Mathematica Slovaca

Roman Ger

Almost approximately convex functions

Mathematica Slovaca, Vol. 38 (1988), No. 1, 61--78

Persistent URL: http://dml.cz/dmlcz/129503

Math. Slovaca 38, 1988, No. 1,

ALMOST APPROXIMATELY CONVEX FUNCTIONS

ROMAN GER

1. Approximately convex functions

Consider a real linear space X and a nonempty convex set $\Delta \subset X$. Assume that we are given a convex functional $g\colon \Delta \to \mathbb{R}$ and a real number $\varepsilon \geqslant 0$. Take an arbitrary bounded function $\varphi\colon \Delta \to \mathbb{R}$ say $|\varphi(x)| < K$ for $x \in \Delta$, and put $f\colon = g + \frac{\varepsilon}{3K}\varphi$. Then, for fixed $x, y \in \Delta$ and $\lambda \in [0, 1]$, we get

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda g(x) + (1-\lambda)g(y) + \frac{\varepsilon}{3K}\varphi(\lambda x + (1-\lambda)y) =$$

$$= \lambda f(x) + (1-\lambda)f(y) + \frac{\varepsilon}{3K}[\varphi(\lambda x + (1-\lambda)y) - \lambda\varphi(x) - (1-\lambda)\varphi(y)] \leqslant$$

$$\leqslant \lambda f(x) + (1-\lambda)f(y) + \varepsilon.$$

This means that a convex functional additively deviated by a suitably bounded function complies with the usual notion of convexity up to the ε-exactness only. This may serve as a motivation for the following definition.

Any functional $f\colon \Delta \to \mathbb{R}$ fulfilling the inequality

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + \varepsilon \tag{1}$$

for all $x, y \in \Delta$ and all $\lambda \in [0, 1]$ will be referred to as ε-convex in the sequel. A functional is termed approximately convex provided it is ε-convex for some $\varepsilon \geqslant 0$.

As we have seen, approximately convex functionals do exist. Quite natural is the question whether any approximately convex functional has to be a deviation of a convex one. In the case where $X = \mathbb{R}^n$ an affirmative answer to that question was given by B. H. Hyers and S. Ulam [7]; they have proved that for any convex domain $\Delta \subset \mathbb{R}^n$ and any ε-convex functional $f\colon \Delta \to \mathbb{R}$ there exists a constant $k_n > 0$ (depending on n only) and a convex functional $g\colon \Delta \to \mathbb{R}$ such that

$$|f(x) - g(x)| \leqslant k_n \varepsilon \qquad \text{for all } x \in \Delta.$$

The constant k_n occurring in [7] ammounts to $\frac{1}{4} n \cdot \frac{n+3}{n+1}$ and is by no means sharp. P. Cholewa (see also J. W. Green [6]) has recently shown in [1] that one may replace k_n by

$$j_n := \min(k_n, l_n) \text{ where } l_n := \frac{1}{2} m \text{ provided that } 2^{m-1} \leqslant n < 2^m.$$

Cholewa's estimation coincides with that of Hyers and Ulam for $n = 1, 2$ and 4 and turns out to be better indeed for any other dimensions (*). Nevertheless, also the j_n's tend to infinity as $n \to \infty$; this does not allow to predict anything regarding the infinite dimension

The next question is whether the Hyers—Ulam result has an analogue in the class of midpoint convex functions. A relevant contribution in the negative direction has also been made by P. Cholewa [1]. He exhibits an example of a function fulfilling the condition

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + 1 \tag{2}$$

and such that there is no modpoint convex function uniformly close to f; in other words, for any midpoint convex function g inequality

$$|f(x) - g(x)| \leqslant M, \; x \in \operatorname{dom} f, \tag{3}$$

fails to hold for any constant $M > 0$. However, one should emphasize that the domain of f in Cholewa's example is $\mathbb{Q}$-convex only, i.e. for any $x, y \in \operatorname{dom} f$ and any rational number $\lambda \in [0, 1]$ one has $\lambda x + (1 - \lambda)y$ in $\operatorname{dom} f$ in turn. This leaves the door open for an investigation of approximate midpoint convexity in the case when the domain of the transformation considered is convex in the usual sense. On the other hand, this concerns a finite dimension only. For infinite dimensional spaces we have the following

2. Counterexample

Let H be any Hamel basis of an infinite dimensional linear space X over the reals and let $\Delta \subset X$ be the collection of all elements of X whose expansions with

---

(*) W. Walter [11] has noticed that $k_1 = j_1 = \frac{1}{2}$ is the best posible constant in the one-dimensional case.

respect to H (over the field $\mathbb{R}$) contain positive coefficients only. Clearly, Δ is a convex subset of X with convexity understood in the usual sense. For $x \in \Delta$, $x = \Sigma \lambda_\alpha h_\alpha$, $\lambda_\alpha \in \mathbb{R}$, $h_\alpha \in H$, we put $m(x) := \max\{\lambda_\alpha : \lambda_\alpha \neq 0\}$ and $f(x) := \min\left\{ \in \mathbb{N} \cup \{0\} : \frac{1}{2^n} \leqslant m(x) \right\}$. Then (2) is satisfied for all $x, y \in \Delta$ and (3) fails to hold for any midpoint convex functional $g: \Delta \to \mathbb{R}$ and any constant $M > 0$. The proof is literally the same as presented in [1].

3. Almost convex functions

Let $\Delta \subset \mathbb{R}^n$ be a nonempty and convex subset of $\mathbb{R}^n$ and let $g: \Delta \to \mathbb{R}$ be any midpoint convex function. Take an arbitrary function $f: \Delta \to \mathbb{R}$ equivalent to g (i.e. such that $f(x) = g(x)$ for almost all $x \in \Delta$ with respect to the n-dimensional Lebesgue measure l_n). Thus, putting $E := \{x \in \Delta : f(x) \neq g(x)\}$ we have $l_n(E) = 0$ and consequently

$$l_{2n}\left(\left\{(x, y) \in \Delta^2 : x \in E \text{ or } y \in E \quad \text{or } \frac{x+y}{2} \in E\right\}\right) = 0.$$

Therefore the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \tag{4}$$

holds for l_{2n}-almost every pair $(x, y) \in \Delta^2$. This may serve as a motivation for the following definition.

Any function $f: \Delta \to \mathbb{R}$ fulfilling (4) for l_{2n}-almost all pairs $(x, y) \in \Delta^2$ will be referred to as almost midpoint convex.

As we have seen, almost midpoint convex functions do exist. Has any almost midpoint convex function to be equivalent to a midpoint convex one? This question was answered in the positive by M. Kuczma [8]; he proved that any almost midpoint convex function is equal almost everywhere to a midpoint convex one (see also M. Kuczma [9] for some generalizations as well as for related problems) is equal to something equals something.

Now, the question arises what can be said about functionals whose convexity has been spoiled in both directions described above simultaneously. This is what we are going to deal with in the rest of this work. For obvious reasons such doubly spoiled convexity is termed as in the title off the present paper. Such a search is inspired by a suggestion made by L. Reich [10] in connection with almost approximate additivity. The results of my endeavours in that direction are contained in [5]. The present paper contains results in line with [5]. We are far from claiming any type of completeness, however.

We shall try to deal with a problem just posed in a slightly more general setting which frequently clarifies things not immediately visible in spaces equipped with numerous and rich structures.

4. Invariant set ideals

The point is to define precisely the set families of "small" ("negligible") sets which would yield a joint generalization of the Haar zero sets and first category sets in locally compact topological groups. In what follows we shall give the definitions only, referring to Kuczma's book [9] and to the paper of J. Dhombres and R. Ger [3] for concrete examples, applications in the theory of functional equuations and an ample bibliography.

Let $(G, +)$ be a group (not necessarily commutative). A nonempty family $\mathscr{I} \subset 2^G$ is termed a proper linearly invariant set ideal (abbreviated to p.l.i. ideal in the sequel) provided that $\mathscr{I} \neq 2^G$, $\mathscr{I}$ is closed under set-theoretical unions, descending inclusions and such that jointly with an $x \in G$ and an $A \in \mathscr{I}$ it contains the set $x - A$. A p.l.i. ideal is said to be a p.l.i. σ-ideal provided that it is closed under countable unions.

Suppose that two ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ in G and G^2 are given, respectively. We shall say that $\mathscr{I}_2$ there exists a set $U \in \mathscr{I}_1$ such that $M_x \in \mathscr{I}_1$ for all $x \in G \setminus U$; the symbol M_x stands here for the vertical section of M through the point x, i.e.

$$M_x := \{y \in G: (x, y) \in M\}$$

(an abstract version of Fubini's theorem in measure theory). Let $\mathscr{I}$ be a p.l.i. ideal in G. The family

$$\Omega(\mathscr{I}) := \{M \subset G^2: \{x \in G: M_x \notin \mathscr{I}\} \in \mathscr{I}\} \tag{5}$$

forms a p.l.i. ideal in G^2; this is the largest p.l.i. ideal in G^2 conjugate with the p.l.i. ideal $\mathscr{I}$ in G.

A property $\mathscr{P}(x)$ is said to hold $\mathscr{I}$-almost everywhere in G whenever $\mathscr{P}(x)$ is satisfied for all $x \in G \setminus E$ where E is a member of $\mathscr{I}$.

Finally if $D \subset G$, $D \notin \mathscr{I}$, and $\varphi: D \to \mathbb{R}$ is a function, then

$$\inf_{x \in D} \operatorname{ess} \varphi(x) := \sup_{E \in \mathscr{I}} \inf_{x \in D \setminus E} \varphi(x). \tag{6}$$

If $\mathscr{I} \subset 2^G$ is a p.l.i. σ-ideal, then it is easy to check that the supremum on the right-hand side of (6) is attained, i.e.

$$\inf_{x \in D} \operatorname{ess} \varphi(x) = \inf_{x \in D \setminus E_0} \varphi(x)$$

for some $E_0 \in \mathscr{I}$.

Now, we are in a position to formulate our

5. Main results

Let $(G, +)$ be an abelian and uniquely 2-divisible group and let $\mathscr{I}_1$, $\mathscr{I}_2$ be two conjugate p.l.i. σ-ideals in G and G^2, respectively. Assume further that

(i) $A \in \mathscr{I}_1$ implies $2A \in \mathscr{I}_1$;

(ii) $\mathscr{I}_2$ is invariant with respect to the transformation

$$G^2 \ni (x, y) \mapsto \frac{1}{2}(x + y, x - y) \in G^2;$$

(iii) Δ is a nonempty midpoint convex subset of G such thhat

$$\Delta(x) := (\Delta - x) \cap (x - \Delta) \notin \mathscr{I}_1 \tag{7}$$

whenever $x \in \Delta$.

To have some exemplification in mind consider any abelian 2-divisible Polish toological group (resp. second countable Baire topological group) $(G, +)$ for which the map $x \mapsto \frac{1}{2}x$ is continuous and take as $\mathscr{I}_1$ and $\mathscr{I}_2$ the σ-ideals of all first category sets in G and G^2, respectively. With Δ being any nonempty midpoint convex open subset of G all the assumptions described above become satisfied.

Alternatively, take $(G, +)$ to be any 2-divisible locally compact abelian group with continuous division by 2 and with the Haar measure h. If the transformation spoken of in (ii) is a nullset preserving for the completed product measure $(h \otimes_\sigma h)' =: H$ in G^2 (this is, for instance, the case when h is the Lebesgue measure in $\mathbb{R}^n$), then setting

$$\mathscr{I}_1 := \{T \subset G : h(T) = 0\}, \quad \mathscr{I}_2 := \{M \subset G^2 : H(M)) = 0\}$$

and $G \supset \Delta$ — any nonempty midpoint convex open set, we obtain another important accomplishment of the assumptions.

For further examples the reader is referred to [3].

Theorem 1. *For each $\mathscr{I}_2$-almost approximately midpoint convex function f: $\Delta \to \mathbb{R}$ there exists an approximately midpoint convex function g: $\Delta \to \mathbb{R}$ such that the difference $f - g$ is uniformly bounded $\mathscr{I}_1$-almost everywhere in Δ.*

For further statements we have to specify the requirements concerning the group $(G, +)$. From now on $(G, +\ \cdot)$ is a real linear space, Δ is a convex subset of G whereas (i) is replaced by a stronger assumption

(i′) $A \in \mathscr{I}_1$ implies $\lambda A \in \mathscr{I}_1$ for all $\lambda \in \mathbb{R}$.

To prevent possible misapprehensions let us formulate here explicitely the following definition.

A functional f: $\Delta \to \mathbb{R}$ is called $\mathscr{I}_2$-almost ε-convex if and only if for any

$\lambda \in [0, 1]$ there exists a set $N(\lambda) \in \mathscr{I}_2$ such that inequality ((1) holds for all pairs $(x, y) \in \Delta^2 \backslash N\lambda)$.

Let us emphasize that the exceptional set $N(\lambda)$ is allowed to depend on $\lambda \in [0, 1]$.

A real functional f on Δ is called $\mathscr{I}_2$-almost approximately convex provided that it is $\mathscr{I}_2$-almost ε-convex for some $\varepsilon \geqslant 0$. Finally, f is termed almost approximately convex whenever it is $\mathscr{I}_2$-almost approximately convex for some p.l.i. σ-ideal $\mathscr{I}_2$ in G^2.

Almost approximately convex functionals do exist. To see this take an arbitrary bounded function $\varphi: \Delta \to \mathbb{R}$ (say, $|\varphi(x)| \leqslant K > 0, x \in \Delta$), any $\varepsilon \geqslant 0$ and any convex functional $g_0: \Delta \to \mathbb{R}$. Fix a member E of the p.l.i. σ-ideal $\mathscr{I}_1$ in G and any function $g: \Delta \to \mathbb{R}$ such that $g|_{\Delta \setminus E} = g_0|_{\Delta \setminus E}$. Then the functional f: $g + \frac{\varepsilon}{3K}\varphi$ is $\Omega(\mathscr{I}_1)$-almost ε-convex. Indeed, for $\lambda \in [0, 1]$ arbitrarily fixed, we have (1) outside the set

$$N(\lambda) := \{(x, y) \in G^2 : \lambda x + (1 - \lambda) y \in E\} \cup (E \times G) \cup (G \times E).$$

It remains to observe that $N(\lambda) \in \Omega(\mathscr{I}_1)$ for all $\lambda \in [0, 1]$. Actually, for $\lambda = 1$ this is trivial since $N(1) = (E \times G) \cup (G \times E) \in \Omega(\mathscr{I}_1)$, whereas for $\lambda \in [0, 1)$ the x-section $(N(\lambda))_x$ of $N(\lambda)$ through any $x \in G$ is equal to $\frac{1}{1-\lambda}(E - \lambda x) \cup (N(1))_x$ and the first summand belongs to $\mathscr{I}_1$ because of (i′).

Theorem 2. *If $f: \Delta \to \mathbb{R}$ is an $\mathscr{I}_2$-almost approximately convex functional, then there exists an approximately convex functional $g: \Delta \to \mathbb{R}$ such that the difference $f - g$ is uniformly bounded $\mathscr{I}_1$-almost everywhere in Δ.*

The crucial step in the proof of Theorem 2 is the following

Proposition. *Anly almost approximately convex and approximately midpoint convex functional is approximately convex.*

As a corollary we get

Theorem 3. *Suppose $\mathscr{I}_2$ and $\mathscr{I}_1$ to be two conjugate p.l.i. σ-ideals in $\mathbb{R}^{2n}$ and $\mathbb{R}^n$, respectively. Given a nonempty convex set $\Delta \subset \mathbb{R}^n$ and a nonnegative number ε assume that a function $f: \Delta \to \mathbb{R}$ satisfies the inequality*

$$f(\lambda x + (1 - \lambda) y) \leqslant \lambda f(x) + (1 - \lambda) f(y) + \varepsilon$$

for all $\lambda \in [0, 1]$ and all pairs $(x, y) \in \Delta^2$ beyond a member $N(\lambda)$ of thhe σ-ideal $\mathscr{I}_2$ in $\mathbb{R}^{2n}$. If assumptions (i′), (ii) *and* (iii) *are satisfied with $G = \mathbb{R}^n$, then there exists a constant $p_n > 0$ (depending exclusively on the dimension of the space) and a function $g: \Delta \to \mathbb{R}$ which is convex (and hence continuous) in Δ such that*

$$|f(x) - g(x)| \leqslant p_n \varepsilon$$

for $\mathscr{I}_1$-almost all $x \in \Delta$.

6. Some comments

(a) Setting $\varepsilon = 0$ in Theorem 3 one gets an analogue of Kuczma's main result from [8] including its generalization presented in [9]. Usual convexity is however considered in place of midpoint convexity for the reasons explained in (d).

Setting $\mathscr{I}_1 = \{\emptyset\}$ (which causes $\mathscr{I}_2$ to equal $\{\emptyset\}$ as well) one gets the Hyers-Ulam main result from [7] (see also Cholewa [1]) up to the numerical value of the coefficient p_n. Recall, however, that neither the Hyers-Ulam coefficients k_n nor Cholewa's j_n are sharp. The same applies to the coefficients occurring in Green's paper [6]. We have $j_n \leqslant k_n \leqslant p_n$, $n \in \mathbb{N}$ (see the proofs in Section 7 below).

(b) Why we deal with σ-ideals instead of ideals? The proof methods are strongly based upon the countable summation of negligible sets. Although till now we do not have any suitable counter-example it seems doubtful for the results to be longer valid without the σ-additivity assumption. None of the guestions of this type have been investigated in literature even in the "almost" case alone (see Kuczma [8] and [9]). On the other hand, in the most important cases (measure and category) we deal just with σ-ideals.

(c) A very interesting σ-ideal of the so–called Christensen zero sets in an Abelian Polish topological group is worthy to be mentioned here (see J. P. R. Christensen's paper [2] and also P. Fischer — Z. Słodkowsksi [4]). The Christensen zero sets serve well as a substitute of the Haar nullsets and they coincide in the case when the topological group in question is locally compact. Unfortunately, in general, the σ-ideal of Christensen zero sets, although proper and linearly invariant, is not conjugate with its product counterpart. The problem of finding conditions under which the conjugacy relation holds true remains open; it seems to be both interesting and difficult.

(d) As we have remarked in Sections 1 and 2 the stability problem (an alternative term for the question whether an approximately (midpoint) convex function is uniformly approximated by a (midpoint) convex one) for midpoint convex functionals with a convex domain remains open in the finite dimensional case. The more so does the same question for almost approximately convex functionals.

7. Proofs

We proceed with the following

Lemma 1. *Let* $(G, +)$ *be an Abelian and uniquely* 2*-divisible group and let* $\mathscr{I}_1$, $\mathscr{I}_2$ *be two conjugate p.l.i.* σ*-ideals in* G *and* G^2, *respectively, and suppose the assumptions (i)—(iii) to be satisfied. If a function* $f\colon \Delta \to \mathbb{R}$ *and a number* $\varepsilon \geqslant 0$ *are given such that*

(8) $$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + \varepsilon$$

for all $(x, y) \in \Delta^2 \setminus M$ *and some* $M \in \mathscr{I}_2$, *then the function* $g\colon \Delta \to [-\infty, \infty)$ *given by formula*

(9) $$g(x)) := \inf_{h \in \Delta(x)} \operatorname{ess} \frac{1}{2}[f(x+h) + f(x-h)], \; x \in \Delta,$$

is well defined and satisfies estimations

(10) $$-2\varepsilon \leqslant f(x) - g(x) \leqslant \varepsilon$$

for $\mathscr{I}_1$*-almost every* $x \in \Delta$. *In particular,* g *is* $\mathscr{I}_1$*-almost every-where finite.*

Proof. Definition (9) is correct because of (7), (6) and the obvious implication

(11) $$h \in \Delta(x) \Rightarrow x + h, \; x - h \in \Delta.$$

Since the ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ are conjugate there exists a set $U \in \mathscr{I}_1$ such that for each $x \in G \setminus U$ the vertical section M_x of M through x falls into $\mathscr{I}_1$. First we shall show that there exists a set $S \in \mathscr{I}_1$ such that

(12) $$\begin{cases} U(x) := \{h \in \Delta(x) \colon (x, x+h) \in M\} \\ \text{and} \\ V(x) := \{h \in \Delta(x) \colon (x+h, x-h) \in M\} \end{cases}$$ belong to $\mathscr{I}_1$ provided that $x \notin S$.

In fact, for any $x \in G \setminus U$ we have $U(x) \subset M_x - x \in \mathscr{I}_1$. On the other hand putting

$$T(x, y) := \frac{1}{2}(x + y, x - y), \quad x, y \in G,$$

on account of (ii) we infer that $T(M)$ belongs to $\mathscr{I}_2$ and therefore, there exists a set $V \in \mathscr{I}_1$ such that the sections $(T(M))_x$ are in $\mathscr{I}_1$ whenever $x \in G \setminus V$. Thus

$$h \in V(x) \Rightarrow T(x+h, x-h) = (x, h) \in T(M) \Rightarrow h \in h \in (T(M))_x \in \mathscr{I}_1$$

provided that $x \notin V$. Consequently, (12) is fulfilled with $S := U \cup V$. Fix arbitrarily a set $W \in \mathscr{I}_1$ and an $x \in G \setminus S$. Write

$$B(x) := \bigcup_{n=0} 2^n (W \cup U(x) \cup (-U(x)) \cup V(x)).$$

By means of (i) and the fact that $\mathscr{I}_1$ is a σ-ideal we have $B(x)$ in $\mathscr{I}_1$. Recalling (iii) take any $h \in \Delta(x) \setminus B(x)$. Then, in partiicular, $h \notin V(x)$ whence $(x + h, x - h) \notin M$ and

$$f(x) \leqslant \frac{f(x+h) + f(x-h)}{2} + \varepsilon$$

in view of (8). Since h was taken arbitrarily from $\Delta(x)\backslash B(x)$ we get

$$f(x) \leqslant \inf_{h \in \Delta(x)\backslash B(x)} \frac{f(x+h)+f(x-h)}{2} + \varepsilon \leqslant \operatorname{inf\,ess}_{h \in \Delta(x)} \frac{f(x+h)+f(x-h)}{2} + \varepsilon =$$

$$= g(x) + \varepsilon$$

because of (6) and (9). This shows that

$$(13) \qquad f(x) - g(x) \leqslant \varepsilon \qquad \text{for } x \in G\backslash S,$$

i.e. the right-hand side of (10). In particular, g is $\mathscr{I}_1$-almost everywhere finite.

To prove the lower estimation indicated by (10), fix an $x \in G\backslash S$ again and note that for $h \in \Delta(x)\backslash B(x)$ we have also $\frac{1}{2^n} h \notin [U(x) \cup (-U(x))]$ for $n \in \mathbb{N} \cup \{0\} = : \mathbb{N}_0$, which means that the pairs $\left(x, x \pm \frac{1}{2^n} h\right) \notin M$ for all $n \in \mathbb{N}_0$ (see (12)). Henceforward,

$$f\left(x \pm \frac{1}{2^{n+1}} h\right) = f\left(\frac{x + \left(x \pm \frac{1}{2^n} h\right)}{2}\right) \leqslant \frac{f(x) + f\left(x \pm \frac{1}{2^n} h\right)}{2} + \varepsilon, \quad n \in \mathbb{N}_0,$$

on account of (8), which proves that the sequences

$$\alpha_n(x) : = f\left(x - \frac{1}{2^n} h\right) - f(x) \quad \text{and} \quad \beta_n(x) : = f\left(x + \frac{1}{2^n} h\right) - f(x), \; n \in \mathbb{N}_0$$

are both solutions of the recurrence relation

$$\gamma_{n+1}(x) \leqslant \frac{1}{2} \gamma_n(x) + \varepsilon, \; n \in \mathbb{N}_0.$$

The latter implies easily (induction) that

$$\gamma_n(x) \leqslant \frac{1}{2^n} \gamma_0(x) + \sum_{k=0}^{n-1} \frac{1}{2^k} \varepsilon, \; n \in \mathbb{N},$$

whence

$$(14) \qquad \limsup_{n \to \infty} \gamma_n(x) \leqslant \sum_{k=0}^{\infty} \frac{1}{2^k} \varepsilon = 2\varepsilon.$$

As a result of (14) aplied for $\gamma_n = \alpha_n$, $n \in \mathbb{N}_0$, and $\gamma_n = \beta_n$, $n \in \mathbb{N}_0$, we obtain the inequalities

$$\limsup_{n\to\infty} f\left(x - \frac{1}{2^n} h\right) \leqslant f(x) + 2\varepsilon \text{ and } \limsup_{n\to\infty} f\left(x + \frac{1}{2^n} h\right) \leqslant f(x) + 2\varepsilon$$

getting

$$(15) \qquad \limsup_{n\to\infty} \frac{f\left(x - \frac{1}{2^n} h\right) + f\left(x + 12^n h\right)}{2} \leqslant f(x) + 2\varepsilon.$$

In particular, making use of the fact that $\frac{1}{2^n} h \notin W$ for all $n \in \mathbb{N}_0$ inequality (15) implies that

Since the set $W \in \mathscr{I}_1$ was taken arbitrarily definitions (6) and (9) lead to the inequality

$$(16) \qquad g(x) \leqslant f(x) + 2\varepsilon$$

valid for every $x \in G \setminus S$.

Now, our assertion (10) results directly from (13) and (16), which finishes the proof.

Lemma 2. *Under the assumptions and denotations of Lemma* 1 *the estimations*

$$(17) \qquad -2\varepsilon \leqslant g(x) - \operatorname*{inf\,ess}_{h \in \Delta(x)} \frac{g(x+h) + g(x-h)}{2} \leqslant \varepsilon$$

hold true for all $x \in \Delta$. *Moreover, there exists a set* N *in the* σ*-ideal* $\Omega(\mathscr{I}_1)$ *defined by* (5), *such that*

$$g\left(\frac{x+y}{2}\right) \leqslant \frac{g(x) + g(y)}{2} + 4\varepsilon$$

for all $(x, y \in \Delta^2 \setminus N$; *in other words,* g *is* $\Omega(\mathscr{I}_1)$*-almost* 4ε*-approximately midpoint convex.*

Proof. Fix an $x_0 \in \Delta$ and put

$$\varphi(h) := \frac{1}{2}\,[f(x_0 + h) + f(x_0 - h)], \quad \psi(h) := \frac{1}{2}\,[g(x_0 + h) + g(x_0 - h)],$$

for $h \in \Delta(x_0)$. Take a set $S \in \mathscr{I}_1$ such that (10) holds true for $x \in G \setminus S$ and write

$$S(x_0) := (S - x_0) \cup (x_0 - S).$$

Then, for any $h \in \Delta(x_0) \setminus S(x_0$ the points $x_0 + h$ do not belong to S whence

$$\varphi(h) \leqslant \psi(h) + \varepsilon \qquad \text{for} \quad h \in \Delta(x_0) \setminus S(x_0).$$

Now, recalling (9) and keeping the observation at the end of Section 4 in mind, one may find a set $W(x_0)\in\mathscr{I}_1$ such that $g(x_0)=\inf\limits_{h\in\Delta(x_0)\setminus W(x_0)}\varphi(h)$ whence

$$g(x_0)\leqslant\inf\{\varphi(h)\colon h\in\Delta(x_0)\setminus[S(S(x_0)\cup W(x_0)\setminus\}\leqslant$$
$$\leqslant\inf\{\psi(h)\colon h\in\Delta(x_0)\setminus[S(x_0)\cup W(x_0)]\}+\varepsilon\leqslant$$
$$\leqslant\operatorname*{inf\,ess}_{h\in\Delta(x_0)}\psi(h)+\varepsilon=\inf_{h\in\Delta(x_0)}\frac{g(x_0+h)+g(x_0-h)}{2}+\varepsilon.$$

The left estimation in (17) may be derived quite analogously.

To prove the second assertion of the lemma write

$$K\colon=\bigcup_{x\in\Delta}[\{x\}\times(S\cup(2S-x))].$$

Clearly, $K\in\Omega(\mathscr{I}_1)$ (see (5)). Put

$$N\colon=(S\times G)\cup M\cup K;$$

then $N\in\Omega(\mathscr{I}_1)$ and taking any pair $(x,y)\in\Delta^2\setminus N$ one has

$x\notin S$, $(x,y)\notin M$, $y\notin S$ as well as $\frac{1}{2}(x+y)\notin S$.

In particular, each of the values $g(x)$, $g(y)$, $g\left(\frac{x+y}{2}\right)$ is finite and with the aid of (10) and (8) one gets

$$g\left(\frac{x+y}{2}\right)-\frac{g(x)+g(y)}{2}\leqslant f\left(\frac{x+y}{2}\right)+2\varepsilon+\frac{-f(x)+\varepsilon-f(y)+\varepsilon}{2}=$$
$$=f\left(\frac{x+y}{2}\right)-\frac{f(x)+f(y)}{2}+3\varepsilon\leqslant\varepsilon+3\varepsilon=4\varepsilon,$$

which ends the proof.

Lemma 3. *Under the assumptions and denotations of Lemma* 1 *the function g is* 7ε*-midpoint convex and everywhere finite on* Δ.

Proof. According to the remark at the end of Section 4, for each $x\in\Delta$ one may find a member $W(x)$ of the σ-ideal $\mathscr{I}_1$ such that

$$\operatorname*{inf\,ess}_{h\in\Delta(x)}\frac{g(x+h)+g(x-h)}{2}=\inf_{h\in\Delta(x)\setminus W(x)}\frac{g(x+h)+g(x-h)}{2}.$$

Lemma 2 gives now

$$g(x)\leqslant\inf_{h\in\Delta(x)\setminus W(x)}\frac{g(x+h)+g(x-h)}{2}+\varepsilon\leqslant\frac{g(x+h)+g(x-h)}{2}+\varepsilon$$

for all $h \in \Delta(x) \setminus W(x)$ and all $x \in \Delta$. Write

$$Z:=\left\{(x, y) \in \Delta^2 \colon g\left(\frac{x+y}{2}\right) > \frac{g(x)+g(y)}{2} + 4\varepsilon\right\}.$$

Making use of Lemma 2 again we infer that $Z \in \Omega(\mathscr{I}_1)$, which implies the existence of a set $E \in \mathscr{I}_1$, such that $Z_x \in \mathscr{I}_1$, provided that $x \in G \setminus E$ (cf (5)). Fix an $x \in \Delta$ and choose $\alpha(x) > g(x)$ arbitrarily. The set

$$B(x):=\left\{h \in \Delta(x) \colon \frac{g(x+h)+g(x-h)}{2} < \alpha(x) + 2\varepsilon\right\} \tag{18}$$

does not belong to $\mathscr{I}_1$; indeed, otherwise

$$\operatorname*{inf\,ess}_{h \in \Delta(x)} \frac{g(x+h)+g(x-h)}{2} \geqslant \inf_{h \in \Delta(x) \setminus B(x)} \frac{g(x+h)+g(x-h)}{2} \geqslant$$

$$\geqslant \alpha(x) + 2\varepsilon > g(x) + 2\varepsilon,$$

which contradicts Lemma 2.

Now, fix a pair $(x, y) \in \Delta^2$ and put $z:=\frac{1}{2}(x+y)$. Since $B(x)$ as well as $B(y)$ do not fall into $\mathscr{I}_1$ we are able to choose the elements

$$h \in B(x) \setminus [(E-x) \cup (x-E)]$$

and

$$k \in B(y) \setminus [(Z_{x+h} - y) \cup (y - Z_{x-h}) \cup (2W(z) - h)]$$

Then the pairs $(x+h, y+k)$ and $(x-h, y-k)$ belong to $\Delta^2 \setminus Z$ and henceforth

$$g\left(z+\frac{h+k}{2}\right) = g\left(\frac{(x+h)+(y+k)}{2}\right) \leqslant \frac{g(x+h)+g(y+k)}{2} + 4\varepsilon$$

and similarly

$$g\left(z-\frac{h+k}{2}\right) \leqslant \frac{g(x-h)+g(y-k)}{2} + 4\varepsilon.$$

Finally, since $\frac{1}{2}(h+k) \notin W(z)$ and $h \in B(x)$, $k \in B(y)$, we get

$$g(z) \leqslant \frac{g\left(z+\frac{1}{2}(h+k)\right) + g\left(z-\frac{1}{2}(h+k)\right)}{2} + \varepsilon \leqslant$$

$$\leqslant \frac{1}{2}\left[\frac{g(x+h)+g(y+k)}{2}+4\varepsilon+\frac{g(x-h)+g(y-k)}{2}+4\varepsilon\right]+\varepsilon=$$

$$=\frac{1}{2}\left[\frac{g(x+\ +h)+g(x-h)}{2}+\frac{g(y+k)+g(y-k)}{2}+8\varepsilon\right]+\varepsilon<$$

$$<\frac{1}{2}\left[(\alpha(x)+2\varepsilon)+(\alpha(y)+2\varepsilon)+8\varepsilon\right]+\varepsilon=\frac{1}{2}\left[\alpha(x)+\alpha(y)\right]+7\varepsilon.$$

Letting $\alpha(x)$ tend to $g(x)$ and $\alpha(y)$ to $g(y)$ we obtain the inequality

$$g\left(\frac{x+y}{2}\right)\leqslant\frac{g(x)+g(y)}{2}+7\varepsilon,$$

as claimed.

In order to prove that g is everywhere finite, fix arbitrarily an $x\in\Delta$, recall that $g(z)$ is finite for $z\notin S\in\mathscr{I}_1$ and choose a $y\in\Delta\setminus(S\cup(2S-x))$. Then $y\notin S$ as well as $\frac{1}{2}(x+y)\notin S$ whence

$$g(x)\geqslant 2g\left(\frac{x+y}{2}\right)-g(y)-7\varepsilon>-\infty;$$

the value $+\infty$ was excluded before (see Lemma 1). This completes the proof.

Proof of Theorem 1. Assume (8) to hold for $(x,y)\notin M\in\mathscr{I}_2$ and define $g:\Delta\to[-\infty,\infty)$ by (9). Actually, g is finite and 7ε-approximately midpoint convex (see Lemma 3). Using (10), which is satisfied $\mathscr{I}_1$-almost everywhere, we infer in particular that

$$|f(x)-g(x)|\leqslant 2\varepsilon$$

for $\mathscr{I}_1$-almost every $x\in\Delta$, thereby finishing the proof.

Proof of the Proposition. Now, G is a real linear space and $\Delta\subset G$ is convex in the usual sense. Let a functional $g\colon\Delta\to\mathbb{R}$ and a number $\eta\geqslant 0$ be given such such that

(˙9) $$g(\lambda x+(1-\lambda)y)\leqslant\lambda g(x)+(1-\lambda)g(y)+\eta$$

for all $\lambda\in[0,1]$ and all pairs $(x,y)\in\Delta^2\setminus M(\lambda)$ is an element of the σ-ideal $\mathscr{I}_2$. Let us emphasize that $M(\lambda)$ is allowed to depend on $\lambda\in[0,1]$. Moreover, let

(20) $$g\left(\frac{x+y}{2}\right)\leqslant\frac{g(x)+g(y)}{2}+\delta$$

hold true for some $\delta\geqslant 0$ and all pairs $(x,y)\in\Delta^2$.

Applying the method similar to that used in the proof of Lemma 1 one may show that (20) implies the inequality

$$G(x) := \inf_{h \in \Delta(x)} \operatorname{ess} \frac{g(x+h) + g(x-h)}{2} \leqslant g(x) + 2\delta, \; x \in \Delta. \tag{21}$$

Now, for arbitrarily fixed $x \in \Delta$ choose any $\alpha(x) > g(x)$ and put

$$B(x) := \left\{ h \in \Delta(x) : \frac{g(x+h) + g(x-h)}{2} < \alpha(x) + 2\delta \right\}.$$

Then

$$B(x) \notin \mathscr{I}_1. \tag{22}$$

Actually, if we had $B(x)$ in $\mathscr{I}_1$ for some $x \in \Delta$, then we would get

$$G(x) \geqslant \inf_{h \in \Delta(x) \setminus B(x)} \frac{g(x+h) + g(x-h)}{2} \geqslant \alpha(x) + 2d > g((x) + 2\delta$$

contradicting (21).

Fix any pair $(x, y) \in \Delta^2$ and a $\lambda \in [0, 1]$. Write $z := \lambda x + (1 - \lambda)y$ and observe that

$$\lambda \Delta(x) + (1 - \lambda)\, \Delta(y) \subset \Delta(\lambda x + (1 - \lambda)y) = \Delta(z) \tag{23}$$

with $\Delta(x)$ defined by (7) straightforward verification based upon the convexity of Δ).

Let $U(\lambda) \in \mathscr{I}_1$ be a set associated with $M(\lambda)$ according to the conjugacy relation. Choose an

$$h \in B\, x) \setminus [(U(\lambda) - x) \cup (x - U(\lambda))] \tag{24}$$

which is possible in view of (22). Then, in particular, $x + h \notin U(\lambda)$ and $x - h \notin U(\lambda)$ whence the vertical sections $(M(\lambda))_{x+h}$ and $(M(\lambda))_{x-h}$ are both in $\mathscr{I}_1$. This and a repeated resort to (22) allows one to choose a

$$k \in B(y) \setminus [((M(\lambda))_{x+h} - y) \cup (y - (M(\lambda))_{x-h})]. \tag{25}$$

Then, in particular,

$$(x + h, y + k) \notin M(\lambda) \quad \text{and} \quad (x - h, y - k) \notin M(\lambda). \tag{26}$$

Finally, putting $l := \lambda h + (1 - \lambda)k$ note that by means of the definition of $B(x)$ one has h in $\Delta(x)$ and k in $\Delta(y)$, respectively, whence by (23) $l \in \Delta(z)$, i.e. $z + l$ as well as $z - l$ are in Δ.

Now, we are in a position to carry out the following calculation:

$$g(\lambda x + (1-\lambda)y) = g(z) \leqslant \frac{g(z+l) + g(z-l)}{2} + \delta =$$

$$= \frac{1}{2}[g(\lambda(x+h) + (1-\lambda)(y+k)) + g(\lambda(x-h) + (1-\lambda)(y-k))] + \delta \leqslant$$

$$\leqslant \frac{1}{2}[\lambda g(x+h) + (1-\lambda)g(y+k) + \eta + \lambda g(x-h) +$$

$$+ (1-\lambda)g(y-k) + \eta] + \delta = \lambda\frac{g(x+h)+g(x-h)}{2} +$$

$$+ (1-\lambda)\frac{g(y+k)+g(y-k)}{2} + \eta + \delta \leqslant \lambda(\alpha(x) + 2\delta) +$$

$$+ (1-\lambda)(\alpha(y) + 2\delta) + \eta + \delta = \lambda\alpha(x) + (1-\lambda)\alpha(y) + \eta + 3\delta,$$

with the aid of the subsequential use of (20), (26), (19), (24) and (25). Letting $\alpha(x)$ *and* $\alpha(y)$ tend to $g(x)$ and $g(y)$, respectively, we come to the inequality

$$g(\lambda x + (1-\lambda)y) \leqslant \lambda g(x) + (1-\lambda)g(y) + (\eta + 3\delta)$$

valid for all $x, y \in \Delta$ and all $\lambda \in [0, 1]$. Thus we have proved that g is $(\eta + 3\delta)$-convex on Δ, which finishes the proof.

Summarizing, we have proved a little more than what a slightly enigmatic statement of the Proposition says. Namely, we have the following.

Proposition (*). *Suppose that G is a real linear space and Δ is a nonempty convex subset of G. Let $\mathscr{I}_2$ and $\mathscr{I}_1$ be two conjugate p.l.i. σ-ideals in G^2 and G, respectively. Given a δ-midpoint convex functional $g\colon \Delta \to \mathbb{R}$ assume that for each $\lambda \in [0, 1]$ there exists a set $M(\lambda) \in \mathscr{I}_2$ such that the inequality*

$$g(\lambda x + (1-\lambda)y) \leqslant \lambda g(x) + (1-\lambda)g(y) + \eta$$

holds for all $(x, y) \in \Delta^2 \setminus M(\lambda)$. If the assumptions (i′), (ii) and (iii) are satisfied, then g is $(\eta + 3\delta)$-convex in Δ; here δ and η stand for any a priori given nonnegative numbers.

Proof of Theorem 2. Now again G is a real linear space, $\Delta \subset G$ is convex and, moreover, assumption (i) is replaced by (i′). Let a functional $f\colon \Delta \to \mathbb{R}$ and a number $\varepsilon \geqslant 0$ be given such that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + \varepsilon \tag{27}$$

for all $\lambda \in [0, 1]$ and all pairs $(x, y) \in \Delta^2 \setminus N(\lambda)$ where $N(\lambda)$ is a certain element of the σ-ideal $\mathscr{I}_2$, possibly depending on λ.

In particular, f satisfies (8) $\left(\text{take } \lambda = \frac{1}{2}\right)$ and, evidently, all the assumptions of Lemma 1 are fulfilled. Therefore, on account of Lemmas 1 and 3, the functional g defined by (9) is everywhere finite and the inequalities

$$g\left(\frac{x+y}{2}\right) \leqslant \frac{g(x)+g(y)}{2} + 7\varepsilon \tag{28}$$

and

$$-2\varepsilon \leqslant f(x) - g(x) \leqslant \varepsilon \tag{10}$$

hold true for all pairs $(x, y) \in \Delta^2$ and for $\mathscr{I}_1$-almost all $x \in \Delta$, respectively. Assume that (10) is satisfied except for a set $S \in \mathscr{I}_1$ and observe that for any $\lambda \in \mathbb{R}\setminus\{1\}$ the set

$$K(\lambda) := \{(x, y) \in G^2 \colon \lambda x + (1-\lambda)y \in S\}$$

belongs to $\Omega(\mathscr{I}_1)$ (cf. (5)). Indeed, for any $x \in G$ one has

$$(K(\lambda))_x = \frac{1}{1-\lambda}(S - \lambda x)$$

which belongs to $\mathscr{I}_1$ in view of (i′). Consequently, the union

$$M(\lambda) := N(\lambda) \cup (S \times G) \cup (G \times S) \cup K(\lambda)$$

belongs to $\Theta(\mathscr{I}_1)$ because of the inclusion $\mathscr{I}_2 \subset \Omega(\mathscr{I}_1)$ (cf. Section 4).

Now, fix any $\lambda \in [0, 1)$ and a pair $(x, y) \in \Delta^2 \setminus M(\lambda)$. Then $(x, y) \notin N(\lambda)$, $x \notin S$, $y \notin S$ and $\lambda x + (1-\lambda)y \notin S$ and henceforward

$$\begin{aligned} g(\lambda x + (1-\lambda)y) &\leqslant f(\lambda x + (1-\lambda)y) + 2\varepsilon \leqslant \lambda f(x) + (1-\lambda)f(y) + 2\varepsilon \leqslant \\ &\leqslant \lambda(g(x)+\varepsilon) + (1-\lambda)(g(y)+\varepsilon) + 2\varepsilon = \lambda g(x) + (1-\lambda)g(y) + 3\varepsilon, \end{aligned}$$

on account of the subsequential use of (10), (27) and (10) again. This means that g satisfies all the assumptions of Proposition (*) with $\delta := 7\varepsilon$, $\eta := 3\varepsilon$ and $\mathscr{I}_2 = \Omega(\mathscr{I}_1)$. Consequently, g is (24ε)-approximately convex. Thus the proof has been completed.

Finally, the

Proof of Theorem 3. is a straightforward consequence of Theorem 2 and the Hyers-Ulam theorem spoken of in Section 1. Actually, Theorem 2 guarantees the existence of a (24ε)-convex function $g_0 \colon \Delta \to \mathbb{R}$ such that

$$|f(x) - g_0(x)| \leqslant 2\varepsilon$$

for $\mathscr{I}_1$-almost all $x \in \Delta \subset \mathbb{R}^n$. On the other hand, the Hyers-Ulam theorem gives a convex function $g \colon \Delta \to \mathbb{R}$ such that

$$|g_0(x) - g(x)| \leqslant 24 j_n \varepsilon$$

for all $x \in \Delta$ (we have used Cholewa's coefficients j_n instead of k_n; see Section 1). Consequently, we get the assertion desired with $p_n := 2(12 j_n + 1)$.

Remark. The coefficients p_n just obtained are by no means sharp. With an additional proof (omitted here) they may be diminished to $13 j_n + 2$. However, the latter are still far from the best.

We condude with the observation that function $g$occurring in Theorem 2 has to be continuous since each convex functional on an open convex and nonvoid subset of a finite dimensional Banach space has to be continuous.

REFERENCES

[1] CHOLEWA, P. W.: Remarks on the stability of functional equations, Aequationes Math. 27(1984), pp. 76—86.

[2] CHRISTENSEN, J. P. R.: On sets of Haar measure zero in Abelian Polish groups, Israel Journal Math. 13(1972), pp. 255—260.

[3] DHOMBRES, J. and GER, R.: Conditional Cauchy equations, Glasnik Mat. 13(1978), pp. 39—62.

[4] FISCHER, P. and SŁODKOWSKI, Z.: Christensen zero sets and measurable convex functions, Proc. Amer. Math. Soc. 79(1980), pp. 449—453.

[5] GER, R.: Almost approximately additive mappings, Proceedings of the Third International Conference on General Inequalities. Edited by E. F. Beckenbach and W. Walter, ISNM 64, Birkhäuser Verlag, Basel und Stuttgart, 1983, pp. 263—276.

[6] GREEN, J. W.: Approximately convex functions, Duke Math. J. 19 (1952), pp. 499—504.

[7] HYERS, B. H. and ULAM, S.: Approximately convex functions, Proc. Amer. Math. Soc. 3 (1952), pp. 821—828.

[8] KUCZMA, M.: Almost convex functions, Colloquium Math. 21 (1970), pp. 279—284.

[9] KUCZMA, M.: An introduction to the theory of functional equations and inequalities, Polish Scientific Publishers and Uniwersytet Sląski, Warszawa–Kraków–Katowice, 1985.

[10] REICH, L.: Oral Communication.

[11] WALTER, W.: Oral communication.

Received March 3, 1986

ul. B. Chrobrego 9/31
40-881 Katowice
Poland

ПОЧТИ АППРОКСИМАТИВНЫЕ ВЫПУКЛЫЕ ФУНКЦИИ

Roman Ger

Резюме

Под аппроксимативной выпуклостью понимается такая модуфикация обыкновенного понятия выпуклости, в которой определяющее неравенство удовлетворяется только с некоторой степенью точности. Почти выпуклость составляет другую модификацию: определяющее неравенство предлагается относительно данного аксиоматическом путем семейства «малых» множеств. Целью настоящей работы является описание поведения функционалов, которых выпуклость ослаблена в обоих этих направлениях одновременно.