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ON THE CANCELLATION LAW FOR DISCONNECTED PARTIALLY ORDERED SETS

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ABSTRACT. In this paper there are given sufficient conditions for the validity of a type of cancellation rule concerning direct product decompositions of partially ordered sets.

1. Introduction

Basic results on direct product decompositions of connected partially ordered sets have been proved by Hashimoto [3], [4]. In the present note we deal with direct product decompositions of partially ordered sets which need not be connected.

We apply the standard notation (cf. Birkhoff [2]); the direct product of partially ordered sets $\mathbb{P}_i$ $(i \in I)$ is denoted by $\prod_{i\in I} \mathbb{P}_i$. If $I = \{1, 2, \dots, n\}$, then we write also $\mathbb{P}_1\mathbb{P}_2 \cdots \mathbb{P}_n$. For the further terminology concerning direct products, cf. Section 2 below.

We will deal with the validity of the implication (*cancellation law*)

$$\mathbb{A}\mathbb{B} \cong \mathbb{A}\mathbb{C} \implies \mathbb{B} \cong \mathbb{C}; \tag{1}$$

further, we consider the implication

$$\mathbb{A}^k \cong \mathbb{B}^k \implies \mathbb{A} \cong \mathbb{B}, \tag{2}$$

where $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are partially ordered sets and k is a positive integer.

Consider the following conditions for a partially ordered set $\mathbb{P}$:

(i) The number of connected components of $\mathbb{P}$ is finite.

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(ii) Each connected component $\mathbb{K}$ of $\mathbb{P}$ containing more than one element is isomorphic to a direct product $\prod\limits_{\lambda\in\Lambda} \mathbb{K}_\lambda$ of indecomposable factors $\mathbb{K}_\lambda$ such that for each $\lambda_0 \in \Lambda$ the set $\{\lambda \in \Lambda : \mathbb{K}_\lambda \cong \mathbb{K}_{\lambda_0}\}$ is finite.

We will prove the following results:

(*) The implication (1) holds for any $\mathbb{A}, \mathbb{B}, \mathbb{C}$ satisfying the conditions (i), (ii).

(**) The implication (2) holds for any positive integer k and $\mathbb{A}, \mathbb{B}$ satisfying (i), (ii).

By examples we will show that without the assumptions (i), (ii) the implications (1) and (2) fail to hold.

The cancellation rule in the class of directed sets of finite length has been dealt with in [7]. Another type of the cancellation rule (dealing with internal direct product decompositions) has been investigated in [5] and [6]. For the case of finite algebras and finite relational structures, several results on the cancellation law have been proved in [8], [9] and [1]; for a survey concerning the implications (1) and (2), cf. [10; Section 5.7]. The particular case of unary algebras has been dealt with in [11] and [12].

2. Preliminaries

Each partially ordered set under consideration is assumed to be nonempty.

We recall that a partially ordered set $\mathbb{P}$ is called *indecomposable* if it has more than one element and if it cannot be written as a direct product $\mathbb{A}\mathbb{B}$ with $|A| > 1$, $|B| > 1$.

Let $\mathbb{P}$ be any partially ordered set and let α be a cardinal number, $\alpha \neq 0$. The symbol $\mathbb{P}^\alpha$ will be used for $\prod\limits_{i\in I} \mathbb{P}_i$, where $|I| = \alpha$ and $\mathbb{P}_i = \mathbb{P}$ for each $i \in I$. By $\mathbb{P}^0$ a one-element partially ordered set will be meant.

Assume that $\mathbb{P}_i = (P_i, \leq_i)$ $(i \in I)$ are partially ordered sets such that $P_{i(1)} \cap P_{i(2)} = \emptyset$ whenever $i(1)$ and $i(2)$ are distinct elements of I. Put $P = \bigcup\limits_{i\in I} P_i$. For $x, y \in P$ we define $x \leq y$ if there exists $h \in I$ such that $x, y \in P_h$ and $x \leq_h y$. Then the partially ordered set $\mathbb{P} = (P, \leq)$ is called the *sum* of the system $(\mathbb{P}_i)_{i\in I}$ and it is denoted by $\sum\limits_{i\in I} \mathbb{P}_i$.

A partially ordered set $\mathbb{S}$ is said to be *connected* if it cannot be expressed as a sum of two its subsets. Connected summands of a partially ordered set $\mathbb{P}$ will be referred to as *connected components* of $\mathbb{P}$.

If $\mathbb{P} = (P, \leq)$ is a partially ordered set, $a, b \in P$, by a *zigzag* connecting a with b, a finite sequence $x_0{=}a, x_1, \dots, x_n{=}b$ in P such that any two adjoining elements are comparable, will be meant. The number n will be called the length of this zigzag.

The following statement is evident.

2.1. Lemma. *A partially ordered set $\mathbb{P} = (P, \leq)$ is connected if and only if any two elements of P can be connected by a zigzag.*

Let us suppose that $\mathbb{S} = (S, \leq)$ is a connected partially ordered set. Let us define the *distance* of two elements of S as follows:
if $a, b \in S$, $d(a,b)$ will be the length of the shortest zigzag connecting a with b.

It is easy to see that d is a metric in S. Now we can define

$$d(\mathbb{S}) = \begin{cases} n & \text{if } n = \max\{d(a,b) :\ a, b \in S\}, \\ \infty & \text{if the set } \{d(a,b) :\ a, b \in S\} \text{ is not bounded.} \end{cases}$$

The following lemma can be proved easily.

2.2. Lemma. *Let $(\mathbb{P}_i : i \in I)$ be a nonempty system of partially ordered sets. If $\prod_{i \in I} \mathbb{P}_i$ is connected, then all $\mathbb{P}_i$ are also connected.*

2.3. Lemma. *Let $(\mathbb{S}_i : i \in I)$ be a nonempty system of connected partially ordered sets. Then $\mathbb{S} = \prod_{i \in I} \mathbb{S}_i$ is connected if and only if for each $a, b \in S$ the set $\{d(a(i), b(i)) :\ i \in I\}$ is bounded.*

Proof. First assume that $\mathbb{S}$ is connected. Take any $a, b \in S$ and suppose that $d(a,b) = n$. Then there exists a zigzag $x_0{=}a, x_1, \dots, x_n{=}b$ in S. Evidently $x_0(i){=}a(i), x_1(i), \dots, x_n(i){=}b(i)$ is a zigzag in S_i, so that $d(a(i), b(i)) \leq n$ for each $i \in I$.

Conversely, suppose that $a, b \in S$, $a \neq b$, $d(a(i), b(i)) \leq n$ for a positive integer n and for each $i \in I$. We will show that there exists a zigzag in S connecting a and b. Without loss of generality we can suppose that n is odd (in the case of n even the method is analogous). The assumption $d(a(i), b(i)) \leq n$ yields that there exists a zigzag $x_0^i{=}a(i), x_1^i, \dots, x_n^i{=}b(i)$ in S_i such that either $x_0^i \leq x_1^i \geq \dots \leq x_n^i$ or $x_0^i \geq x_1^i \leq \dots \geq x_n^i$ holds. Let I_1 be the set of all $i \in I$ such that the first possibility occurs. If $I_1 = I$, we have $a = x_0 \leq x_1 \geq \dots \leq x_n = b$ for x_j defined by $x_j(i) = x_j^i$ for all $i \in I$. If $I_1 \neq I$, define y_j for $j \in \{0, \dots, n{+}1\}$ in such a way that

$$y_j(i) = \begin{cases} x_j^i & \text{if } i \in I_1,\ j \leq n, \\ x_n^i & \text{if } i \in I_1,\ j = n+1, \\ x_{j-1}^i & \text{if } i \in I - I_1,\ j > 0, \\ x_0^i & \text{if } i \in I - I_1,\ j = 0. \end{cases}$$

Then it is easy to see that $a = y_0 \leq y_1 \geq \dots \leq y_n \geq y_{n+1} = b$. □

Looking at the previous proof we obtain:

2.3.1. Corollary. *Let* $(\mathbb{S}_i : i \in I)$ *be a nonempty system of partially ordered sets,* $\mathbb{S} = \prod_{i\in I} \mathbb{S}_i$ *and let* $\mathbb{S}$ *be connected. If* $a, b \in S$, *then*

(i) $d\big(a(i), b(i)\big) \le d(a,b)$ *for each* $i \in I$;

(ii) *if* $d\big(a(i), b(i)\big) \le n$ *for each* $i \in I$, *then* $d(a,b) \le n+1$.

Using 2.3 we obtain:

2.4. Proposition. *Let* $(\mathbb{S}_i : i \in I)$ *be a nonempty system of connected partially ordered sets. Then* $\prod_{i\in I} \mathbb{S}_i$ *is connected if and only if the set* $I_1 = \{i \in I : d(\mathbb{S}_i) = \infty\}$ *is finite and the set* $\{d(\mathbb{S}_i) : i \in I - I_1\}$ *is bounded.*

Proof. First let us suppose that $I_1 = \{i \in I : d(\mathbb{S}_i) = \infty\}$ is finite and $n' = \max\{d(\mathbb{S}_i) : i \in I - I_1\}$. Take any a, b belonging to the Cartesian product of the sets S_i $(i \in I)$; put $n = \max\big(\{n'\} \cup \{d\big(a(i), b(i)\big) : i \in I_1\}\big)$. Evidently $d\big(a(i), b(i)\big) \le n$ holds for each $i \in I$, so that the set $\{d\big(a(i), b(i)\big) : i \in I\}$ is bounded. We have proved that $\prod_{i\in I} \mathbb{S}_i$ is connected. To prove the converse, let us suppose that either I_1 is infinite or $\{d(\mathbb{S}_i) : i \in I - I_1\}$ is unbounded. In both cases we can find an infinite sequence $\{i_n\}_{n=1}^{\infty}$ of distinct elements of I and a sequence $\big\{(a_{i_n}, b_{i_n})\big\}_{n=1}^{\infty}$ such that $a_{i_n}, b_{i_n} \in S_{i_n}$, $d(a_{i_n}, b_{i_n}) > n$ for each positive integer n. Now take any $a, b \in \prod_{i\in I} S_i$ with $a(i_n) = a_{i_n}$, $b(i_n) = b_{i_n}$. Evidently the set $\{d\big(a(i), b(i)\big) : i \in I\}$ is not bounded. Hence $\prod_{i\in I} \mathbb{S}_i$ is not connected. The proof is finished. □

We can prove easily:

2.5. Lemma. *Let* $(\mathbb{A}_i : i \in I)$, $(\mathbb{B}_j : j \in J)$ *be two nonempty systems of partially ordered sets. Then*

$$\left(\sum_{i\in I} \mathbb{A}_i\right)\left(\sum_{j\in J} \mathbb{B}_j\right) = \sum_{i\in I}\sum_{j\in J} \mathbb{A}_i \mathbb{B}_j .$$

We will use the following theorems (cf [4]):

2.6. Theorem. *Any two direct product decompositions of a connected partially ordered set have a common refinement.*

2.7. Theorem. *The representation of a connected partially ordered set as a direct product of indecomposable factors, if it exists, is unique up to isomorphism of the factors.*

In what follows, the symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ will be used for the set of all positive integers, nonnegative integers and integers, respectively.

For dealing with direct products of partially ordered sets which may have an infinite number of direct factors we need a slight generalization of the notion of a polynomial over an integrity domain.

2.8. Definition. Let $\mathcal{O}$ be an integrity domain and let α be an infinite cardinal number. Suppose that I is a well-ordered set with $|I| = \alpha$. For each $i \in I$ let x_i be a symbol not belonging to $\mathcal{O}$ such that $x_{i(1)} \neq x_{i(2)}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. The symbols x_i will be called *indeterminates* over $\mathcal{O}$. Let $n \in \mathbb{N}$; for $k \in \{1, \dots, n\}$ let $\alpha_{ki} \in \mathbb{N}_0$. Consider the expressions

$$p_k = \prod_{i \in I} x_i^{\alpha_{ki}},$$

and assume that $p_{k(1)} \neq p_{k(2)}$ if $k(1), k(2) \in \{1, \dots, n\}$, $k(1) \neq k(2)$. The symbol

$$c_1 p_1 + \dots + c_n p_n$$

with $c_1, \dots, c_n \in \mathcal{O} - \{0\}$ will be called a *generalized polynomial* over $\mathcal{O}$ with the indeterminates x_i $(i \in I)$.

The system consisting of all such generalized polynomials and of the zero polynomial will be denoted by $\mathcal{O}[x_i : i \in I]$.

For $f, g \in \mathcal{O}[x_i : i \in I]$ we can define the relation $f = g$ and the operations $f + g$, $f \cdot g$ analogously as in the case of polynomials over $\mathcal{O}$.

By using the well-known fact that the ring of polynomials $\mathcal{O}[x]$ is an integrity domain and by applying the transfinite induction (with respect to the elements i of the well-ordered set I) we obtain:

2.9. Proposition. *Let $\mathcal{O}$ be an integrity domain. Then $\mathcal{O}[x_i : i \in I]$ is an integrity domain as well.*

3. Cancellation law

In this section we will deal with the validity of the implications

$$\mathbb{A}\mathbb{B} \cong \mathbb{A}\mathbb{C} \implies \mathbb{B} \cong \mathbb{C}, \tag{1}$$

$$\mathbb{A}^k \cong \mathbb{B}^k \implies \mathbb{A} \cong \mathbb{B} \tag{2}$$

with $\mathbb{A}, \mathbb{B}, \mathbb{C}$ being partially ordered sets, $k \in \mathbb{N}$.

First we will consider the implication (1). It is easy to see that this implication doesn't hold in general. If, e.g., $\mathbf{2}$ is a two-element chain and we take $\mathbb{A} = \mathbf{2}^{\aleph_0}$, $\mathbb{B} = \mathbf{2}$, $\mathbb{C} = \mathbf{2}^2$, then $\mathbb{A}\mathbb{B} \cong \mathbf{2}^{\aleph_0} \cong \mathbb{A}\mathbb{C}$, but $\mathbb{B} \not\cong \mathbb{C}$.

Assume that $\mathcal{P}$ is a nonempty class of partially ordered sets such that for each $\mathbb{P} \in \mathcal{P}$ the conditions (i) and (ii) from Introduction are satisfied. Let us suppose that $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$ and that at least one of them has a connected component containing more than one element. Consider all connected components of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ containing more than one element and their decompositions as direct products of indecomposable factors (by 2.7 these decompositions are determined uniquely up to isomorphisms).

Let $\{\mathbb{Y}_i\}_{i\in I}$ be a system of partially ordered sets such that:

(a) if $i(1), i(2)$ are distinct elements of I, then $\mathbb{Y}_{i(1)} \not\cong \mathbb{Y}_{i(2)}$;

(b) if $\mathbb{X} \in \{\mathbb{A}, \mathbb{B}, \mathbb{C}\}$, $\mathbb{K}$ is a connected component of $\mathbb{X}$ and $\mathbb{K}_1$ is an indecomposable factor of $\mathbb{K}$, then there exists $i \in I$ such that $\mathbb{K}_1 \cong \mathbb{Y}_i$;

(c) if $i \in I$, then there exist $\mathbb{X} \in \{\mathbb{A}, \mathbb{B}, \mathbb{C}\}$, a connected component $\mathbb{K}$ of $\mathbb{X}$ and an indecomposable factor $\mathbb{K}_1$ of $\mathbb{K}$ such that $\mathbb{K}_1 \cong \mathbb{Y}_i$.

Since the direct product of all connected components of $\mathbb{A}, \mathbb{B}, \mathbb{C}$, as the product of finitely many connected partially ordered sets, is connected, so is the product $\prod\limits_{i\in I} \mathbb{Y}_i$. Hence $I_1 = \{i \in I : d(\mathbb{Y}_i) = \infty\}$ is finite and the set $\{d(\mathbb{Y}_i) : i \in I - I_1\}$ is bounded by 2.4. Now let $\alpha_i \in \mathbb{N}_0$ for each $i \in I$. Then $\prod\limits_{i\in I} \mathbb{Y}_i^{\alpha_i}$ is connected, too. Namely, if we define $\mathbb{S}_{ij} = \mathbb{Y}_i$ for each $i \in I$ with $\alpha_i > 0$ and $j \in \{1, \dots, \alpha_i\}$, then $\prod\limits_{i\in I} \mathbb{Y}_i^{\alpha_i} \cong \prod\limits_{\substack{i\in I\\ \alpha_i>0}} \prod\limits_{j=1}^{\alpha_i} \mathbb{S}_{ij}$ and the factors $\mathbb{S}_{ij}$ also satisfy the conditions concerning $d(\mathbb{S}_{ij})$ given in 2.4.

Let $\mathbb{Q}$ be any partially ordered set and $c \in \mathbb{N}$. If $c = 1$, we put $c\mathbb{Q} = \mathbb{Q}$. If $c > 1$, we define $c\mathbb{Q}$ to be the sum of c copies of $\mathbb{Q}$.

In view of 2.7 we can state:

3.1. Lemma. *If α_i, β_i $(i \in I)$ are any nonnegative integers, then $\prod\limits_{i\in I} \mathbb{Y}_i^{\alpha_i}$ is isomorphic to $\prod\limits_{i\in I} \mathbb{Y}_i^{\beta_i}$ if and only if $\alpha_i = \beta_i$ for each $i \in I$ (under an appropriate notation of the indices).*

So we have:

3.2. Lemma. *Each connected component of any of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ (including the one-element ones) is isomorphic to $\prod\limits_{i\in I} \mathbb{Y}_i^{\alpha_i}$ for a unique system $(\alpha_i)_{i\in I}$ of nonnegative integers.*

Now let $f = f\big((x_i)_{i\in I}\big)$ be a generalized polynomial belonging to $\mathcal{O}[x_i : i \in I]$ with $\mathcal{O} = \mathbb{Z}$ (cf. Definition 2.8). Assume that all coefficients $c_1, \dots, c_n$ standing in f belong to $\mathbb{N}$. Then we will say that f is a *generalized polynomial over* $\mathbb{N}$. If

$$f\big((x_i)_{i\in I}\big) = \sum_{t=1}^{n} c_t \prod_{i\in I} x_i^{\alpha_{ti}},$$

then we put

$$f\big((\mathbb{Y}_i)_{i\in I}\big) = \sum_{t=1}^{n} c_t \prod_{i\in I} \mathbb{Y}_i^{\alpha_{ti}} .$$

The following lemma is evident.

3.3. Lemma. *If $f = f\big((x_i)_{i\in I}\big)$, $g = g\big((x_i)_{i\in I}\big)$ are generalized polynomials over $\mathbb{N}$, then $f\big((\mathbb{Y}_i)_{i\in I}\big)$ is isomorphic to $g\big((\mathbb{Y}_i)_{i\in I}\big)$ if and only if $f = g$.*

In view of 3.2 and 3.3 we have:

3.4. Lemma. *Each of the partially ordered sets $\mathbb{A}, \mathbb{B}, \mathbb{C}$ is isomorphic to $f\big((\mathbb{Y}_i)_{i\in I}\big)$ for a unique generalized polynomial $f\big((x_i)_{i\in I}\big)$ over $\mathbb{N}$.*

Let $\mathbb{A} \cong f_A\big((\mathbb{Y}_i)_{i\in I}\big)$, $\mathbb{B} \cong f_B\big((\mathbb{Y}_i)_{i\in I}\big)$, $\mathbb{C} \cong f_C\big((\mathbb{Y}_i)_{i\in I}\big)$.
Using 2.5 we obtain:

3.5. Lemma. *The product $\mathbb{AB}$ $(\mathbb{AC})$ is isomorphic to $(f_A \cdot f_B)\big((\mathbb{Y}_i)_{i\in I}\big)$ $\big((f_A \cdot f_C)\big((\mathbb{Y}_i)_{i\in I}\big)\big)$.*

Let $(*)$ be as in Introduction.

Proof of $(*)$. Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be partially ordered sets satisfying (i) and (ii), $\mathbb{AB} \cong \mathbb{AC}$. First suppose that all connected components of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are one-element sets, i.e., $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are (finite) antichains. Then $\mathbb{AB}, \mathbb{AC}$ are also antichains and they are of the same cardinality. Then evidently $\mathbb{B}, \mathbb{C}$ are of the same cardinality, too, so that they are isomorphic.

Now let at least one of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ have a connected component containing more than one element. In view of 3.5 and 3.3 we have $f_A\big((x_i)_{i\in I}\big) \cdot f_B\big((x_i)_{i\in I}\big) = f_A\big((x_i)_{i\in I}\big) \cdot f_C\big((x_i)_{i\in I}\big)$. Since evidently f_A fails to be a zero polynomial, using the cancellation law in the integrity domain $\mathbb{Z}\big[(x_i)_{i\in I}\big]$ (cf. 2.9) we obtain $f_B\big((x_i)_{i\in I}\big) = f_C\big((x_i)_{i\in I}\big)$. The last equality implies $\mathbb{B} \cong \mathbb{C}$. □

Now we want to show that if some of the conditions (i), (ii) from $(*)$ is omitted, then the implication (1) need not hold.

3.6. Example. Let $\mathbf{2}$ be as above and let $\mathbb{A} = \mathbf{2}^0 + \mathbf{2}^0 + \mathbf{2}^0 + \dots$, $\mathbb{B} = \mathbf{2} + \mathbf{2}^2 + \mathbf{2}^3 + \dots$, $\mathbb{C} = \mathbb{B} + \mathbb{B}$. So $\mathbb{A}, \mathbb{B}, \mathbb{C}$ don't satisfy (i). Using 2.5 we obtain

$$\mathbb{AB} \cong \mathbb{B} + \mathbb{B} + \dots, \qquad \mathbb{AC} = \mathbb{A}(\mathbb{B} + \mathbb{B}) = \mathbb{AB} + \mathbb{AB} \cong \mathbb{B} + \mathbb{B} + \dots .$$

Hence $\mathbb{AB} \cong \mathbb{AC}$, but evidently $\mathbb{B} \not\cong \mathbb{C}$. Let us notice that the partially ordered set $\mathbb{A}$ is indecomposable.

3.7. EXAMPLE. Let $\mathbf{2}$ be as above and let $\mathbf{3}$ be a three-element chain. Further, let us denote $\alpha = \aleph_0$, $\beta = 2^{\aleph_0}$ and take $\mathbb{A} = \mathbf{2}^\beta + \mathbf{3}^\beta$, $\mathbb{B} = \mathbf{2}^\alpha\mathbf{3}^\beta + \mathbf{2}^\beta\mathbf{3}^\alpha$, $\mathbb{C} = \mathbf{2}^\alpha\mathbf{3}^\alpha + \mathbf{2}^\beta\mathbf{3}^\beta$. Since $\alpha + \alpha = \alpha$, $\beta + \beta = \beta$, $\alpha + \beta = \beta$, we have $\mathbb{AB} \cong \mathbf{2}^{\beta+\alpha}\mathbf{3}^\beta + \mathbf{2}^{\beta+\beta}\mathbf{3}^\alpha + \mathbf{2}^\alpha\mathbf{3}^{\beta+\beta} + \mathbf{2}^\beta\mathbf{3}^{\beta+\alpha} \cong \mathbf{2}^\beta\mathbf{3}^\beta + \mathbf{2}^\beta\mathbf{3}^\alpha + \mathbf{2}^\alpha\mathbf{3}^\beta + \mathbf{2}^\beta\mathbf{3}^\beta$, $\mathbb{AC} \cong \mathbf{2}^{\beta+\alpha}\mathbf{3}^\alpha + \mathbf{2}^{\beta+\beta}\mathbf{3}^\beta + \mathbf{2}^\alpha\mathbf{3}^{\beta+\alpha} + \mathbf{2}^\beta\mathbf{3}^{\beta+\beta} \cong \mathbf{2}^\beta\mathbf{3}^\alpha + \mathbf{2}^\beta\mathbf{3}^\beta + \mathbf{2}^\alpha\mathbf{3}^\beta + \mathbf{2}^\beta\mathbf{3}^\beta$, so that $\mathbb{AB} \cong \mathbb{AC}$, but evidently $\mathbb{B} \not\cong \mathbb{C}$. We will show that $\mathbb{A}$ is indecomposable. Let us suppose that this is not true. Then $\mathbb{A} \cong \mathbb{UV}$ for some partially ordered sets $\mathbb{U}, \mathbb{V}$ with $|U| > 1$, $|V| > 1$. As $\mathbb{A}$ has two connected components, just one of $\mathbb{U}, \mathbb{V}$ has two connected components, the other is connected. Assume that $\mathbb{U} = \mathbb{U}_1 + \mathbb{U}_2$ and $\mathbb{U}_1\mathbb{V} \cong \mathbf{2}^\beta$, $\mathbb{U}_2\mathbb{V} \cong \mathbf{3}^\beta$. As $\mathbb{U}_1\mathbb{V}, \mathbb{U}_2\mathbb{V}$ are connected, 2.7 yields $\mathbb{V} \cong \mathbf{2}^\gamma$, $\mathbb{V} \cong \mathbf{3}^\delta$ for some cardinal numbers $\gamma, \delta \le \beta$. Using again 2.7 we obtain $\gamma = \delta = 0$, so that $\mathbb{V}$ is a one-element set, a contradiction.

Now let us deal with the statement (**) from Introduction; for proving it, we use the argument similar to that applied above.

Proof of (**). Let $\mathbb{A}, \mathbb{B}$ be partially ordered sets satisfying (i), (ii) and let $\mathbb{A}^k \cong \mathbb{B}^k$ for some $k \in \mathbb{N}$, $k > 1$. If all connected components of $\mathbb{A}, \mathbb{B}$ are one-element sets, then evidently $\mathbb{A} \cong \mathbb{B}$. Now let at least one of $\mathbb{A}, \mathbb{B}$ have a connected component containing more than one element. Then $f_A^k = f_B^k$ for f_A and f_B being generalized polynomials belonging to $\mathbb{A}$ and $\mathbb{B}$, respectively (in the sense of 3.4). We have $f_A^k - f_B^k = (f_A - f_B)(f_A^{k-1} + f_A^{k-2}f_B + \cdots + f_B^{k-1})$. The relation $f_A^k - f_B^k = 0$ yields $f_A - f_B = 0$ or $f_A^{k-1} + f_A^{k-2}f_B + \cdots + f_B^{k-1} = 0$. The latter case is impossible because f_A and f_B are generalized polynomials over $\mathbb{N}$. So we have $f_A = f_B$ and this implies $\mathbb{A} \cong \mathbb{B}$. □

The following example shows that without the conditions (i) and (ii) the implication (2) does not hold in general.

3.8. EXAMPLE. Let α be any infinite cardinal number. Take $\mathbb{A} = \mathbf{2}^\alpha + \mathbf{2}^0 + \mathbf{2}^0 + \ldots$, $\mathbb{B} = \mathbf{2}^\alpha + \mathbb{A}$. Using 1.5 we obtain $\mathbb{A}^2 = \mathbf{2}^\alpha + \mathbf{2}^\alpha + \cdots + \mathbf{2}^0 + \mathbf{2}^0 + \cdots = \mathbb{B}^2$. But evidently $\mathbb{A} \not\cong \mathbb{B}$.

The partially ordered sets $\mathbb{A}, \mathbb{B}$ in the previous example satisfy neither (i) nor (ii). It can be proved that if $\mathbb{A}, \mathbb{B}$ consist of two or three connected components and each of these connected components containing more than one element is a direct product of indecomposable factors (the number of mutually isomorphic factors can be arbitrary), then $\mathbb{A}^2 \cong \mathbb{B}^2$ implies $\mathbb{A} \cong \mathbb{B}$. The question, if this implication holds also in the case when $\mathbb{A}, \mathbb{B}$ consist of more than three connected components (but finitely many), is open.

REFERENCES

[1] APPLESON, R. R.—LOVÁSZ, L.: *A characterization of cancellable k-ary structures*, Period. Math. Hungar. **6** (1975), 17–19.

[2] BIRKHOFF, B.: *Lattice Theory* (Rev. ed.), Amer. Math. Soc., New York, 1948.

[3] HASHIMOTO, J.: *On the product decomposition of partially ordered sets*, Math. Japon. **1** (1948), 120–123.

[4] HASHIMOTO, J.: *On direct product decomposition of partially ordered sets*, Ann. of Math. **54** (1951), 315–318.

[5] JAKUBÍK, J.—CSONTÓOVÁ, M.: *Convex isomorphisms of directed multilattices*, Math. Bohem. **118** (1993), 359–379.

[6] JAKUBÍK, J.—CSONTÓOVÁ, M.: *Cancellation rule for internal direct product decompositions of a connected partially ordered sets*, Math. Bohem. **125** (2000), 115–122.

[7] JAKUBÍK, J.: *On direct and subdirect product decompositions of partially ordered sets*, Math. Slovaca **52** (2002), 377–395.

[8] LOVÁSZ, L.: *Operations with structures*, Acta Math. Acad. Sci. Hungar. **18** (1967), 321–328.

[9] LOVÁSZ, L.: *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1** (1971), 145–156.

[10] McKENZIE, R.—McNULTY, G.—TAYLOR, W.: *Algebras, Lattices, Varieties, Vol. I*, Wadsworth, Belmont, 1987.

[11] NOVOTNÝ, J.: *On the characterization of a certain class of monounary algebras*, Math. Slovaca **40** (1990), 123–126.

[12] PLOŠČICA, M.—ZELINA, M.: *Cancellation among finite unary algebras*, Discrete Math. **159** (1996), 191–198.

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