{L}_1\cong\mathcal{A}\times\mathcal{B}^{\sim}$; but, in fact, the stronger result $(\alpha_1)\Rightarrow(\alpha_2)$ was proved in [4]. If (α_2) holds, then h is a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$.)

1. Lemma. *Let $\mathcal{T}=(T;\leqq)$ be a lattice of type C. Then $\mathcal{T}$ is subdirectly irreducible.*

The proof is simple; it will be omitted.

Now let $\mathcal{L}$, $\mathcal{L}_1$ and h be as above. Assume that (α_2) holds. We denote $\mathcal{A}=(A;\leqq)$, $\mathcal{B}=(B;\leqq)$. In view of the assumption, there exists an isomorphism f of $\mathcal{L}$ onto $\mathcal{A}\times\mathcal{B}$. If $x\in L$ and $f(x)=(a,b)$, then we write also $a=x(A)$, $b=x(B)$. For $M\subseteq L$ we put $M(A)=\{x(A)\colon x\in M\}$, $M(B)=\{x(B)\colon x\in M\}$.

2. Lemma. *Let $\mathcal{T}=(T;\leqq)$ be a sublattice of $\mathcal{L}$ and suppose that $\mathcal{T}$ is of type C. Then we have either* (i) $\operatorname{card} T(A)=1$, *or* (ii) $\operatorname{card} T(B)=1$.

Proof. Put $\mathcal{T}_1=(T(A);\leqq)$, $\mathcal{T}_2=(T(B);\leqq)$. The injection defined by $f|_T\colon \mathcal{T}\to\mathcal{T}_1\times\mathcal{T}_2$ is a subdirect product representation of $\mathcal{T}$; in view of Lemma 1 we infer that either (i) of (ii) is valid

If (i) holds, then clearly T is reversed under f; if (ii) is valid, then T is preserved under f.

3. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be semimodular lattices. Then* $(\alpha_2) \Rightarrow (\alpha_1)$.

Proof. Let h: $L \to L_1$ be a bijection. Assume that (α_2) is valid. Then $h = g^{-1}f$, and as already remarked above, h is a graph isomorphism. By way of contradiction, suppose that there is a sublattice $\mathscr{T}$ in $\mathscr{L}$ such that $\mathscr{T}$ is of type C and T is not preserved under h. (If in this supposition $\mathscr{L}$ and $\mathscr{H}$ are replaced by $\mathscr{L}_1$ and h^{-1}, then we proceed analogously.) Thus the condition (i) of Lemma 2 holds and hence $\mathscr{T}$ is reversed under h. Also, from (α_2) we easily obtain that $(h(T); \leqq_1) = \mathscr{T}_1$ is a sublattice of $\mathscr{L}_1$ which is dually isomorphic to C. By using [8], § 45 it is easy to verify that $\mathscr{L}_1$ is not semimodular, which is a contradiction.

Theorem (A) and Lemma 3 yield:

4. Theorem. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be semimodular lattices and let h be a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$. Then the conditions (α_1) and (α_2) are equivalent.*

Let $\mathscr{T} = (T; \leqq)$ be a sublattice of a lattice $\mathscr{L} = (L; \leqq)$. Assume that there exists an isomorphism φ of C onto $\mathscr{T}$ such that $\varphi(u) < \varphi(x_1) < \varphi(v)$, $\varphi(u) < \varphi(y_1) < \varphi(v)$, $\varphi(x) < \varphi(z)$ and $\varphi(y) < \varphi(z)$. Then $\mathscr{T}$ will be called a C_1-sublattice of $\mathscr{L}$. If, moreover, $\varphi(x_1) < \varphi(x)$, $\varphi(v) < \varphi(z)$ and $\varphi(y_1) < \varphi(y)$, then $\mathscr{T}$ is said to be a C_2-sublattice of $\mathscr{L}$.

Let $\mathscr{L}_1 = (L_1; \leqq_1)$ be a lattice and let h: $\mathscr{L} \to \mathscr{L}_1$ be a bijection. Consider the following conditions $(i = 1, 2)$:

(α_{1i}) All C_i-sublattices of $\mathscr{L}$ are preserved under h and all C_i-sublattices of $\mathscr{L}_1$ are preserved under h^{-1}.

Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of L such that $u < x_1 < x_2 < \ldots < x_m < v$, $u < y_1 < y_2 < \ldots < y_n < v$ and either (i) $x_1 \vee y_1 = v$, or (ii) $x_m \wedge y_n = u$. Then the set $\{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is said to be a cycle in $\mathscr{L}$; if moreover, $m > 1$ or $n > 1$, then this cycle is called proper.

From [6] (Thm. 3.7 and Lemma 2.3) we obtain:

5. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be lattices and let h be a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$. Then the condition* (α_2) *is equivalent with the condition*

(α_3) *if C_0 is a proper cycle of $\mathscr{L}$ (of $\mathscr{L}_1$), then C_0 is either preserved or reversed under h (or h^{-1}, respectively).*

6. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be semimodular lattices and let h be a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$. Then* $(\alpha_{11}) \Rightarrow (\alpha_2)$.

Proof. In establishing the proof of Theorem 2 in [4] the condition (α_1) was used in the proofs of the lemmas 9 and 10 only; now for proving that $(\alpha_{11}) \Rightarrow (\alpha_2)$ is valid it suffices to replace the expresion 'a lattice of type C' by 'a C_1-sublattice' in these lemmas.

7. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be semimodular lattices and let h be a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$. Then* $(\alpha_2) \Rightarrow (\alpha_{11})$.

Proof. According to Lemma 3 we have $(\alpha_2) \Rightarrow (\alpha_1)$, and clearly $(\alpha_1) \Rightarrow (\alpha_{11})$.

Alternative proof: Let $\mathcal{T}$ be a C_1-sublattice of $\mathcal{L}$. Under the denotations as above, there exist elements $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in L$ such that $\varphi(x_1) = a_0 < a_1 < \ldots < a_m = \varphi(x)$, $\varphi(y_1) = b_0 < b_1 < b_2 < \ldots < b_n = \varphi(y)$. Then $\{\varphi(u), \varphi(z), a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n\}$ is a proper cycle in $\mathcal{L}$ (because $a_m \wedge b_n = \varphi(u)$). Hence in view of Lemma 5, the interval $J = [\varphi(u), \varphi(z)]$ is either preserved or reversed under h. If J is reversed under h, then we easily obtain from (α_2) that $h|_J$ is a dual isomorphism of J onto the interval $[h(\varphi(z)), h(\varphi(u))]$ of $\mathcal{L}_1$, but this interval fails to be semimodular; thus $\mathcal{L}_1$ is not semimodular, which is a contradiction. Hence T is preserved under h. Analogously we verify that each C_1-sublattice of $\mathcal{L}_1$ is preserved under h^{-1}.

Theorem 4, Lemma 6 and Lemma 7 yield:

8. Corollary. *Let $\mathcal{L}$ and $\mathcal{L}_1$ be semimodular lattices and let h be a graph isomorphism of $\mathcal{L}$ onto $\mathcal{L}_1$. Then* $(\alpha_2) \Leftrightarrow (\alpha_{11}) \Leftrightarrow (\alpha_1)$.

The following question remains open:

Let $\mathcal{L}$, $\mathcal{L}_1$ and h be as in Corollary 8; are the conditions (α_2) and (α_{12}) equivalent?

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ИЗОМОРФИЗМЫ ГРАФОВ ПОЛУДЕДЕКИНДОВЫХ РЕШЕТОК

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Резюме

В статье автора [4] найдено достаточное условие, при котором полудедекиндовы решетки $\mathcal{L}$ и $\mathcal{L}_1$ локально конечной длины с изоморфными графами отличаются только двоиственностью некоторого прямого сомножителя; в предлагаемой заметке доказано, что это условие является тоже необходимым.