Dariusz Bugajewski

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ON FIXED POINT THEOREMS FOR ABSOLUTE RETRACTS

Dariusz Bugajewski

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ABSTRACT. In this paper we prove a general fixed point principle for mappings defined on absolute retracts. In particular, as a corollary from this principle, we obtain the answer to the open problem concerning validity of the Darbo-Sadovski fixed point theorem for absolute retracts. Our results are illustrated by suitable examples and compared with other of this type.

1. Introduction

One of the main results of nonlinear functional analysis is the famous Schauder fixed point theorem. It was proved in 1930 (see [9]) for compact mappings defined on closed and convex subsets of a Banach space. This theorem has many applications in the theory of differential and integral equations and has many generalizations. These generalizations can be divided, roughly speaking, into three groups. The first group contains results which weaken the assumption of compactness of a mapping and the second group contains ones which do not contain the assumption of the convexity of the domain. Finally, the third group contains results in which one does not assume that a mapping maps its domain into itself.

The aim of this paper is to prove very general Schauder type fixed point theorem for mappings defined on absolute retracts without the assumption of their compactness. In this place recall that every convex subset of a Banach space is an absolute retract (see [9; p. 93, Theorem 10.5]). Moreover, we prove fixed point theorems for the special classes of absolute retracts. In particular, we formulate the condition for absolute retracts under which the well-known Darbo-Sadovski fixed point theorem is valid.

Our paper is organized as follows: In Section 2 we recall some basic definitions and theorems which will be useful in the sequel. Section 3 contains our general

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fixed point principle and Darbo-Sadovski's type fixed point theorem for absolute retracts. Section 4 contains a different type fixed point theorems for absolute retracts, examples and a comparison with other fixed point results. Finally, in Section 5 we formulate some open problems.

For the basic concepts concerning the fixed point theory which appear in our paper, we refer e.g. [9].

2. Preliminaries

DEFINITION 1. A space Y is called an *absolute retract* (briefly: AR) whenever:

(i) Y is metrizable,

(ii) for any metrizable space X and closed set $A \subset X$ each mapping $f \colon A \to Y$ is extendable over X.

THEOREM 1. (Arens-Eells, [9; p. 158]) *Any metric space Y can be isometrically embedded as a closed subset in a normed linear space.*

THEOREM 2. ([9; pp. 93–94, Theorem 10.6]) *A metrizable space Y is an AR if and only if it is a retract of every metrizable space in which it is embedded as a closed set.*

DEFINITION 2. A metric space (X,d) is called *hyperconvex* if, for any index class of closed balls in X, $\overline{B}(x_i,r_i)$, $i \in I$, satisfying the condition that $d(x_i,x_j) \le r_i + r_j$ for all $i,j \in I$, the intersection $\bigcap_{i\in I} \overline{B}(x_i,r_i)$ is nonempty.

Recall that the above notion of hyperconvexity was introduced by N. Aronszajn and P. Panitchpakdi [1] in 1956.

DEFINITION 3. For any bounded subset A of a metric space (X,d) the *Kuratowski measure of noncompactness* — $\alpha(A)$ — is defined as the infimum of all positive numbers ε such that A can be covered by a finite number of sets of diameter $\le \varepsilon$.

For the properties and examples of index α, we refer e.g [3] or [4].

THEOREM 3. (Darbo-Sadovski) *Let D be a closed convex and bounded subset of a given Banach space and let $f \colon D \to D$ be a continuous and α-condensing mapping, i.e.,*

$$\alpha\big(f(A)\big) < \alpha(A) \qquad \text{for each} \quad A \subset D \ \text{ such that } \alpha(A) > 0 . \tag{1}$$

Then f has at least one fixed point in D.

Recall that R. Espinola-Garcia has proved that the Darbo-Sadovski fixed point theorem is valid in the case when D is a hyperconvex bounded metric space (see [10]).

3. Fixed point theorems

Let D be an AR. In view of Theorem 1 we infer that $i(D) = K$, where K is a closed subset of a normed space $(E, \|\cdot\|)$ and i is an isometry. Let $K^c = \operatorname{conv} K$. In view of Theorem 2 there exists a retraction $r\colon K^c \to K$.

Now we are able to formulate the general fixed point theorem for absolute retracts, namely:

THEOREM 4. *Let D be an AR, $x_0 \in D$ and let $f\colon D \to D$ be a continuous mapping of D into itself. If the following implication*

$$\begin{aligned}\Big(\big(V = \operatorname{conv}(i \circ f \circ i^{-1} \circ r(V)) \wedge V \subset K^c\big)\\ \vee \big(V = f(V) \cup \{x_0\} \wedge V \subset D\big)\Big)\end{aligned} \tag{2}$$

$$\Longrightarrow V \text{ is relatively compact}$$

holds for every subset V of K^c or D, respectively, then f has a fixed point.

In the case when i and r are identity maps, Theorem 4 practically reduces to [17; Theorem 1]. However, note that in the case when one imposes on f a condition formulated e.g. in terms of the Kuratowski measure of noncompactness, the isometry i is not important, because this index does not depend on an isometry.

In Remark 1 we consider particular cases encompassed by Theorem 4 and such that r is not an identity map.

Proof of Theorem 4. Consider the mapping $F = i \circ f \circ i^{-1} \circ r\colon K^c \to K^c$. Let $F(V) \cup \{y_0\} = V$, where $V \subset K^c$ and $y_0 = i(x_0)$. In fact $F(K^c) \subset K$, so $F(V) \subset K$. Further, $y_0 \in K$, so $V \subset K$. Then, we have

$$V = F(V) \cup \{y_0\} = i \circ f \circ i^{-1} \circ r(V) \cup \{i(x_0)\} = i \circ f \circ i^{-1}(V) \cup \{i(x_0)\},$$

so $i^{-1}(V) = f \circ i^{-1}(V) \cup \{x_0\}$. By the assumption $i^{-1}(V)$ is relatively compact, and therefore V is relatively compact. Hence, it is clear that F satisfies all assumptions of [17; p. 1, Theorem 1] (actually this result is proved for a Banach space but the completeness in this theorem is not necessary; cf. [5; p. 35, Theorem 1]), and therefore it has a fixed point. To end our proof it is enough to remark that $x \in D$ is a fixed point of f if and only if $i(x)$ is a fixed point of F. Indeed, let $x \in D$ be such that $f(x) = x$. Then we have $i^{-1}\big(r\big(i(x)\big)\big) = x$ and therefore $i\big(f\big(i^{-1}\big(r\big(i(x)\big)\big)\big)\big) = i\big(f(x)\big) = i(x)$. Thus $i(x)$ is a fixed point of F. Conversely, let $y \in K^c$ be such that $F(y) = y$. Then $y \in K$, and therefore $x = i^{-1}(y)$ for some $x \in D$. Then, we have $i \circ f \circ i^{-1}(y) = y$, so $f(x) = x$. □

Remark 1. Let D be an AR and $f\colon D \to D$ be a continuous mapping satisfying the Darbo-Sadovski condition (1). Let r, K, K^c denote the same as at the beginning of this section. Assume that K^c is complete and the retraction r satisfies the following condition

$$r(V) \subset \bigcup_{0 \le \lambda \le 1} \lambda V \qquad \text{for any} \quad V \subset K^c . \tag{3}$$

Now we show that under the above assumptions f satisfies the assumptions of Theorem 4. First, let $A \subset D$ be such that $A = f(A) \cup \{x_0\}$. If $\alpha(A) > 0$, then $\alpha(A) = \alpha\big(f(A)\big) < \alpha(A)$, what gives a contradiction. Now assume that $V \subset K^c$ is such that $V = \operatorname{conv}\big(i \circ f \circ i^{-1} \circ r(V)\big)$. Denote by α_E the Kuratowski measure of noncompactness in $\big(E, \|\cdot\|\big)$. If $\alpha_E\big(r(V)\big) = 0$, then

$$\alpha_E(V) = \alpha\big(f \circ i^{-1} \circ r(V)\big) \le \alpha\big(f \circ i^{-1}\big(\overline{r(V)}\,\big)\big) = 0 ,$$

so V is relatively compact. On the other hand, if $\alpha_E\big(r(V)\big) > 0$, then by (1), (3) and by the properties of α we obtain

$$\alpha_E(V) = \alpha\big(f \circ i^{-1} \circ r(V)\big) < \alpha\big(i^{-1} \circ r(V)\big) \le \alpha\Big(i^{-1}\Big(\bigcup_{0 \le \lambda \le 1} \lambda V\Big)\Big) = \alpha_E(V) ,$$

what gives a contradiction. Hence f satisfies the assumptions of Theorem 4 and therefore it has a fixed point.

It is quite clear that instead of (3) one can assume that

$$r \text{ is nonexpansive.} \tag{4}$$

In view of (4) and [1; p. 422, Theorem 8] as a corollary of Theorem 4 we obtain the Darbo-Sadovski fixed point theorem for hyperconvex metric spaces (see [10; p. 135]).

The cases considered in Remark 1 are special ones of the following theorem.

THEOREM 5. *Let D be a bounded AR and let $f \colon D \to D$ be a continuous mapping satisfying* (1). *If*

$$\alpha_E\big(r(V)\big) \le \alpha_E(V) \qquad \textit{for any} \quad V \subset K^c$$

(*briefly: if r is α_E-nonexpansive*) *and K^c is complete, then f has a fixed point.*

P r o o f. It is enough to apply similar arguments as in Remark 1. □

4. Examples and other results

At the beginning of this section we compare Theorem 4 with another extension of the Schauder fixed point theorem for absolute retracts, namely:

THEOREM 6. ([12; p. 215, Theorem 5.1]) *Let D be an AR and let $f: D \to D$ be a compact absorbing contraction* (see [12] for the definition). *Then f has a fixed point.*

In this place recall only that a compact absorbing contraction is assumed to be locally compact, that is, for every point of the domain there exists such its neighbourhood that the restriction of the mapping to this neighbourhood is compact.

EXAMPLE 1. For simplicity consider first the set $D = \{x = (x_1, x_2, \dots, x_n, \dots) \in l^\infty : \|x\| \le 1, \ x_n \ge 0 \text{ for } n \in \mathbb{N}\}$ and let $(q_n)_{n\ge1}$ be a sequence such that $q_n \in (0,1)$ for $n \in \mathbb{N}$ and $q_n \to 1$. Define

$$f(x) = (q_1 x_1, q_2 x_2, \dots, q_n x_n, \dots), \qquad x \in D.$$

It is well known that every convex subset of a locally convex linear space is an AR (see [9; p. 93, Theorem 10.5]), so D is an AR as a convex set. Further, it is not difficult to verify that if $V \subset D$ is such that $V = \operatorname{conv} f(V)$ or $V = f(V) \cup \{(0)_{n\in\mathbb{N}}\}$, then $V = \{(0)_{n\in\mathbb{N}}\}$, so V is relatively compact. Hence f satisfies the assumptions of Theorem 4. Now we verify that f is not a locally compact mapping. Indeed, let $D_\varepsilon = \{x \in D : \|x\| < \varepsilon\}$, where $\varepsilon \in (0,1)$ and let $C = \{(x_n^k) \in D : x_n^k = 0 \text{ if } k \ne n, \ x_n^k = \frac{\varepsilon}{2} \text{ if } k = n, \ k, n \in \mathbb{N}\}$. We have $f(C) = \{(x_n^k) \in D : x_n^k = 0 \text{ if } k \ne n, \ x_n^k = \frac{\varepsilon}{2} q_k \text{ if } k = n, \ k, n \in \mathbb{N}\}$. It is quite clear that $\alpha(f(C)) = \frac{\varepsilon}{2}$, so $f(C)$ is not a relatively compact set. Therefore for any open subset U of D such that $(0)_{n\in\mathbb{N}} \in U$ the restriction $f|_U$ of f to U is not compact, so f is not a locally compact mapping. Hence f does not satisfy the assumptions of Theorem 5.

Remark 2. Note that D from Example 1 is convex and hyperconvex, f from the same example is linear and nonexpansive. In particular, to establish the existence of a fixed point of f one can apply Baillon's fixed point theorem [2; p. 14, Theorem 5] for nonexpansive mappings defined on a bounded hyperconvex space. But using the idea from Example 1 one can consider more sophisticated situations. For example, let $D = \{x \in c_0 : \|x\| \le 1, \ |x_1| = |x_2|\}$ and let

$$f(x) = \left(\sqrt{|x_1|}, \sqrt{|x_2|}, q_3 x_3, q_4 x_4, \dots\right), \qquad x \in D,$$

where $(q_n)_{n\ge1}$ is the sequence from Example 1.

In this case the mapping f does not satisfy also the assumptions of [17; p. 1, Theorem 1] and Theorem 8 (see Section 5).

Now, we formulate two another fixed point theorems for mappings defined on an AR. First is the following Krasnoselski-type.

THEOREM 6. *Let E be a Banach space and let $K \subset E$ be an AR. Assume that*

1° $f_1 \colon K \to E$ *is a contractive map,*
2° $f_2 \colon K \to E$ *is a compact map,*
3° $f_1(x) + f_2(y) \in K$ *for any* $x, y \in K$.

Then $f = f_1 + f_2$ *has a fixed point.*

Proof. It is enough to apply similar arguments as in [14; p. 125, Theorem 2] and the generalized Schauder fixed point theorem [9; p. 94, Theorem 10.8] instead of the classical one. □

The second result is the following:

THEOREM 7. *Let E be a Banach space and let $K \subset E$ be a bounded AR such that $\lambda K \subset K$ for every $0 < \lambda \leq 1$. Assume that*

i) $f_1 \colon K \to E$ *is nonexpansive,*
ii) $f_2 \colon K \to E$ *is a compact map,*
iii) $f_1(x) + f_2(y) \in K$ *for any* $x, y \in K$,
iv) *every sequence* (x_n) *such that* $x_n \in K$ *for* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \big(x_n - f(x_n)\big) = 0 \,,$$

where $f = f_1 + f_2$, *has a limit point.*

Then f *has a fixed point.*

Proof. In this case it is enough to apply similar arguments as in [6; Theorem 1] and a generalized Schauder fixed point theorem instead of the classical one. □

Remark 3. It is clear that in Theorem 6 and Theorem 7 one can assume that E is a Fréchet space. Further note that for a bounded AR, $K \subset E$, where E is a Banach space and $\lambda K \subset K$ for $\lambda \in (0,1]$, Theorem 6 is a special case of Theorem 7. Indeed, assume that the conditions $1^\circ - 3^\circ$ in Theorem 6 are satisfied and let $(x_n)_{n\in\mathbb{N}}$, $x_n \in K$ for $n \in \mathbb{N}$, be a sequence satisfying the equality in the condition iv) of Theorem 7. Let id denote the identity map and $V = \{x_n : n \in \mathbb{N}\}$. In view of the properties of the measure α we obtain

$$\begin{aligned} \alpha(V) &\leq \alpha\big((\mathrm{id} - f)(V) + f(V)\big) \leq \alpha\big((\mathrm{id} - f)(V)\big) + \alpha\big(f(V)\big) \\ &\leq \alpha\big(f_1(V)\big) + \alpha\big(f_2(V)\big) \leq q\alpha(V) \,, \end{aligned}$$

where q is the contractive constant of f_1. Thus $\alpha(V) = 0$, so V is relatively compact. Hence the sequence (x_n) has a convergent subsequence and therefore it satisfies the condition iv) in Theorem 7. The above observation remains valid for the case when E is a Fréchet space. In this case one can also apply the Sadovski measure of noncompactness (see [16] for the definition and the properties) instead of the index α.

To this end remark that using similar arguments as above one can infer that Theorem 6 is a special case of Theorem 4 if we assume additionally that the retraction r, defined before Theorem 4, is α-nonexpansive.

Now we illustrate Theorem 7 by the following example.

EXAMPLE 2. Let $K = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in l^\infty : \ \|x\| \le 1, \ |x_1| = |x_2|\}$. It is known (see [1; p. 423, Theorem 9]) that a subset A of a given hyperconvex space X is hyperconvex if and only if it is a retract of X by a contracting retraction.

Using this fact it can be easily verified that K is a hyperconvex set, so it is an AR ([1; p. 422, Corollary 4]). Let

$$f_1(x) = (0, 0, -x_3, -x_4, \ldots), \qquad f_2(x) = \big(|x_1|^p, |x_2|^p, 0, 0, \ldots\big), \quad p \in \mathbb{N}, \ p > 1,$$
$$f(x) = f_1(x) + f_2(x), \qquad x \in K.$$

Obviously f_1 is nonexpansive and f_2 is a compact map. Moreover, it is clear that $f_1(x) + f_2(y) \in K$ for any $x, y \in K$.

Now, let $(x^k)_{k\in\mathbb{N}}$ be a sequence satisfying the condition in iv). Then we have

$$x^k - f(x^k) = \big(x_1^k - |x_1^k|^p, x_2^k - |x_2^k|^p, 2x_3, 2x_4, \ldots\big).$$

Thus $\lim\limits_{k\to\infty} \sup\limits_{n\ge 3} |x_n^k| = 0$, $\lim\limits_{k\to\infty} \big(x_1^k - |x_1^k|^p\big) = 0$, and $\lim\limits_{k\to\infty} \big(x_2^k - |x_2^k|^p\big) = 0$. It is clear that such a sequence $(x^k)_{k\in\mathbb{N}}$ has a convergent subsequence. Hence the mapping f satisfies all assumptions of Theorem 7 and therefore it has a fixed point in K.

Note that also in this example one can replace the space l^∞ by c_0.

In the last remark we compare Theorem 4 and Theorem 7.

Remark 4. Let K and f denote the same as in Example 2 and let $C = \{x \in K : \ x_1 = x_2 = 0\}$. Then we have $C = f(C) \cup \{(0)_{n\in\mathbb{N}}\}$. But C is not a compact set; actually, in view of the well-known result of Furi and Vignoli [11], $\alpha(C) = 2$, so f does no satisfy condition (2) in Theorem 4. Therefore Theorem 4 and Theorem 7 are independent.

5. Open problems

1. In [7] we have proved the following fixed point theorem for hyperconvex metric spaces, namely:

THEOREM 8. *Let X be a hyperconvex metric space, $x_0 \in X$ and let f be a continuous mapping of X into itself. If the following implication*

$$\bigl(V \text{ is isometric to } \varepsilon f(V) \ \vee \ V = f(V) \cup \{x_0\}\bigr) \implies \alpha(V) = 0\,,$$

where $\varepsilon f(V)$ denotes the hyperconvex hull of $f(V)$, holds for every subset V of X, then f has a fixed point.

An interesting problem is to compare Theorem 4 and Theorem 8 for hyperconvex metric space.

2. Condition (2) in Theorem 4 depends on the retraction r and the choice of the isometric embedding. An open problem is to prove an analogous result to Theorem 4 with the condition formulated only by using the mapping f.

3. The third problem is to find (if exists) an example of a mapping which satisfies the assumptions of Theorem 5 and does not satisfy the assumptions of Theorem 4.

4. In Theorem 5 the assumption that r is α_E-nonexpansive is a sufficient condition to establish the existence of a fixed point. Discuss possibilities to weaken this condition.

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Faculty of Mathematics
and Computer Science
A. Mickiewicz University
Matejki 48/49
PL-60-769 Poznań
Poland
E-mail: ddbb@amu.edu.pl