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GENERALIZED CONTINUITY AND SEPARATE CONTINUITY

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The relation between the separate continuity of a function of two variables and its continuity depends on the type of continuity. It is well known that the ordinary separate continuity does not imply continuity, while the continuity implies the separate continuity. When quasicontinuity is considered, then the converse is true (See [4], [6], [7]). The situation between the separate somewhat continuity and continuity was studied in [7]. This paper in its first part gives counterexamples showing that the separate almost continuity does not imply the almost continuity, as well as the almost continuity does not imply the separate almost continuity. In the second part some results of [6] and [7] are extended to more general theorems and examples are given, showing that the assumptions, which restrict the component spaces spaces in these generalized theorems, are essential.

1. Preliminaries

We shall denote by X a topological space, without writing $(X, \mathscr{G})$ where $\mathscr{G}$ denotes a topology on X. When X and Y are topological spaces, $X \times Y$ will denote the topological space with the usual product topology. For a function $f: X \times Y \to Z$ the symbols f_x, f^y denote its x-section or y-section, respectively, i.e., f_x for any $x \in X$ is the function defined on Y such that $f_x(y) = f(x, y)$. The y-section is defined analogically. $\mathrm{Cl}(A)(\mathrm{int}(A))$ stands for closure (interior) of A, respectively, $f(A)(f^{-1}(A))$ denotes the image (inverse image) of A.

Definition 1. *If X, Y are topological spaces, then a function $f: X \to Y$ is said to be quasicontinuous at $x_0 \in X$ if for any open U containing x_0, and any open V containing $f(x_0)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. The function f is said to be quasicontinuous if it is quasicontinuous at any $x \in X$.*

Definition 2. *A function $f: X \to Y$ is said to be somewhat continuous if for any open $G \subset Y$ such that $f^{-1}(G) \neq 0$, $\mathrm{int}\, f^{-1}(G) \neq \emptyset$ holds.*

Definition 3. *A function $f: X \to Y$ is said to be almost continuous at $x_0 \in X$ if for any V open, $V \subset Y$, containing $f(x_0)$, the set* $\mathrm{Cl}(f^{-1}(V))$ *contains a neighbourhood of x_0. We say that f is almost continuous if it is almost continuous at any $x \in X$.*

As to the almost continuity, various results concerning this type of functions were obtained in [2], [3], and elsewhere. But the notion itself apears as "nearly continuity" in [10], where it was used in connection with the problems concerning the open mapping theorem.

The quasicontinuity was discussed in [4]. Now it is well known that it is equivalent with the semicontinuity in the sense of Levine [5]. The equivalence was proved in [9].

The somewhat continuity as a generalization of the quasicontinuity was introduced in [1].

The following results may be easily obtained using the mentioned equivalence, or directly from the definition. (See also [8].)

Lemma 1. *A function $f: X \to Y$ is quasicontinuous at x_0 if and only if for any open set U containing x_0 and any open V containing $f(x_0)$,* $\mathrm{int}\, f^{-1}(V) \cap U \neq \emptyset$.

Lemma 2. *A function f is quasicontinuous on X if and only if it is somewhat continuous with respect to any open $U \subset X$, i.e. if its restriction to any open $U \subset X$ is somewhat continuous.*

2. Almost continuity and separate almost continuity.

Example 1. On the interval $\langle -1,1 \rangle \times \langle -1,1 \rangle$ in R^2 define a real function f as

$$f(x, y) = \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are irrational or } (x, y) = (0,0), \\ 0 & \text{if at least one of } x, y \text{ is rational and } (x, y) \neq (0, 0). \end{cases}$$

Then f is almost continuous at each point (x, y), but the sections f_{x_0}, f^{y_0} are not almost continuous when $(x_0, y_0) = (0, 0)$, because none of them is almost continuous at the point 0.

Theorem 1. *Let X, Y be separable metric spaces without isolated points. Then there exists a real function $f: X \times Y \to R$ such that f is almost continuous at each $(x, y) \in X \times Y$, and a dense set $C \subset X \times Y$ such that for each $(x_0, y_0) \in C$, the sections f_{x_0} and f^{y_0} are not almost continuous.*

Lemma 3. *Let X be a separable metric space without isolated points. There exists a countable dense set $D \subset X$ such that $X - D$ is dense in X.*

Proof. Let $B_1, B_2, \ldots, B_n \ldots$ be a countable basis of nonempty open sets in X. Choose $x_1 \in B_1$, $x_1^* \in B_1$, $x_1 \neq x_1^*$. Suppose that a sequence $x_1, x_1^*, x_2, x_2^*, \ldots, x_n, x_n^*$ is constructed such that $x_i \in B_i$, $x_i^* \in B_i$, $x_i \neq x_i^*$ for $i = 1, 2, \ldots, n$. Take x_{n+1}, x_{n+1}^* such that $x_{n+1} \neq x^*_{n+1}$, $x_{n+1} \in B_{n+1}$. Evidently the sets $\{x_n : n = 1, 2, \ldots\}$, $\{x^*_n : n = 1, 2, \ldots\}$ are dense in X. We may put $D = \{x_n : n = 1, 2, \ldots\}$.

Lemma 4. *Let X be a separable metric space without isolated points. Let D be a dense set in X. Then $D = \bigcup_{n=1}^{\infty} D_n$, where $\{D_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets each of which is dense in X.*

Proof. Consider D as a subspace of the space X. Then D is separable and without isolated points. In view of Lemma 1, there exists a set $E_1 \subset D$, E_1 dense in D and $D - E_1$ dense in D. Suppose that for $n \geqq 1$, $E_1, E_2, \ldots, E_n$ have been constructed such that E_i is dense in D (and hence in X) for $i = 1, 2, \ldots, n$, and such that $D - \bigcup_{i=1}^{n} E_i$ is dense in D. If we consider $D - \bigcup_{i=1}^{n} E_i$ as a subspace of X, then it is a separable subspace without isolated points. Again, according to Lemma 1, there exists $E_{n+1} \subset D - \bigcup_{i=1}^{n} E_i$ dense in X and $D - \bigcup_{i=1}^{n+1} E_i$ dense in X too. Thus a sequence $E_1, E_2, \ldots, E_n, \ldots$ of pairwise disjoint sets, which are dense in X, is constructed. Put $D_n = E_n$, $n = 2, 3, \ldots$, $D_1 = E_1 \cup (D - \bigcup_{n=1}^{\infty} E_n)$. Then D_i are pairwise disjoint and $D = \bigcup_{n=1}^{\infty} D_n$.

Proof of Theorem 1. The space $X \times Y$ is a separable metric space without isolated points. Choose a countable set C, dense in $X \times Y$, such that $D = X \times Y - C$ is dense in $X \times Y$ too. Denote $\{(x_n, y_n)\}_{n=1}^{\infty}$ a sequence, the set of points of which is C. We may suppose $(x_n, y_n) \neq (x_m, y_m)$ if $n \neq m$. Let $D = \bigcup_{n=0}^{\infty} D_n$, where D_n are pairwise disjoint dense in $X \times Y$ (see Lemma 4).

Put $D^*_0 = D_0$ and

$$D^*_n = D_n - ((\{x_n\} \times Y) \cup X \times \{y_n\}))$$

for $n = 1, 2, \ldots$, ($\{a\}$ denotes the one point set the element of which is a). The sets D_n are dense in $X \times Y$. Let $f: X \times Y \to R$ be defined as

$$f(x, y) = \begin{cases} k & \text{if } (x, y) \in D^*_k \text{ or } (x, y) = (x_k, y_k),\ k = 1, 2, \ldots \\ 0 & \text{if } (x, y) \notin ((\bigcup_{k=1}^{\infty} D^*_k) \cup C) \end{cases}$$

The function f is almost continuous. In fact, if f assumes the value 0 at some point (x_0, y_0), then the set of all (x, y) for which $f(x, y) = 0$ is dense in each neighbourhood of (x_0, y_0), because $f(x, y) = 0$ in each point of D_0. If f assumes a value k, then the almost continuity at any point, in which this value is assumed, follows from the density of D^*_k.

Now let n be any positive integer. Choose the point (x_n, y_n). Since $(x_n, y) \notin D^*_n$, we have for $y \neq y_n$, $f(x_n, y) \neq n$. But $f(x_n, y)$ is a positive integer. Since

$f(x_n, y_n)=n$, the section f_{x_n} is not almost continuous at y_n. Similarly we can prove that f^{y_n} is not almost continuous at x_n. The theorem is proved.

If a function is separately almost continuous, it need not be almost continuous.

Example 2. On the interval $\langle -1,1\rangle \times \langle -1,1\rangle$ consider the set $F=\{(x, y): 0\leqq x\leqq 1, \frac{1}{2}x\leqq y\leqq x$.

Define

$$f: \langle -1,1\rangle \times \langle -1,1\rangle \rightarrow R, \quad \text{as}$$

$$f(x, y)=\begin{cases} 0 & \text{if } (x, y)\in F-\{(0, 0)\} \\ 0 & \text{if bot } x, y \text{ are simultaneously rational or irrational and } (x, y)\notin F \\ 1 & \text{if } x \text{ is rational, } y \text{ irrational or conversely and } (x, y)\notin F \end{cases}$$

$$f(0, 0)=1$$

The function f is not almost continuous at $(0, 0)$. The almost continuity of the sections f_{x_0}, f^{y_0} may be easily verified for each $x_0\in X$, $y_0\in Y$, respectively.

3. Quasicontinuity, somewhat continuity and the corresponding separate continuities

Separate quasicontinuity implies quasicontinuity, as it was proved by Kempisty for the case of functions of two real variables. An abstract version was given in [6] and it is as follows.

Theorem A. *Let X be a Baire space, Y second countable and Z metric. Let $f: X\times Y\rightarrow Z$ be se separately quasicontinuous. Then f is quasicontinuous.*

In [7] we proved that separate somewhat continuity does not imply somewhat continuity but the following is true.

Theorem B. *Let X be a Baire space, Y second countable and Z regular. Let $f: X\times Y\rightarrow Z$ have all the x-sections somewhat continuous and all the y-sections quasicontinuous. Then f is somewhat continuous.*

It seems to be interesting to find out if the assumptions on the component spaces may be weakened in the theorems A and B. Before the discussion of this problem we give firstly a slight generalization of the mentioned Theorems. Since their proofs are similar, we shall prove only one of them (Theorem 2).

Theorem 2. *Let X be a Baire space, Y such that each point $y\in Y$ possesses a neighbourhood satisfying the second countability axiom and Z a regular space. Let $f: X\times Y\rightarrow Z$ be such that f^y is quasicontinuous for each $y\in Y$ and the* *x-sections f_x are quasicontinuous with the exception of a set of the first category. Then f is quasicontinuous.*

Theorem 3. *Let X be a Baire space, Y a space satisfying the second countability axiom and Z a regular space. Let $f: X\times Y\to Z$ be such that for each $y\in Y$ the section f^y are quasicontinuous and the x-sections f_x, with the exception of a set of the first category are somewhat continuous. Then f is somewhat continuous.*

Lemma 5. *Let $g: X\to Y$ (X, Y arbitrary topological spaces) be quasicontinuous. Let $G\subset Y$, $V\subset X$ be any open sets in X and Y, respectively, such that $g^{-1}(G)\cap V\neq 0$. Then* $\operatorname{int} g^{-1}(G)\cap V\neq\emptyset$.

Proof. Let $z\in g^{-1}(G)\cap V$. Then, using the quasicontinuity at z we obtain from Lemma 1 the desired result.

Proof of Theorem 2. Suppose that f is not quasicontinuous. Then there exists a point (x_0, y_0), an open set $G\subset Z$ containing $f(x_0, y_0)$, and a neighbourhood $U\times V$ of (x_0, y_0), where U is a neighbourhood of x_0 and V a neighbourhood of y_0 such that (Lemma 1)

$$\operatorname{int}(f^{-1}(G)\cap(U\times V))=\emptyset. \tag{1}$$

Without loss of generality we may suppose that V satisfies the second countability axiom. Denote by $\{V_n\}$ the countable basis (in V) of open sets. Let G_1 be open such that $f(x_0, y_0)\in G_1\subset \operatorname{cl}(G_1)\subset G$. The quasicontinuity of f^{y_0} at x_0 gives

$$W=\operatorname{int}((f^{y_0})^{-1}(G_1))\cap U\neq\emptyset$$

Put $T=\{x: x\in W;\ f_x \text{ is quasicontinuous}\}$. Define

$$A_n=\{x: x\in T;\ V_n\subset \operatorname{int}((f_x)^{-1}(G_1))\}.$$

There is

$$T=\bigcup_{n=1}^{\infty} A_n. \tag{2}$$

The inclusion $\bigcup_{n=1}^{\infty} A_n\subset T$ is obvious. If $x\in T$, then $x\in W$ hence $f^{y_0}(x)\in G_1$. Thus $f_x(y_0)\in G_1$. The last gives $f_x^{-1}(G_1)\cap V\neq 0$. From Lemma 5 we get $\operatorname{int} f_x^{-1}(G_1)\cap V\neq 0$, hence $x\in A_n$ for some n and (2) is proved.

We shall prove that for each n the set A_n is nowhere dense in W. Let $S\subset W$ be any open set. Since $S\times V_n\subset U\times V$ is a nonempty open set, we have, according to (1) that there exists a point $(u, v)\in S\times V_n$ such that $f(u, v)\notin G$. Let G_2 be a neighbourhood of $f(u, v)$ such that $G_2\cap G_1=\emptyset$. From the quasicontinuity of f^v at the point u there exists a nonempty open set $S_1\subset S$ such that $f(x, v)\in G_2$ for any $x\in S_1$. Hence $f(x, v)\notin G_1$. Thus $v\notin f_x^{-1}(G_1)$ for any $x\in S_1$. The last gives $V_n\not\subset \operatorname{int}((f_x)^{-1}(G_1))$. And so $S_1\cap A_n=\emptyset$ and the fact that A is nowhere dense in W is proved. This and (2) imply that T is of the first category, which is a contradiction.

What is interesting is the fact that in Theorem 3 we can not substitute the assumption of the second countability by a "locally" second countability as it was done in Theorem 2.

Example 3. $T=(0, 1)$ will serve as an index set. To each $t \in T$ an isometric image of the metric space $X=(0, 1)$ with the usual metric will be associated. We may suppose $Y_t \cap Y_{t'} = \emptyset$ for $t \neq t'$. If necessary we shall denote by y_t the corresponding image of $y \in (0, 1)$ in the space Y_t. We write simply y instead of y_t. The sets Y_t are supposed to have the order structure taken over from $(0, 1)$.

Put $Y = \bigcup_{t \in T} Y_t$. As to the topology, G is open in Y if $G = \bigcup_{t \in T} G_t$ where G_t are open in Y_t. $R = (-\infty, \infty)$ is considered to have the usual topology. We can see that in Y any point y posesses a neighbourhood satisfying the second countability axiom. For any $t \in T$ the function ${}^{(t)}f\colon X \times Y_t \to R$ is defined as:

$$
{}^{(t)}f(x, y) = \begin{array}{l}
0 \text{ if } x<t,\ y(=y_t), \text{ rational} \\
1 \text{ if } x<t,\ y \text{ irrational} \\
0 \text{ if } x=t,\ 0<y\leqq \tfrac{1}{2} \\
1 \text{ if } x=t,\ \tfrac{1}{2}<y<1 \\
0 \text{ if } x>t,\ y \text{ irrational} \\
1 \text{ if } x>t,\ y \text{ rational}
\end{array}
$$

On the product $X \times Y$ define $f\colon X \times Y \to R$ as:

$$f(x, y) = {}^{(t)}f(x, y) \text{ if } y \in Y_t$$

For any $y \in Y$, f^y is a quasicontinuous function. In fact, if $y \in Y$, then $y(=y_t) \in Y_t$ for exactly one $t \in T$. Hence $f^y(x) = {}^{(t)}f^y(x)$. Thus if y is rational

$$
f^y(x) = \begin{array}{l}
0 \text{ if } x<t \\
1 \text{ if } x>t \\
0 \text{ if } x=t,\ 0<y\leqq \tfrac{1}{2} \\
1 \text{ if } x=t,\ \tfrac{1}{2}\leqq y<1
\end{array}
$$

The quasicontinuity of f^y may be easily veriffied. The situation for y irrational is similar.

For any $x \in X$ the function f_x is somewhat continuous. Take any $G \subset R$ open and such that $f^{-1}(G) \neq \emptyset$. Then G contains at least one of the numbers 0, 1. Since

$$f_x^{-1}(G) = \bigcup_{t \in T} {}^{(t)}f_x^{-1}(G)$$

and for $t=x$ the set ${}^{(t)}f_x^{-1}(G)$ contains at least one of the open intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, we have $\operatorname{int} f_x^{-1}(G) \neq \emptyset$. Thus f_x is somewhat continuous.

The function f is not somewhat continuous as a function of two variables. In fact, if $G = (\frac{1}{2}, \frac{3}{2})$, then $f^{-1}(G) \neq \emptyset$, but $\operatorname{int} f^{-1}(G) = \emptyset$.

The following example shows that the assumption of X being a Baire space, in Theorem 3, is also essential.

Example 4. In what follows let X be the set of all rational numbers in $(0, 1)$ with the usual topology. Let $Y=(1, \infty)$ be the set of all real numbers greater than 1, again with the usual topology. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of all the rational numbers of $X (r_n \neq r_m$ for $n \neq m)$.

Put

$$
{}^{(n)}f(x, y) = \begin{cases} 0 & \text{if } x<r_n,\ n<y\leqq n+1,\ y \text{ rational} \\ 1 & \text{if } x<r_n,\ n<y\leqq n+1,\ y \text{ irrational} \\ 0 & \text{if } x=r_n,\ n<y\leqq n+\frac{1}{2} \\ 1 & \text{if } x=r_n,\ n+\frac{1}{2}<y\leqq n+1 \\ 0 & \text{if } x>r_n,\ n<y\leqq n+1,\ y \text{ irrational} \\ 1 & \text{if } x>r_n,\ n<y\leqq n+1,\ y \text{ rational} \end{cases}
$$

Define $f: X\times Y\to R$:

$$f(x,y)={}^{(n)}f(x, y), \text{ if } n<y\leqq n+1.$$

The quasicontinuity of the sections f^y and the somewhat continuity of f_x may be veriffied similarly as in the preceding example. Similarly as in example 3 we can check that f is not somewhat continuous.

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ОБОБЩЕННАЯ НЕПРЕРЫВНОСТЬ И НЕПРЕРЫВНОСТЬ СЕЧЕНИЙ

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Резюме

В работе исследуется связь между непрерывностью функции двух переменных на произведении топологических пространств и непрерывностью ей сечений в случае непрерывности в некотором обобщенном смысле. С этой точки зрения исследуются квазинепрерывность, почти непрерывность и другие (смотри определения в работе). Даются обобщения некоторых результатов, касающихся именно квазинепрерывности, и даны новые результаты для некоторых других типов непрерывностей.