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INTERSECTION GRAPHS OF TREES AND TREE ALGEBRAS

BOHDAN ZELINKA

The concept of an intersection graph of an algebra was introduced by J. Bosák [1], at first for semigroups. Intersection graphs of various types of algebras were studied by various authors. The author of this paper has studied intersection graphs of graphs [3] which were defined analogously to the intersection graphs of algebras.

Here we shall study intersection graphs of trees and of tree algebras. The special character of a tree allows us to define intersection graphs of trees in somewhat different way than the intersection graphs of graphs were defined.

The intersection graph of a tree T is the undirected graph whose vertices are all proper subtrees of T and in which two vertices are joined by an edge if and only if the corresponding subtrees have a non-empty intersection. We consider also subtrees consisting of only one vertex.

Three algebras were introduced by L. Nebeský [2]. A tree algebra (M, P) is an algebra with the set M of elements and with a ternary operation P satisfying the following axioms:

I. $P(u, u, v)=u$;
II. $P(u, v, w)=P(v, u, w)=P(u, w, v)$:
III. $P(P(u, v,w), v, x)=P(u, v, P(w, v, x))$;
IV. if $P(u, v, x)\neq P(v, w, x)\neq P((u, w, x)$, then $P(u, v, x)=P(u, w, x)$.

L. Nebeský has proved that there exists a one-to-one correspondence between tree algebras and trees; to a tree algebra (M, P) a tree T corresponds whose vertex set is M and $x=P(u, v, w)$ if and only if the vertex x of T is the common vertex of the path connecting u and v, the path connecting u and w and the path connecting v and w.

The intersection graph of a tree algebra (M, P) is the undirected graph whose vertices are all proper tree subalgebras of the algebra (M, P) and in which two vertices are joined by an edge if and only if the corresponding subalgebras have a non-empty intersection.

At first we shall study intersection graphs of trees.

Lemma 1. *Let T be a finite tree, let $G(T)$ be the intersection graph of T. The vertices of $G(T)$ corresponding to the subtrees of T which consist only of one vertex form an independent subset of the vertex set of $G(T)$ of the greatest cardinality.*

Proof. Two distinct one-vertex subtrees of T have always an empty intersection, therefore the set of vertices corresponding to them is an independent set. If n is the number of vertices of T, then this set has the cardinality n. Let some independent set in $G(T)$ contain a vertex corresponding to a subtree of T with at least two vertices. Then there exist at most $n-2$ subtrees of T which have empty intersections with this subtree and with each other and the considered set has at most $n-1$ vertices.

Theorem [illegible]. *Let the intersection graph $G(T)$ of a finite tree be given. Then the tree T is uniquely determined up to isomorphism.*

Proof. We shall describe a reconstruction of T from $G(T)$. Find the independent set V_0 in $G(T)$ with the greatest cardinality; according to Lemma 1 it is unique and consists of the vertices corresponding to one-vertex subrees. A subtree of T has non-empty intersections with m one-vertex subtrees if and only if it has m vertices. Thus if v is a vertex of $G(T)$ not belonging to V_0, then v corresponds to a subtree of T with m vertices if and only if it is adjacent in $G(T)$ to m vertices of V_0. Two vertices of T are adjacent if and only if they belong to the same subtree of T with two vertices. Thus two vertices v, w of V_0 correspond to one-vertex subtrees of T whose vertices are adjacent in T if and only if there exists a vertex u of $G(T)$ not belonging to V_0 such that u is adjacent to both v and w and to no other vertex of V_0. This enables us to reconstruct T.

Theorem [illegible]. *Let $G(T)$ be the intersection graph of a tree T with n vertices. Then $G(T)$ can be obtained from the intersection graph $G(T')$ of some tree T' with $n-1$ vertices by the following procedure P:*

(1) *Choose a vertex u of the independent set of $G(t')$ whose cardinality is the greatest. By H denote the subgraph of $G(T')$ induced by the set consisting of u and of all vertices adjacent to u.*

(2) *To $G(T')$ add a graph H' isomorphic to H and vertex-disjoint with $G(T')$ and two distinct vertices a, b belonging neither to $G(T')$ nor to H'.*

(3) *Let φ be an isomorphic mapping φ of H' onto H. Join every vertex x of H' by edges with all vertices of H and with all vertices of $G(T')$ which are joined with $\varphi(x)$ in $G(T')$.*

(4) *Join a by edges with all vertices of H'.*

(5) *Join b by edges with all vertices of $G(T') \cup H'$.*

Proof. Any tree T with n vertices can be obtained from a tree T' with $n-1$ vertices by adding a new vertex a_0 to T' and joining it with some vertex of T'. Choosing a vertex u according to (1) means choosing a one-vertex subtree of T'

consisting of the vertex u_0 with which the new vertex a_0 is joined. The graph H' from (2) is the subgraph of $G(T)$ consisting of all subtrees of T which contain the new vertex a_0 and the vertex u_0. Any of these subtrees is obtained by adding a_0 to some subtree of T' containing u_0 and joining it with u_0. Therefore H' is isomorphic to the subgraph H of $G(T)$ consisting of all subtrees containing u_0. The vertex a corresponds to the subtree of T consisting only of a_0. The vertex b corresponds to the subtree T' of T (it is not in $G(T)$, because it is not a proper subtree of T'). From this the steps (4) and (5) follow. Any vertex of H' is joined according to (3) with all vertices of H, because all subtrees corresponding to vertices of H and of H' contain u_0. Thus they have a non-empty intersection. To each subtree of T containing u_0 and a_0 the isomorphic mapping φ assigns the tree obtained from it by deleting a_0 and the edge A_0u_0. Thus $\varphi(x)$ has a non-empty intersection with some subtree of T' if and only if so has x.

Thus by repeating the procedure ***P*** we may obtain intersection graphs of all finite trees with at least three vertices, starting from the intersection graph of the unique tree with three vertices. This tree T_0 and its intersection graph $G(T_0)$ are in Fig. 1.

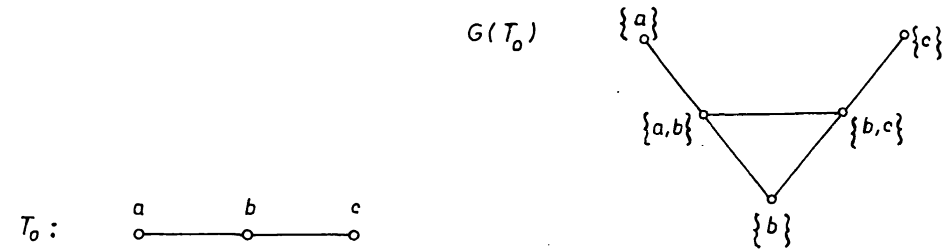

Fig. 1

Now we shall consider another type of intersection graphs of trees which will be denoted by $G'(T)$. The vertices of $G'(T)$ are all proper subtrees of T which contain at least one edge, two vertices are joined by an edge if and only if the corresponding subtrees have a non-empty intersection, that is, at least one vertex in common.

Before proving a theorem analogous to Theorem 1 we shall introduce some notions.

In the following by the word subtree we shall always mean a subtree having at least one edge.

Let T be a finite tree. We define the sequence $T(0)$, $T(1)$, $T(2)$, ... as follows:
(a) $T(0)=T$;
(b) if $k \geqq 1$ and the tree $T(k-1)$ has at least two edges, then $T(k)$ is obtained from $T(k-1)$ by deleting all terminal vertices and all edges incident to them; if $T(k-1)$ has at most one edge, then $T(k)$ is an empty graph. As is well-known, if δ

is the diameter of T, then th reate k for h ( ) i n] $(\delta - 1)$[, this means the least integer which gr ter than or e ua to $(\delta$ 1 . L K be the set of all k for which $T(k)$ is non-empty.

Now let $E(k)$ be the set of edges of T which belong to $T(k)$, but not to $T(k+1)$. Evidently $E(k_1) \cap E(k_2) = \emptyset$ for $k_1 \neq k_2$. Further let $F(k)$ $\bigcup_{i\ 0}^{k} E(i)$ for each $k \in K$. The subgraph of T induced by the edge set $E(k)$ (or $F(k)$) l be d noted by $R(k)$ (or $S(k)$ respectively) for each $k \in K$. The subgraph of a r ph induced by some set of edges is the subgraph of this graph consi ting of e ges of thi t and their end vertices. We see that $E(0) = F(0)$ $R(0) - S(0)$ nd $S(0)$ f t all of whose connected components are stars.

If T' is some subtree of T not containing two centres of T, then by $d(T')$ we hall denote the vertex of T' whose dist nce from t e ne rer c r f m nim l; this vertex is determined uniquely.

Also $R(k)$ for each $k \in K$ is a forest, all of who e c n d c m o ents are stars. The graph $S(k)$ for $k \in K$, $k \geqq 1$ is a forest suc that ea h f it connected components either does not contain edg s of $E($), or th btree of this component induced by the edges belonging to $E(k)$ is s . I] $(\delta$ 1)[, then $S(k) = T$.

Now let us have a connected component C o $S(k)$ f some $k \in K$ which contains an edge of $E(k)$; all of these d s are inc d n w h $d(C$. Each end vertex of an edge of $E(k)$ different from $d(C)$ oinc de i h $d(C)$ her C i some connected component of $S(k-1)$ cont ined in C (oth r i e thi ed e would be in $S(0)$). The vertex $d(C)$ may coincide th such v rt x $d(C')$ ut not necessarily.

Lemma 2. *Let T_0 be a subtree of a finite tree T. Then T_0 i a ubtree of some connected component of $S(0)$ if and only if any two subtree of T which have non-empty intersections with T have al o a n n-empty inters ction with each* other.

Proof. Let T_0 be a subtree of some connected compon nt of $S(0)$. Then T_0 is a star consisting of terminal edges of T. Let u be the vertex of T_0 which is not a terminal vertex of T. Then each subtree of T which h s a non empty intersection with T_0 must contain u and thus any two uch subtr es have a non-empty intersection with each other. Now let T_0 contain n edge e which i not a terminal edge of T. Let u_1 u_2 be its end vertices; as e s non terminal, there exists an edge $e_1 \neq e$ incident with u_1 and an edge $e_2 \neq e$ inci ent with u_2. Th dges e_1, e_2 cannot have a common end vertex; otherwise the ed es e, e_1, e_2 wou d form a triangle. Let E_1 (or E_2) be the subtree of T formed by e_1 (or e re p ctively) and its end vertices. Then E_1, E_2 are vertex-disjoint and they both h ve n n- mpty in ersections with T_0.

Lemma 3. *Let T_0 be a subtree of a finite tree t, let $k \in K$. Then T_0 is a subtree of some connected component of $S(k)$ if and only if any two subtrees of T which have non-empty intersection with T_0 and are not subtrees of $S(k-1)$ have also a non-empty intersection with each other.*

Proof. Let T_0 be a subtree of $S(k)$. Then it is a subtree of some connected component C of $S(k)$. The vertex $d(C)$ separates any other vertex of C from all vertices of T not belonging to C. Thus each subtree of T which is not a subtree of $S(k)$ and has a non-empty intersection with T_0 contains $d(C)$ and any two such subtrees have a non-empty intersection with each other. Now let T_0 contain an edge e not belonging to $S(k)$, let u_1, u_2 be the end vertices of e. Then e is not a terminal edge of $T(k)$ and there exist two subtrees T_1, T_2 of $T(k)$ such that T_1 contains u_1 and not u_2, T_2 contains u_2 and not u_1; they both have non-empty intersections with T_0, but the intersection of T_1 and T_2 is empty. The trees T_1, T_2, being subtrees of $T(k)$, are not subtrees of $S(k-1)$.

Lemma 4. *Let T be a finite tree, let $k \in K$. Let C be a connected component of $S(k)$, let C' be a connected component of $S(k-1)$. Let C' be a subtree of C. Then $d(C') = d(C)$ if and only if each subtree of T which is not a subtree of $S(k)$ and has a non-empty intersection with some subtree of C has also a non-empty intersection with some subtree of C'.*

Proof follows from the fact that each subtree of T which is not a subtree of $S(k)$ and has a non-empty intersection with some subtree of C contains $d(C)$ and there exists at least one such subtree which does not contain any vertex of C except for $d(C)$ (for example the subtree formed by an edge incident with $d(C)$ but not belonging to C and its end vertices).

We shall introduce an auxiliary symbol $G''(H)$. If H is a proper subtree of T, then $G''(H)$ is the subgraph of $G'(T)$ induced by the set consisting of all vertices of $G'(H)$ (this is a subgraph of $G'(T)$) and of the vertex of $G'(T)$ corresponding to H. If H is a subgraph of T which is not a proper subtree of T, then $G''(H) = G'(H)$.

Theorem 3. *Let the intersection graph $G'(T)$ of a finite tree T be given. Then the tree T is uniquely determined up to isomorphism.*

Proof. According to Lemma 2 we can find the subgraph of $G'(T)$ which is $G''(S(0))$. Then recurrently according to Lemma 3 we may find $G\mu(S(k))$ for each $k \in K$. The graph $G''(S(0))$ consists of connected components which are cliques. If $G''(S(0)) = G'(T)$, then $S(0) = T$ and T is a star; it has m edges if and only if $G'(T)$ has $2^m - 2$ vertices, because any proper non-empty subset of the edge set of a star induces a proper subtree of this star and vice versa. If $G''(S(0)) \neq G'(T)$, then each connected component of $G''(S(0))$ is $G''(C)$ for some connected component C of $S(0)$. This component C is a star and has m edges if and only if $G''(C)$ has $2^m - 1$ vertices (in $G''(C)$ we have also the vertex corresponding to the whole C). Thus we can reconstruct $S(0)$ and for each connected component of $S(0)$ we can find $d(C)$; this is the centre of C. Now suppose we have reconstructed $S(k-1)$ and $G''(S(k))$ for some $k \in K$ and assume that in each connected component C of $S(k-1)$ we have found $d(C)$. Take a connected component of $G''(S(k))$; this is $G''(C)$ for some connected component C of $S(k)$. Consider all connected components of $G''(S(k-1))$ which are subgraphs of this $G''(C)$; any of them is $G''(C')$ for some connected component C' of $S(k-1)$. Let $\mathscr{C}(C)$ be the set of all such C'. According to Lemma 4 we can recognize for which connected component $C'' \in \mathscr{C}(C)$ of $S(k-1)$ we have $d(C') = d(C)$ or whether such a component does not exist. Now we can reconstruct C. If there exists C'' mentioned above, then we put $d(C'') = d(C)$ and join it by edges of $E(k)$ with each $d(C')$ for all $C' \in \mathscr{C}(C) - \{C''\}$. If C'' does not exist, then we take a vertex $d(C)$ not belonging to any graph from $\mathscr{C}(C)$ and join it with all $d(C')$ for $C' \in \mathscr{C}(C)$. Thus we can reconstruct $S(k)$. We proceed so until we come to $k =]\frac{1}{2}(\delta - 1)[$; then $S(k) = T$.

Now let us study intersection graphs of tree algebras. At first we shall prove a theorem which will be convenient for our considerations. We say that a vertex z lies between the vertices x and y in a tree T if and only if z belongs to the path connecting x and y in T.

Theorem 4. *Let T be a finite tree, let β be a ternary relation on the vertex set of T such that $(x, y, z) \in \beta$ if and only if one of the vertices x, y, z lies between the other two. If the vertex set of T and the relation on it is given, then the tree T is determined uniquely up to isomorphism.*

Proof. For any two vertices x, y of the vertex set $V(T)$ of T let $B(x, y)$ be the set of all $z \in V(T)$ such that $(x, y, z) \in \beta$; evidently $x \in B(x, y)$, $y \in B(x, y)$, $y \in B(x, y)$. The set $B(x, y)$ consists of all vertices of the path connecting x and y and of all vertices z such that x lies between y and z or y lies between x and z. Let $\mathscr{B}$ be the family of the sets $B(x, y)$ for all pairs x, y. Let $x_0, y_0)$ is minimal with respect to the set inclusion in $\mathscr{B}$. Let $T(x_0, y_0)$ be the subgraph of T induced by the set $B(x_0, y_0)$; it is evidently a subtree of T. Suppose that $T(x_0, y_0)$ is not a path. Then $T(x_0, y_0)$ contains a vertex u of the degree at least three in $T(x_0, y_0)$. Let v_1, v_2, v_3 be three vertices of $T(x_0, y_0)$ adjacent to u. If u is an inner vertex of the path connecting x_0 and y_0, then at least one of the vertices v_1, v_2, v_3 does not belong to this path, without loss of generality let such a vertex be v_1. If x_0 lies between u and y_0, then at least two of the vertices v_1, v_3, v_3 are separated from y_0 by x_0; let these vertices be v_1, v_2. Analogously, if y_0 lies between u and x_0. Consider the set $B(v_1, y_0)$. Let $z \in B(v_1, y_0)$. If z lies between v_1 and y_0, then either z lies between x_0 and y_0, or x_0 lies between z and y_0, thus $z \in B(x_0, y_0)$. If v_1 lies between z and y_0, then also x_0 lies between z and y_0 and $z \in B(x_0, y_0)$. If y_0 lies between z and v_1, then

y_0 lies also between z and x_0 and again $z \in B(x_0, y_0)$. We have $B(v_1, y_0) \subseteq B(x_0, y_0)$. Now if u is an inner vertex of the path connecting x_0 and y_0, then according to the above considerations $x_0 \notin B(v_1, y_0)$. If x_0 lies between u and y_0, then $v_2 \notin B(v_1, y_0)$. If y_0 lies between x_0 and u, then we consider $B(v_1, x_0)$ instead of $B(v_1, y_0)$ and prove analogously $B(v_1, x_0) \subseteq B(x_0, y_0)$ and $v_2 \notin B(v_1, x_0)$. In all of these cases we have found a proper subset of $B(x_0, y_0)$ which is in $\mathscr{B}$ and this is a contradiction with the minimality of $B(x_0, y_0)$. This means that $T(x_0, y_0)$ must be a path. Let x_1, y_1 be the terminal vertices of this path. Any vertex from $B(x_0, y_0)$ lies between x_1 and y_1, thus $B(x_0, y_0) \subseteq B(x_1, y_1)$. Suppose that there exists some $z \in B(x_1, y_1) - B(x_0, y_0)$. The vertex z cannot lie between x_1 and y_1, otherwise it would belong to $B(x_0, y_0)$. If x_1 lies between y_1 and z, then so do all vertices of the path connecting x_1 and y_1, which is $T(x_0, y_0)$, in particular also x_0 and y_0. As both x_0 and y_0 lie between y_1 and z, either x_0 lies between y_0 and z, or y_0 lies between x_0 and z; in both these cases $z \in B(x_0, y_0)$. Analogously, if y_1 lies between x_1 and z. We have $B(x_1, y_1) \subseteq B(x_0, Y_0)$ and thus $B(x_1, y_1) = B(x_0, y_0)$. Now suppose that x_1 is not a terminal vertex of T. Then there exists a vertex z_1 such that $z_1 \neq x_1$ and x_1 lies between y_1 and z_1. This means $z_1 \in B(x_1, y_1) = B(x_0, y_0)$, but z_1 does not belong to the path connecting x_1 and y_1, which is a contradiction. Thus x_1 is a terminal vertex of T. Analogously we prove that y_1 is a terminal vertex of T. Thus each minimal set in $\mathscr{B}$ is the set of vertices of some path connecting two terminal vertices of T; it is easy to prove also vice versa. Thus let B be a minimal set in $\mathscr{B}$. This means that B is the vertex set of some path P_B in T connecting two terminal vertices of T. Let x_1, y_1 be the terminal vertices of P_B, let X_B (or Y_B) be the set of vertices z in B such that no vertex of T of degree at least three lies between x_1 (or y_1 respectively) and z. We have $x_1 \in x_B$, $y_1 \in Y_B$, therefore $X_B \neq \emptyset$, $Y_B \neq \emptyset$. Further let $Z = B - (X_B \cup Y_B)$. If T is a path, then B is the vertex set of T and we have $X_B = Y_B = B$, $Z = \emptyset$. Let x_2 (or y_2) be the vertex from Z whose distance from x_1 (or y_1 respectively) is minimal. The degree of the vertex x_2 (or y_2) is evidently at least three; let x_3 (or y_3 respectively) be a vertex adjacent to x_2 (or y_2 respectively) which does not belong to B. Now let $x \in B$, $y \in B$. If $x \in X_B$, $y \in Y_B$, then we have $x_3 \in B(x, y)$, $y_3 \in B(x, y)$, thus $B(x, y) \not\subseteq B$. Analogously if $x \in Y_B$, $y \in Y_B$. If $x \in X_B$, $y \in Z$, then $y_3 \in B(x, y)$. If $x \in Y_B$, $y \in Z$, then $x_3 \in B(x, y)$. If $x \in Z$, $y \in Z$, then $x_3 \in B(x, y)$, $y_3 \in B(x, y)$. But if $x \in X_B$, $y \in Y_B$ or $x \in Y_B$, $y \in X_B$, then $B(x, y) = B$. Thus if T is not a path, we find all minimal sets B of $\mathscr{B}$ and in each of them we determine X_B and Y_B. These sets B correspond uniquely to paths in T whose terminal vertices are terminal vertices of T; they are their vertex sets. The sets X_B, Y_B for all such sets B correspond to terminal vertices of T by such a way that each of these sets is a set of all vertices of T with the property that no vertex of degree greater than two lies between such a vertex and the terminal vertex of T corresponding to this set. Thus we determine the number of terminal vertices of T and for any two of them we determine their distance; this is the number of vertices of the set B such that X_B

and Y_B corr po d to th se tı e m Acc rdıng to A. A Zykov [4] the tree T is un q ly d t rm n d

Theorem 5. *Let the inter ction r ph $G(M, P)$ of a finıte tr e algebra (M, P) be given. Then the tree algebra (M, P) ı determined uniquely up to isomorphism.*

Proof. Analogou ly as ın the p oof of Th orem 1 we can determıne the set of vertices of $G(M, P)$ which orre pond to ne lem nt subalgebras of (M, P) and for any other vertex of $G(M, P)$ e can de rmin which elements are contained in the subalgebra correspo dın to thi ve t (e. wıth which one element subalgebras it has a non-empty ınterse tion) Ea h one- 1 ment or two element subset of (M, P) is a subalgebra f (M, P) A th e lement sub t $\{x, y, z\}$ of (M, P) is a subalgebra of (M, P), a d n f $P(x,$ z ı equal to some of the elements x, y, z. This occurs if and onl $(x$ $z) \in \beta$ hu we can recon truct the relation β. According to Th or m we then r n tru t the tree T o which the tree algebra (M, P) cor p n . the c r p n nce b t e n trees and tree algebras is on -to-one, r o t th tr al eb a (M, P).

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ГРА Ы ПЕ ЕСЕЧЕ И ДЕ Е И АЛ ЕБ ДЕ ЬЕВ

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В ста ье изуч ютс д а гр р в гр ф $G'(T)$ Граф пересеч ний $G(T)$ р T о вл дд рев д рева T и в котором д в рш е о гд к г а со тв тст ующие поддере и ею А тс G' T) о ко с тем различи м что его щ н ше˘ мере одно ре ро Далее уча ф $G($ $(M$) алг бр ı дерев ев ввел Л Небески Д з с оим графом $G(T)$ ли $G'(T)$ и что ко а) м граф $G(M$ $P)$