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ON UPPER AND LOWER α-CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. Neubrunn [14] defined a multifunction $F\colon X \to Y$ to be upper (lower) α-continuous if $F^+(V)$ ($F^-(V)$) is α-open in X for every open set V of Y. In this paper, we obtain several characterizations and some basic properties concerning upper (lower) α-continuous multifunctions. An improvement of [14, Theorem 1] is given as follows: if a multifunction is lower α-continuous and upper β-continuous, then it is lower weakly continuous (Theorem 4.3).

1. Introduction

In 1965, Njåstad [15] introduced a weak form of open sets called *α-sets.* In 1982, the second author [18] of the present paper defined a function from a topological space into a topological space to be *strongly semi-continuous* if the inverse image of each open set is an α-set. Mashhour et al. [12] called strongly semi-continuous functions *α-continuous* and obtained several properties of such functions. In 1986, Neubrunn [14] extended these functions to multifunctions and introduced the notion of *upper* (*lower*) *α-continuous multifunctions.* The purpose of the present paper is to obtain several characterizations of upper (lower) α-continuous multifunctions and some basic properties of such multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *α-open* [15] (resp. *semi-open* [7], *preopen* [11]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$). The family of all α-open (resp. semi-open, preopen) sets in X is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$). For these

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three families, it is shown in [19, Lemma 3.1] that $\alpha(X) = \mathrm{SO}(X) \cap \mathrm{PO}(X)$. Since $\alpha(X)$ is a topology for X [15, Proposition 2], by $\alpha\,\mathrm{Cl}(A)$ we shall denote the closure of A with respect to $\alpha(X)$. A subset A is called an *α-neighbourhood* of a point x in X if there exists $U \in \alpha(X)$ such that $x \in U \subset A$. The complement of a semi-open (resp. α-open) set is said to be *semi-closed* (resp. *α-closed*). The intersection of all semi-closed sets of X containing A is called the *semi-closure* [3] and is denoted by $\mathrm{sCl}(A)$. The union of all semi-open sets of X contained in A is called the *semi-interior* of A and is denoted by $\mathrm{sInt}(A)$. A subset A is said to be *feebly open* [10] if there exists an open set U such that $U \subset A \subset \mathrm{sCl}(U)$. The complement of a feebly open set is called *feebly closed.* Since $\mathrm{sCl}(U) = \mathrm{Int}(\mathrm{Cl}(U))$ for any open set U [4, Lemma 2.1], it follows from [19, Lemma 4.12] that the notion of feebly open sets is equivalent to that of α-open sets.

LEMMA 2.1. *The following are equivalent for a subset A of a topological space X:*

(a) $A \in \alpha(X)$.
(b) $U \subset A \subset \mathrm{Int}(\mathrm{Cl}(U))$ *for some open set* U.
(c) $U \subset A \subset \mathrm{sCl}(U)$ *for some open set* U.
(d) $A \subset \mathrm{sCl}(\mathrm{Int}(A))$.

Proof. This follows from [4, Lemma 2.1], [19, Lemma 4.12] and [23, Theorem 1].

LEMMA 2.2. *The following properties hold for a subset A of a topological space X:*

(a) *A is α-closed in X if and only if* $\mathrm{sInt}(\mathrm{Cl}(A)) \subset A$;
(b) $\mathrm{sInt}(\mathrm{Cl}(A)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$;
(c) $\alpha\,\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

Proof. This follows from [23, Theorem 2], [4, Lemma 2.1] and [1, Theorem 2.2].

Maheshwari and Jain [8] defined a function to be *feebly continuous* if the inverse image of every open set is feebly open. However, we realize that feeble continuity is equivalent to α-continuity [12], that is, strong semi-continuity [18]. Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces and $F\colon X \to Y$ (resp. $f\colon X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F\colon X \to Y$, we shall denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is,

$$F^+(G) = \{x \in X \mid F(x) \subset G\} \quad \text{and} \quad F^-(G) = \{x \in X \mid F(x) \cap G \neq \emptyset\}.$$

3. Characterizations

DEFINITION 3.1. *A multifunction $F\colon X \to Y$ is said to be*

(a) *upper α-continuous at a point x of X if for any open set V of Y such that $F(x) \subset V$, there exists $U \in \alpha(X)$ containing x such that $F(U) \subset V$;*

(b) *lower α-continuous at $x \in X$ if for any open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;*

(c) *upper (resp. lower) α-continuous* [14] *if it is upper (resp. lower) α-continuous at every point of X.*

THEOREM 3.1. *The following are equivalent for a multifunction $F\colon X \to Y$:*

(a) *F is upper α-continuous at a point x of X.*
(b) *$x \in \mathrm{sCl}(\mathrm{Int}\,(F^+(V)))$ for any open set V of Y containing $F(x)$.*
(c) *For any $U \in SO(X)$ containing x and any open set V of Y containing $F(x)$, there exists a nonempty open set U_V of X such that $U_V \subset U$ and $F(U_V) \subset V$.*

Proof.

(a) $\Longrightarrow$ (b). Let V be any open set such that $F(x) \subset V$. Then there exists $U \in \alpha(X)$ containing x such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Since U is α-open, by Lemma 2.1 we have

$$x \in U \subset \mathrm{sCl}(\mathrm{Int}(U)) \subset \mathrm{sCl}(\mathrm{Int}\,(F^+(V)))\,.$$

(b) $\Longrightarrow$ (c). Let V be any open set of Y such that $F(x) \subset V$. Then $x \in \mathrm{sCl}(\mathrm{Int}\,(F^+(V)))$. Let U be any semi-open set containing x. Then $U \cap \mathrm{Int}(F^+(V)) \neq \emptyset$ [17, Lemma 3] and $U \cap \mathrm{Int}(F^+(V)) \in SO(X)$ [16, Lemma 1]. Put $U_V = \mathrm{Int}[U \cap \mathrm{Int}(F^+(V))]$, then U_V is a nonempty open set of Y [16, Lemma 4], $U_V \subset U$ and $F(U_V) \subset V$.

(c) $\Longrightarrow$ (a). Let $\mathrm{SO}(X,x)$ be the family of all semi-open sets of X containing x. Let V be any open set of Y containing $F(x)$. For each $U \in \mathrm{SO}(X,x)$, there exists a nonempty open set U_V such that $U_V \subset U$ and $F(U_V) \subset V$. Let $W = \bigcup\{U_V \mid U \in \mathrm{SO}(X,x)\}$. Then W is open in X, $x \in \mathrm{sCl}(W)$ and $F(W) \subset V$. Put $S = W \cup \{x\}$, then $W \subset S \subset \mathrm{sCl}(W)$. Therefore, by Lemma 2.1 $x \in S \in \alpha(X)$ and $F(S) \subset V$. This shows that F is upper α-continuous at x.

THEOREM 3.2. *The following are equivalent for a multifunction $F\colon X \to Y$:*

(a) *F is lower α-continuous at $x \in X$.*

(b) $x \in \mathrm{sCl}(\mathrm{Int}(F^-(V)))$ *for any open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$.

(c) *For any* $U \in \mathrm{SO}(X)$ *containing* x *and any open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$, *there exists a nonempty open set* U_V *such that* $F(u) \cap V \neq \emptyset$ *for every* $u \in U_V$ *and* $U_V \subset U$.

Proof. The proof is similar to that of Theorem 3.1.

THEOREM 3.3. *The following are equivalent for a multifunction* $F\colon X \to Y$:

(a) F *is upper* α*-continuous.*

(b) $F^+(V) \in \alpha(X)$ *for any open set* V *of* Y.

(c) $F^-(V)$ *is* α*-closed in* X *for any closed set* V *of* Y.

(d) $\mathrm{sInt}(\mathrm{Cl}(F^-(B))) \subset F^-(\mathrm{Cl}(B))$ *for any set* B *of* Y.

(e) $\alpha\,\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ *for any set* B *of* Y.

(f) *For each point* x *of* X *and each neighbourhood* V *of* $F(x)$, $F^+(V)$ *is an* α*-neighbourhood of* x.

(g) *For each point* x *of* X *and each neighbourhood* V *of* $F(x)$, *there exists an* α*-neighbourhood* U *of* x *such that* $F(U) \subset V$.

Proof.

(a) $\Longrightarrow$ (b). Let V be any open set of Y and let $x \in F^+(V)$. By Theorem 3.1, $x \in \mathrm{sCl}(\mathrm{Int}(F^+(V)))$. Therefore, we obtain $F^+(V) \subset \mathrm{sCl}(\mathrm{Int}(F^+(V)))$. It follows from Lemma 2.1 that $F^+(V) \in \alpha(X)$.

(b) $\Longleftrightarrow$ (c). This follows from the fact that $F^+(Y-B) = X - F^-(B)$ for any subset B of Y.

(c) $\Longrightarrow$ (d). Let B be any subset of Y. Then $F^-(\mathrm{Cl}(B))$ is α-closed in Y. By Lemma 2.2, we have

$$\mathrm{sInt}(\mathrm{Cl}(F^-(B))) \subset \mathrm{sInt}(\mathrm{Cl}(F^-(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B)).$$

(d) $\Longrightarrow$ (e). Let B be any subset of Y. By Lemma 2.2, we have

$$\alpha\,\mathrm{Cl}(F^-(B)) = F^-(B) \cup \mathrm{sInt}(\mathrm{Cl}(F^-(B))) \subset F^-(\mathrm{Cl}(B)).$$

(e) $\Longrightarrow$ (c). Let V be any closed set of Y. Then we have

$$\alpha\,\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V)) = F^-(V).$$

This shows that $F^-(V)$ is α-closed in X.

(b) $\Longrightarrow$ (f). Let $x \in X$ and V be a neighbourhood of $F(x)$. Then there exists an open set G of Y such that $F(x) \subset G \subset V$. Therefore we obtain

$x \in F^+(G) \subset F^+(V)$. Since $F^+(G) \in \alpha(X)$, $F^+(V)$ is an α-neighbourhood of x.

(f) $\implies$ (g). Let $x \in X$ and V be a neighbourhood of $F(x)$. Put $U = F^+(V)$, then U is an α-neighbourhood of x and $F(U) \subset V$.

(g) $\implies$ (a). Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighbourhood of $F(x)$. There exists an α-neighbourhood U of x such that $F(U) \subset V$. Therefore, there exists $A \in \alpha(X)$ such that $x \in A \subset U$; hence $F(A) \subset V$.

THEOREM 3.4. *The following are equivalent for a multifunction $F\colon X \to Y$:*

(a) *F is lower α-continuous.*
(b) *$F^-(V) \in \alpha(X)$ for any open set V of Y.*
(c) *$F^+(V)$ is α-closed in X for any closed set V of Y.*
(d) *$\mathrm{sInt}(\mathrm{Cl}(F^+(B))) \subset F^+(\mathrm{Cl}(B))$ for any subset B of Y.*
(e) *$\alpha\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for any subset B of Y.*
(f) *$F(\alpha\mathrm{Cl}(A)) \subset \mathrm{Cl}(F(A))$ for any subset A of X.*
(g) *$F(\mathrm{sInt}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(F(A))$ for any subset A of X.*
(h) *$F(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) \subset \mathrm{Cl}(F(A))$ for any subset A of X.*

Proof. The proofs except for the following are similar to those of Theorem 3.3 and are thus omitted.

(e) $\implies$ (f). Let A be any subset of X. Since $A \subset F^+(F(A))$, we have $\alpha\mathrm{Cl}(A) \subset \alpha\mathrm{Cl}(F^+(F(A))) \subset F^+(\mathrm{Cl}(F(A)))$ and $F(\alpha\mathrm{Cl}(A)) \subset \mathrm{Cl}(F(A))$.

(f) $\implies$ (g). This follows immediately from Lemma 2.2.

(g) $\implies$ (h). This is obvious by Lemma 2.2.

(h) $\implies$ (a). Let $x \in X$ and V be any open set such that $F(x) \cap V \neq \emptyset$. Then $x \in F^-(V)$. We shall show that $F^-(V) \in \alpha(X)$. By the hypothesis, we have

$$F(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(Y-V))))) \subset \mathrm{Cl}(F(F^+(Y-V))) \subset Y-V,$$

and hence $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(Y-V)))) \subset F^+(Y-V) = X - F^-(V)$. Therefore, we obtain $F^-(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(V))))$ and hence $F^-(V) \in \alpha(X)$. Put $U = F^-(V)$. We have $x \in U \in \alpha(X)$ and $F(u) \cap V \neq \emptyset$ for every $u \in U$. Thus F is lower α-continuous.

DEFINITION 3.2. *A function $f\colon X \to Y$ is said to be α-continuous* [12] *(resp. feebly continuous* [8], *semi-continuous* [7]*) if for every open set V of Y, $f^{-1}(V)$ is α-open (resp. feebly open, semi-open) in X.*

COROLLARY 3.1. (Popa [23], Mashhour et al. [12]). *The following are equivalent for a function $f\colon X \to Y$:*

(a) *f is feebly continuous.*
(b) *f is α-continuous.*
(c) *$f^{-1}(V)$ is α-closed in X for every closed set V of Y.*
(d) $\operatorname{sInt}(\operatorname{Cl}(f^{-1}(B))) \subset f^{-1}(\operatorname{Cl}(B))$ *for subset B of Y.*
(e) $\alpha\operatorname{Cl}(f^{-1}(B)) \subset f^{-1}(\operatorname{Cl}(B))$ *for any subset B of Y.*
(f) *For each $x \in X$ and each neighbourhood V of $f(x)$, $f^{-1}(V)$ is an α-neighbourhood of x.*
(g) *For each $x \in X$ and each neighbourhood V of $f(x)$, there exists an α-neighbourhood U of x such that $f(U) \subset V$.*
(h) $f(\alpha\operatorname{Cl}(A)) \subset \operatorname{Cl}(f(A))$ *for any subset A of X.*
(i) $f(\operatorname{sInt}(\operatorname{Cl}(A))) \subset \operatorname{Cl}(f(A))$ *for any subset A of X.*
(j) $f(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(A)))) \subset \operatorname{Cl}(f(A))$ *for any subset A of X.*

A multifunction $F\colon X \to Y$ is said to be *upper quasi continuous* [24] (resp. *lower quasi continuous*) if for each $x \in X$, each open set U containing x and each open set V of Y such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists a nonempty open set $G \subset U$ such that $F(G) \subset V$ (resp. $F(g) \cap V \neq \emptyset$ for every $g \in G$). A multifunction $F\colon X \to Y$ is said to be *upper precontinuous* [25] (resp. *lower precontinuous*) if $F^{+}(V) \in \operatorname{PO}(X)$ (resp. $F^{-}(V) \in \operatorname{PO}(X)$) for every open set V of Y.

THEOREM 3.5. *A multifunction $F\colon X \to Y$ is upper α-continuous (resp. lower α-continuous) if and only if it is upper quasi continuous (resp. lower quasi continuous) and upper precontinuous (resp. lower precontinuous).*

Proof. This follows from [24, Theorem 4.1] and [19, Lemma 3.1].

COROLLARY 3.2. (Noiri [19]). *A function $f\colon X \to Y$ is α-continuous if and only if it is precontinuous and semi-continuous.*

DEFINITION 3.3. *A subset A of a topological space X is said to be α-paracompact [27] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X.*

DEFINITION 3.4. *A subset A of topological space X is said to be α-regular [6] if for each point $x \in A$ and each open set U of X containing x, there exists an open set G of X such that $x \in G \subset \operatorname{Cl}(G) \subset U$.*

LEMMA 3.1. (Kovačević [6]). *If A is an α-regular α-paracompact subset of a topological space X and U is an open neighbourhood of A, then there exists an open set G of X such that $A \subset G \subset \operatorname{Cl}(G) \subset U$.*

A multifunction $F\colon X \to Y$ is said to be *punctually α-paracompact* (resp. *punctually α-regular*) if for each $x \in X$, $F(x)$ is α-paracompact (resp. α-regular). By $\alpha\,\mathrm{Cl}(F)\colon X \to Y$, we shall denote a multifunction defined as follows: $[\alpha\,\mathrm{Cl}(F)](x) = \alpha\,\mathrm{Cl}(F(x))$ for each point $x \in X$.

LEMMA 3.2. *If $F\colon X \to Y$ is punctually α-regular and punctually α-paracompact, then $[\alpha\,\mathrm{Cl}(F)]^+(V) = F^+(V)$ for every open set V of Y.*

Proof. Let V be any open set of Y and $x \in [\alpha\,\mathrm{Cl}(F)]^+(V)$. Then $\alpha\,\mathrm{Cl}(F(x)) \subset V$ and hence $F(x) \subset V$. Therefore, $x \in F^+(V)$ and hence $[\alpha\,\mathrm{Cl}(F)]^+(V) \subset F^+(V)$. Conversely, let V be any open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$. Since $F(x)$ is α-regular and α-paracompact, by Lemma 3.1 there exists an open set G such that $F(x) \subset G \subset \mathrm{Cl}(G) \subset V$; hence $\alpha\,\mathrm{Cl}(F(x)) \subset \mathrm{Cl}(G) \subset V$. This shows that $x \in [\alpha\,\mathrm{Cl}(F)]^+(V)$ and hence $F^+(V) \subset [\alpha\,\mathrm{Cl}(F)]^+(V)$. Consequently, we obtain $[\alpha\,\mathrm{Cl}(F)]^+(V) = F^+(V)$.

THEOREM 3.6. *Let $F\colon X \to Y$ be punctually α-regular and punctually α-paracompact. Then F is upper α-continuous if and only if $\alpha\,\mathrm{Cl}(F)\colon X \to Y$ is upper α-continuous.*

Proof.

Necessity. Suppose that F is upper α-continuous. Let $x \in X$ and V be any open set of Y such that $\alpha\,\mathrm{Cl}(F)(x) \subset V$. By Lemma 3.2, we have $x \in [\alpha\,\mathrm{Cl}(F)]^+(V) = F^+(V)$. Since F is upper α-continuous, there exists $U \in \alpha(X)$ containing x such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 3.1 there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset V$. Therefore, we have $\alpha\,\mathrm{Cl}(F(u)) \subset \mathrm{Cl}(H) \subset V$ for each $u \in U$ and hence $\alpha\,\mathrm{Cl}(F)(U) \subset V$. This shows that $\alpha\,\mathrm{Cl}(F)$ is upper α-continuous.

Sufficiency. Suppose that $\alpha\,\mathrm{Cl}(F)\colon X \to Y$ is upper α-continuous. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. By Lemma 3.2, we have $x \in F^+(V) = [\alpha\,\mathrm{Cl}(F)]^+(V)$ and hence $\alpha\,\mathrm{Cl}(F)(x) \subset V$. Since $\alpha\,\mathrm{Cl}(F)$ is upper α-continuous, there exists $U \in \alpha(X)$ containing x such that $\alpha\,\mathrm{Cl}(F)(U) \subset V$; hence $F(U) \subset V$. This shows that F is upper α-continuous.

LEMMA 3.3. *For a multifunction $F\colon X \to Y$, it follows that for each α-open set V of Y $[\alpha\,\mathrm{Cl}(F)]^-(V) = F^-(V)$.*

Proof. Suppose that V is any α-open set of Y. Let $x \in [\alpha\,\mathrm{Cl}(F)]^-(V)$. Then $\alpha\,\mathrm{Cl}(F(x)) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$. Therefore, we obtain

$x \in F^{-}(V)$. This shows that $[\alpha \operatorname{Cl}(F)]^{-}(V) \subset F^{-}(V)$. Conversely, let $x \in F^{-}(V)$. Then we have $\emptyset \neq F(x) \cap V \subset \alpha \operatorname{Cl}(F(x)) \cap V$ and hence $x \in [\alpha \operatorname{Cl}(F)]^{-}(V)$. This shows that $F^{-}(V) \subset [\alpha \operatorname{Cl}(F)]^{-}(V)$. Consequently, we obtain $[\alpha \operatorname{Cl}(F)]^{-}(V) = F^{-}(V)$.

THEOREM 3.7. *A multifunction $F\colon X \to Y$ is lower α-continuous if and only if $\alpha \operatorname{Cl}(F)\colon X \to Y$ is lower α-continuous.*

P r o o f. By utilizing Lemma 3.3, this can be proved similarly to that of Theorem 3.6.

For a multifunction $F\colon X \to Y$, the graph multifunction $G_F\colon X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

LEMMA 3.4. *The following hold for a multifunction $F\colon X \to Y$:*

(a) $G_F^{+}(A \times B) = A \cap F^{+}(B)$,
(b) $G_F^{-}(A \times B) = A \cap F^{-}(B)$,

for any subsets $A \subset X$ and $B \subset Y$.

P r o o f. We shall prove only (b). Let A and B be any subsets of X and Y, respectively. Let $x \in G_F^{-}(A \times B)$. Then

$$\emptyset \neq G_F(x) \cap (A \times B) = (\{x\} \times F(x)) \cap (A \times B) = (\{x\} \cap A) \times (F(x) \cap B).$$

Therefore, we have $x \in A$, and $F(x) \cap B \neq \emptyset$ and hence $x \in A \cap F^{-}(B)$. Conversely, let $x \in A \cap F^{-}(B)$. Then $x \in A$, and $F(x) \cap B \neq \emptyset$ and hence $G_F(x) \cap (A \times B) \neq \emptyset$. Therefore, $x \in G_F^{-}(A \times B)$. This completes the proof.

THEOREM 3.8. *Let $F\colon X \to Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper α-continuous if and only if $G_F\colon X \to X \times Y$ is upper α-continuous.*

P r o o f.

Necessity. Suppose that $F\colon X \to Y$ is upper α-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) \mid y \in F(x)\}$ is an open cover of $F(x)$ and there exist a finite number of points, say, $y_1, y_2, \dots, y_n$ in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) \mid 1 \leqq i \leqq n\}$. Set $U = \bigcap\{U(y_i) \mid 1 \leqq i \leqq n\}$ and $V = \bigcup\{V(y_i) \mid 1 \leqq i \leqq n\}$. Then U and V are open in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$.

Since F is upper α-continuous, there exists $U_0 \in \alpha(X)$ containing x such that $F(U_0) \subset V$. By Lemma 3.4, we have

$$U \cap U_0 \subset U \cap F^+(V) = G_F^+(U \times V) \subset G_F^+(W).$$

Therefore, we obtain $U \cap U_0 \in \alpha(X)$ and $G_F(U \cap U_0) \subset W$. This shows that G_F is upper α-continuous.

Sufficiency. Suppose that $G_F\colon X \to X \times Y$ is upper α-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \alpha(X)$ containing x such that $G_F(U) \subset X \times V$. Therefore, by Lemma 3.4, $U \subset G_F^+(X \times V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper α-continuous.

THEOREM 3.9. *A multifunction $F\colon X \to Y$ is lower α-continuous if and only if $G_F\colon X \to Y$ is lower α-continuous.*

Proof.

Necessity. Suppose that F is lower α-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $G_F(x) \cap W \neq \emptyset$. There exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since $F(x) \cap V \neq \emptyset$, there exists $U_0 \in \alpha(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U_0$; hence $U_0 \subset F^-(V)$. By Lemma 3.4,

$$U \cap U_0 \subset U \cap F^-(V) = G_F^-(U \times V) \subset G_F^-(W).$$

Moreover, $x \in U \cap U_0 \in \alpha(X)$ and hence G_F is lower α-continuous.

Sufficiency. Suppose that G_F is lower α-continuous. Let $x \in X$ and V be an open set in Y such that $F(x) \cap V \neq \emptyset$. Then $X \times V$ is open in $X \times Y$ and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

There exists $U \in \alpha(X)$ containing x such that $G_F(u) \cap (X \times V) \neq \emptyset$ for each $u \in U$. By Lemma 3.4, we obtain $U \subset G_F^-(X \times V) = F^-(V)$. This shows that F is lower α-continuous.

COROLLARY 3.3. (Hasanein et al. [5]). *A function $f\colon X \to Y$ is α-continuous if and only if the graph map $g_f\colon X \to X \times Y$, defined by $g_f(x) = (x, f(x))$ for every $x \in X$, is α-continuous.*

4. Some properties

The following lemma was shown by Mashhour et al. [12] and Reilly and Vamanamurthy [26].

LEMMA 4.1. *Let A and B be subsets of a topological space X.*

(a) *If $A \in \mathrm{SO}(X) \cup \mathrm{PO}(X)$ and $B \in \alpha(X)$, then $A \cap B \in \alpha(A)$.*

(b) *If $A \subset B \subset X$, $A \in \alpha(B)$ and $B \in \alpha(X)$, then $A \in \alpha(X)$.*

THEOREM 4.1. *If a multifunction $F\colon X \to Y$ is upper α-continuous (resp. lower α-continuous) and $X_0 \in \mathrm{PO}(X) \cup \mathrm{SO}(X)$, then the restriction $F|_{X_0}\colon X_0 \to Y$ is upper α-continuous (resp. lower α-continuous).*

Proof. We prove only the assertion for F upper α-continuous, the proof for F lower α-continuous being analogous. Let $x \in X_0$ and V be any open set of Y such that $(F|_{X_0})(x) \subset V$. Since F is upper α-continuous and $(F|_{X_0})(x) = F(x)$, there exists $U \in \alpha(X)$ containing x such that $F(U) \subset V$. Set $U_0 = U \cap X_0$, then by Lemma 4.1 we have $x \in U_0 \in \alpha(X_0)$ and $(F|_{X_0})(U_0) \subset V$. This shows that $F|_{X_0}$ is upper α-continuous.

THEOREM 4.2. *A multifunction $F\colon X \to Y$ is upper α-continuous (resp. lower α-continuous) if for each $x \in X$ there exists $X_0 \in \alpha(X)$ containing x such that the restriction $F|_{X_0}\colon X_0 \to Y$ is upper α-continuous (resp. lower α-continuous).*

Proof. We prove only the assertion for F upper α-continuous, the proof for F lower α-continuous being analogous. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. There exists $X_0 \in \alpha(X)$ containing x such that $F|_{X_0}$ is upper α-continuous. Therefore, there exists $U_0 \in \alpha(X_0)$ containing x such that $(F|_{X_0})(U_0) \subset V$. By Lemma 4.1, $U_0 \in \alpha(X)$ and $F(u) = (F|_{X_0})(u)$ for every $u \in U_0$. This shows that $F\colon X \to Y$ is upper α-continuous.

COROLLARY 4.1. *Let $\{U_\alpha \mid \alpha \in \nabla\}$ be an α-open cover of X. A multifunction $F\colon X \to Y$ is upper α-continuous (resp. lower α-continuous) if and only if the restriction $F|_{U_\alpha}\colon U_\alpha \to Y$ is upper α-continuous (resp. lower α-continuous) for each $\alpha \in \nabla$.*

Proof. This is an immediate consequence of Theorems 4.1 and 4.2.

COROLLARY 4.2. (Mashhour et al. [12]). *Let $\{U_\alpha \mid \alpha \in \nabla\}$ be an α-open cover of X. A function $f\colon X \to Y$ is α-continuous if the restriction $f|_{U_\alpha}\colon U_\alpha \to Y$ is α-continuous for each $\alpha \in \nabla$.*

Abd El-Monsef et al. [13] defined a subset A of a topological space X to be *β-open* if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The family of β-open sets of X contains $\mathrm{PO}(X)$ and $\mathrm{SO}(X)$.

DEFINITION 4.1. *A multifunction $F\colon X \to Y$ is said to be*

(a) *upper β-continuous* [21] *if for each $x \in X$ and each open set V of Y such that $F(x) \subset V$ there exists a β-open set U containing x such that $F(U) \subset V$;*

(b) *lower β-continuous* [21] *if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$ there exists a β-open set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.*

Remark 4.1. For a multifunction $F\colon X \to Y$, the following implications hold:

upper α-continuity ⇒ upper quasi-continuity
upper α-continuity ⇒ upper precontinuity
upper quasi-continuity ⇒ upper β-continuity
upper precontinuity ⇒ upper β-continuity

LEMMA 4.2. (Noiri and Popa [21]). *A multifunction $F\colon X \to Y$ is upper β-continuous (resp. lower β-continuous) if and only if*

$$\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^-(B)))) \subset F^-(\operatorname{Cl}(B))$$

$$(\textit{resp.}\ \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^+(B)))) \subset F^+(\operatorname{Cl}(B)))$$

for every subset B of Y.

DEFINITION 4.2. *A multifunction $F\colon X \to Y$ is said to be*

(a) *upper weakly continuous* [22] *if for each $x \in X$ and each open set V of Y such that $F(x) \subset V$, there exists an open set U containing x such that $F(U) \subset \operatorname{Cl}(V)$;*

(b) *lower weakly continuous* [22] *if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap \operatorname{Cl}(V) \neq \emptyset$ for every $u \in U$.*

THEOREM 4.3. *If a multifunction $F\colon X \to Y$ is lower α-continuous and upper β-continuous, then it is lower weakly continuous.*

Proof. Let V be any open set of Y. Since F is lower α-continuous, $F^-(V) \in \alpha(X)$ by Theorem 3.4. Since F is upper β-continuous, by Lemma 4.2 we have

$$F^-(V) \subset \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^-(V)))) \subset F^-(\operatorname{Cl}(V)).$$

Therefore, we obtain $F^-(V) \subset \operatorname{Int}(F^-(\operatorname{Cl}(V)))$. It follows from [22, Theorem 4] that F is lower weakly continuous.

COROLLARY 4.3. (Neubrunn [14]). *If a multifunction is lower α-continuous and upper quasi-continuous, then it is lower weakly continuous.*

THEOREM 4.4. *If a multifunction $F\colon X \to Y$ is upper α-continuous and lower β-continuous, then it is upper weakly continuous.*

Proof. Let V be any open set of Y. By Theorem 3.3 and Lemma 4.2, we have

$$F^+(V) \subset \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^+(V)))) \subset F^+(\operatorname{Cl}(V)).$$

Therefore, we obtain $F^+(V) \subset \operatorname{Int}(F^+(\operatorname{Cl}(V)))$. It follows from [22, Theorem 6] that F is upper weakly continuous.

COROLLARY 4.4. (Neubrunn [14]). *If a multifunction is upper α-continuous and lower quasi-continuous, then it is upper weakly continuous.*

A topological space X is said to be *α-compact* [9] if every α-open cover of X has a finite subcover.

THEOREM 4.5. *Let $F\colon X \to Y$ be an upper α-continuous surjective multifunction such that $F(x)$ is compact for each $x \in X$. If X is α-compact, then Y is compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be an open cover of Y. For each $x \in X$, $F(x)$ is compact and there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Set $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Since F is upper α-continuous, there exists $U(x) \in \alpha(X)$ containing x such that $F(U(x)) \subset V(x)$. The family $\{U(x) \mid x \in X\}$ is an α-open cover of X and there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup\{U(x_i) \mid 1 \leqq i \leqq n\}$. Therefore, we have

$$Y = F(X) = F\left(\bigcup_{i=1}^{n} U(x_i)\right) = \bigcup_{i=1}^{n} F(U(x_i)) \subset \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \nabla(x_i)} V_\alpha .$$

This shows that Y is compact.

COROLLARY 4.5. (Noiri and Di Maio [20]). *If X is α-compact and $f\colon X \to Y$ is an α-continuous surjection, then Y is compact.*

For a multifunction $F\colon X \to Y$, the *graph* $G(F)$ of F is defined as follows: $G(F) = \{(x, y) \mid x \in X \text{ and } y \in F(x)\}$.

THEOREM 4.6. *If $F\colon X \to Y$ is an upper α-continuous multifunction into a Hausdorff space Y and $F(x)$ is compact for each $x \in X$, then the graph $G(F)$ is α-closed in $X \times Y$.*

P r o o f. Let $(x,y) \in X \times Y - G(F)$. Then $y \in Y - F(x)$. For each $a \in F(x)$, there exist open sets $V(a)$ and $W(a)$ containing a and y, respectively, such that $V(a) \cap W(a) = \emptyset$. The family $\{V(a) \mid a \in F(x)\}$ is an open cover of $F(x)$ and there exist a finite number of points in $F(x)$, say, $a_1, a_2, \dots, a_n$ such that $F(x) \subset \bigcup\{V(a_i) \mid 1 \leqq i \leqq n\}$. Set $V = \bigcup\{V(a_i) \mid 1 \leqq i \leqq n\}$ and $W = \bigcap\{W(a_i) \mid 1 \leqq i \leqq n\}$. Since $F(x) \subset V$ and F is upper α-continuous, there exists $U \in \alpha(X)$ such that $x \in U$ and $F(U) \subset V$. Therefore, we obtain $F(U) \cap W = \emptyset$ and hence $(U \times W) \cap G(F) = \emptyset$. Since $U \times W$ is α-open in $X \times Y$ and $(x,y) \in U \times W$, $(x,y) \notin \alpha \operatorname{Cl}(G(F))$ and $G(F)$ is α-closed in $X \times Y$.

THEOREM 4.7. *If $F, G\colon X \to Y$ are upper α-continuous and Y is Hausdorff, then $A = \{x \in X \mid F(x) \cap G(x) \neq \emptyset\}$ is α-closed in X.*

P r o o f. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper α-continuous if and only if $F_\alpha\colon (X,\tau^\alpha) \to (Y,\sigma)$ is upper semi-continuous, where F_α is the multifunction defined by $F_\alpha(x) = F(x)$ for every $x \in X$ and τ^α denotes the family of α-open sets of (X,τ). Since (Y,σ) is Hausdorff, A is closed in (X,τ^α) [2, Theorem 3.3]. Therefore, A is α-closed in (X,τ).

COROLLARY 4.6. (N o i r i [19]). *If $f, g\colon X \to Y$ are α-continuous and Y is Hausdorff, then the set $\{x \in X \mid f(x) = g(x)\}$ is α-closed in X.*

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REFERENCES

[1] ANDRIJEVIĆ, D.: *Some properties of the topology of α-sets*, Mat. Vesnik **36** (1984), 1–10.

[2] BÂNZARU, T.: *On some properties of multifunctions.* (Romanian), Stud. Cerc. Mat. **24** (1972), 1503–1510.

[3] CROSSLEY, S. G.—HILDEBRAND, S. K.: *Semi-closure*, Texas J. Sci. **22** (1971), 99–112.

[4] DI MAIO, G.—NOIRI, T.: *On s-closed spaces*, Indian J. Pure Appl. Math. **18** (1987), 226–233.

[5] HASANEIN, I. A.—LONG, P. E.—MASHHOUR, A. S.: *Functions with α-closed graphs and preclosed graphs*, (Reprint).

[6] KOVAČEVIĆ, I.: *Subsets and paracompactness*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **14** (1984), 79–87.

[7] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[8] MAHESHWARI, S. N.—JAIN, P. C.: *Some new mappings*, Mathematica (Cluj) **24(47)** (1982), 53–55.

[9] MAHESHWARI, S. N.—THAKUR, S. S.: *On α-compact spaces*, Bull. Inst. Math. Acad. Sinica **13** (1985), 341–347.

[10] MAHESHWARI, S. N.—TAPI, U. D.: *Note on some applications on feebly open sets*, Madhya Bharati, J. Univ. Saugar, (To appear).

[11] MASHHOUR, A. S.—ABD EL-MONSEF, M. E.—EL-DEEB, S. N.: *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[12] MASHHOUR, A. S.—HASANEIN, I. A.—EL-DEEB, S. N.: *α-continuous and α-open mappings*, Acta Math. Hungar. **41** (1983), 213–218.

[13] ABD EL-MONSEF, M. E.—EL-DEEB, S. N.—MAHMOUD, R. A.: *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[14] NEUBRUNN, T.: *Strongly quasi-continuous multivalued mappings.* In: General Topology and its Relations to Modern Analysis and Algebra VI (Prague 1986), Heldermann, Berlin, 1988, pp. 351–359.

[15] NJÅSTAD, O.: *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

[16] NOIRI, T.: *On semi-continuous mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **54** (1973), 210–214.

[17] NOIRI, T.: *A note on semi-continuous mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **55** (1973), 400–403.

[18] NOIRI, T.: *A function which preserves connected spaces*, Časopis Pěst. Mat. **107** (1982), 393–396.

[19] NOIRI, T.: *On α-continuous functions*, Časopis Pěst. Mat. **109** (1984), 118–126.

[20] NOIRI, T.—DI MAIO, G.: *Properties of α-compact spaces.* In: Third National Conference on Topology (Trieste, 1986) (Italian), Rend. Circ. Mat. Palermo (2) Suppl. **18** (1988), 359–369.

[21] NOIRI, T.—POPA, V.: *On upper and lower β-continuous multifunctions*, (Preparing).

[22] POPA, V.: *Weakly continuous multifunctions*, Boll. Un. Mat. Ital. A (5) **15** (1978), 379–388.

[23] POPA, V.: *On characterizations of feebly continuous functions*, Mathematica (Cluj) **27(52)** (1985), 167–170.

[24] POPA, V.: *Sur certaines formes faibles de continuité pour les multifonctions*, Rev. Roumaine Math. Pures Appl. **30** (1985), 539–546.

[25] POPA, V.: *Some properties of H-almost continuous multifunctions*, Problemy Mat. **10** (1988), 9–26.

[26] REILLY, I. L.—VAMANAMURTHY, M. K.: *Connectedness and strong semi-continuity*, Časopis Pěst. Mat. **109** (1984), 261–265.

[27] WINE, J. D.: *Locally paracompact spaces*, Glasnik Mat. **10(30)** (1975), 351–357.

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