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THE DISTRIBUTIVITY PROPERTY OF VALUATION RINGS

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In order to make this paper self — contained we repeat some basic facts about valuation rings (see [1]).

Let K be a field, Γ an additive abelian totally ordered group. By a valuation (of the field K) we mean a mapping v: $K' = K - \{0\} \twoheadrightarrow \Gamma$ such that $v(xy) = v(x) + v(y)$ and $v(x+y) \geq \min\{v(x), v(y)\}$. The set of all x such that $v(x) \geq 0$ or $x = 0$ forms a ring V. This ring is said to be the valuation ring of v.

We shall say that $A \subset K$ is a valuation ring if A is the valuation ring for some valuation v of K.

This valuation v is uniquely determined by the ring A up to equivalence. This means that if A is a valuation ring for two valuations v_1, v_2 with the valuation groups Γ_1, Γ_2 respectively, then there exists such an isomorphism Φ: $\Gamma_1 \rightarrow \Gamma_2$ that $\Phi \circ v_1 = v_2$.

The set of all subrings of a field K forms a lattice if the lattice operations are $\cap$ and $\vee$. Hereby $A \vee B$ denotes the ring generated by the $A \cup B$.

It is easy to show that every overring of a valuation ring is again a valuation ring. Thus together with A, B the join $A \vee B$ is also a valuation ring.

If v_1, v_2 are valuations corresponding to the rings A_1, A_2 respectively, then we denote by $v_1 \vee v_2$ the valuation which corresponds to the ring $A \vee B$.

Let v_i, v_j be the valuations of the field K and A_i, A_j their valuation rings. If there is no relation of inclusion between the rings A_i, A_j, we shall say that the valuations v_i, v_j are incomparable.

Let P be the greatest common prime ideal of the rings A_i, A_j. Let $\Delta_{ij} \subset \Gamma_i$ be the set $\Gamma_i - \{\pm v_i(x) | x \in P\}$. Then it is easy show that Δ_{ij} is such a subgroup of Γ_i that a factorgroup Γ_i/Δ_{ij} is a naturally ordered group. Moreover, we can identify the group of values of valuation $v_1 \vee v_2$ with this factorgroup. Let Θ_{ij} be a canonical homomorphism $\Gamma_i \rightarrow \Gamma_i/\Delta_{ij}$. Then we shall say that a pair $(\alpha_i, \alpha_j) \in \Gamma_i \times \Gamma_j$ is compatible if $\Theta_{ij}(\alpha_i) = \Theta_{j'i}(\alpha_j)$.

Finally we can formulate the Ribenboim approximation theorem. (Theorem 1, chapter E, [1]) Let $v_1, v_2, \ldots, v_s$ be pairwise incomparable valuations of the field K, and let $(\alpha_1, \ldots, \alpha_s) \in \Gamma_1 \times \ldots \times \Gamma_s$. Then there exists an element $x \in K$ such that $v_i(x) = \alpha_i$ $(i = 1, \ldots, s)$ if and only if every pair (α_i, α_j) $(i, j = 1, 2, \ldots, s\ i \neq j)$ is compactible.

In this paper we show that every triple of valuation rings has the distributivity property. More precisely, there holds:

Theorem. *Let K be a field and A_1, A_2, A_3 be valuation rings of the field K. Then we have*

$$A_1 \cap (A_2 \vee A_3) = (A_1 \cap A_2) \vee (A_1 \cap A_3). \tag{1}$$

Moreover, we shall show that this need not be true if at least one of the rings A_1, A_2, A_3 is not a valuation ring. Before proving the Theorem, we prove the following Lemma.

Lemma. *If A is a valuation ring and B a ring with a unit, then $A \vee B = \{a \cdot b / a \in A,\ b \in B\}$.*

Proof. Since $A \vee B = \left\{\sum_{i=1}^{n} a_i b_i \,|\, a_i \in A,\ b_i \in B,\ n \in N\right\}$, ($N$ is the set of natural numbers), it is sufficient to show that every element of the form $a_1 b_1 + \ldots + a_n b_n$ can be written as ab, where $a \in A$, $b \in B$.

Let v be a valuation of the field K for which A is its valuation ring. Then let $b_j \in \{b_1, \ldots, b_n\}$ be an element such that $v(b_j) = \min_{1 \leq i \leq n} \{v(b_i)\}$. Then we have

$$v(a_i b_i b_j^{-1}) = v(a_i) + v(b_i) - v(b_j) \geqq 0$$

Hence $a_1 b_1 b_j^{-1} + \ldots + a_n b_n b_j^{-1} \in A$ and the element $a_1 b_1 + \ldots + a_n b_n$ has the required form $b_j(a_1 b_1 b_j^{-1} + \ldots + a_n b_n b_j^{-1})$. The lemma is proved.

Proof of the Theorem. First at all we exclude some trivial cases. If $A_2 \supset A_3$ or $A_3 \supset A_2$, the distributivity equality holds, since both sides of (1) are equal to $A_2 \cap A_1$ or $A_3 \cap A_1$, respectively. If any of the rings A_2, A_3 is an overring of A_1, then we have also the distributive identity, since both sides of (1) are equal to A_1. From now we assume that the rings A_1, A_2, A_3 are not in the above inclusions.

Since $(A_1 \cap A_2) \vee (A_1 \cap A_3) \subset (A_2 \vee A_3) \cap A_1$ holds in every lattice, it is sufficient to prove the converse inclusion.

Thus we want to show that every element $a_2 a_3 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$ is contained in $(A_1 \cap A_2) \vee (A_1 \cap A_3)$.

From $a_2 a_3 \in A_1$ we have either $a_2 \in A_1$ or $a_3 \in A_1$. Let e.g., $a_2 \in A_1$. Now if we find an element $d \in K - \{0\}$ such that $a_2 d^{-1} \in A_2 \cap A_1$, and $a_3 d \in A_3 \cap A_1$ then $a_2 a_3 = (a_2 d^{-1})(d a_3) \in (A_2 \cap A_1) \vee (A_3 \cap A_1)$ and the Theorem will be proved.

(Further we shall assume that $a_3 \notin A_1$ (since otherwise it is sufficient to put $d = 1$) and $a_2 \neq 0$ (since if $a_2 = 0$ and $a_3 \notin A_1$, we can put $d = a_3^{-1}$).)

Let v_1, v_2, v_3 be valuations on the field K which corresponds to the valuation rings A_1, A_2, A_3, respectively. We wish to find an element d for which

1. $v_1(d) \leqq v_1(a_2)$
2. $-v_1(d) \leqq v_1(a_3)$
3. $v_2(d) \leqq v_2(a_2)$
4. $-v_3(d) \leqq v_3(a_3)$

We can even find such an element d for which there holds:

1′. $v_1(d) = v_1(a_2)$
2′. $v_2(d) = v_2(a_2)$
3′. $v_3(d) \geqq 0.$

These conditions are stronger, for 1′, 2′, 3′ imply 1, 2, 3, 4. Indeed $1' \Rightarrow 1$, $2' \Rightarrow 3$, $3' \Rightarrow 4$, since $v_3(a_3) \geqq 0 \geqq -v_3(d)$. We further get 2 from $v_1(d) = v_1(a_2) \geqq -v_1(a_3)$.

By the above hypothesis we have as the only possible inclusion among the rings A_1, A_2, A_3 the inclusion $A_2 \subset A_1$. We shall show that in this case condition 2′ implies 1′. Indeed, we have the following chain of implications:

$$v_2(d) = v_2(a_2) \Rightarrow v_2(da_2^{-1}) = 0 \Rightarrow v_2(d^{-1}a_2) = 0 \Rightarrow da_2^{-1}$$

has an inverse element in $A_1 \Rightarrow v_1(da_2^{-1}) = 0 \Rightarrow v_1(d) = v_1(a_2)$. Hence in this case it is sufficient to find an element d satisfying conditions 2′, 3′. Since A_2, A_3 are incomparable rings we can apply the Ribenboim Theorem in the same way as in the following case, when the rings A_1, A_2, A_3 are pairwise incomparable.

To apply the Ribenboim Theorem we must, for the sake of 3′ add a non-negative element, namely c from the valuation group of v_3 in such a way that the triple $(v_1(a_2), v_2(a_2), c)$ becomes compatible. If $v_3\ (a_2) \geqq 0$, we can take $v_3(a_2)$, if not, then we can take the zero element, since in this case

$$(v_3 \vee v_1)(a_2) = 0, \quad (v_3 \vee v_2)(a_2) = 0.$$

Now by the Ribenboim Theorem we know that there exists an element d for which the conditions 1′, 2′, 3′ are satisfied. This proves our Theorem.

Remark. Now what about the dual distributive property? In an arbitrary lattice both distributive identities required for all triples are equivalent. Hence we expect that also the dual condition holds for any triple of valuation rings. This is true, but we must be a little careful. In the proof of the dual identity we have to use the fact that every overring of a valuation ring is a valuation ring. We begin with an element $(A_1 \vee A_3) \cap (A_1 \vee A_2)$ and we can use the theorem, since $A_1 \vee A_2$ is also a valuation ring. Hence

$$\begin{aligned}(A_1 \vee A_3) \cap (A_1 \vee A_2) &= [A_1 \cap (A_1 \vee A_2)] \vee [A_3 \cap (A_1 \vee A_2)] \\ &= A_1 \vee [(A_3 \cap A_1) \vee (A_3 \cap A_2)] \qquad (2) \\ &= A_1 \vee (A_3 \cap A_2).\end{aligned}$$

We now show that if we require that only two of the rings are valuation rings, then the identity (1) need not be true. It is interesting that the dual identity (2) is true even if only the two rings, A_2 and A_3 are valuation rings. (A_1 being any subring.)

Example 1. Let x, y be independent variables over a field L. We consider the rational function field $K = L(x, y)$ and define the following subrings A_1, A_2, A_3 of the field K.

$$A_1 = \left\{ \left(\frac{p_0(x)}{q_0(x)} + y\frac{p_1(x)}{q_1(x)} + \ldots + y^n \frac{p_n(x)}{q_n(x)} \right/ \right.$$

$$\left. \left/ \frac{p_{n+1}(x)}{q_{n+1}(x)} + y\frac{p_{n+2}(x)}{q_{n+2}(x)} + \ldots + y^{m-1}\frac{p_{n+m}(x)}{q_{n+m}(x)} \right), \right.$$

where $p_i(x)$, $q_i(x) \neq 0$, are polynomials in the variable x over L, and x does not divide $q_0(x)$, $q_{n+1}(x)$, $p_{n+1}(x)$.$\}$

A_3 is defined by interchanging x and y in A_1.

$A_2 = L\left[\frac{1}{x}\right]$ — the ring of polynomials in the variable $\frac{1}{x}$.

The ring A_1 is a valuation ring for the valuation $v_1\colon K - \{0\} \to Z \times Z$ (Z is the set of integers), where the set $Z \times Z$ is lexicographically ordered, (this means that $(a, b) \leqq (c, d)$ if and only if either $a < c$ or $a = c$ and $b \leqq d$) and v_1 is defined by

$$v_1\left(\frac{y^{m_1}x^{n_1}p_0(x) + y^{m_1+1}p_1(x) + \ldots + y^{m_1+k}p_k(x)}{y^{m_2}x^{n_2}q_0(x) + y^{m_2+1}q_1(x) + \ldots + y^{m_2+s}q_s(x)}\right) = (m_1 - m_2, n_1 - n_2),$$

where $p_i(x)$, $q_i(x)$ are polynomials in the variable x and x does not divide $p_0(x)$, $q_0(x)$.

Thus the rings A_1, A_3 are valuation rings, but A_2 is not a valuation ring, since y, $\frac{1}{y} \notin A_2$ and it is impossible for any nontrivial valuation v to have negative values on both y and $\frac{1}{y}$.

Now we have $\frac{y}{x} \in (A_2 \vee A_3) \cap A_1$ and since $A_2 \cap A_1 = L$, we have

$$\frac{y}{x} \notin A_3 = A_3 \vee L = A_3 \vee (A_2 \cap A_1) \supset (A_3 \cap A_1) \vee (A_2 \cap A_1)$$

Hence the distributivity does not hold.

Example 2. We change the role of A_i in the preceding Example by putting $B_1 = A_2$, $B_2 = A_1$, $B_3 = A_3$. Then again the identity $(B_1 \cap B_2) \vee (B_1 \cap B_3) = B_1 \cap (B_2 \vee B_3)$ does not hold since $\frac{1}{x} \in B_1 \cap (B_2 \vee B_3)$, indeed $\frac{1}{x} = \frac{y}{x^2} \cdot \frac{x}{y}$, but

$$\frac{1}{x} \notin (B_1 \cap B_2) \vee (B_1 \cap B_3) = L.$$

Example 3. The rings in Example 1 do not satisfy the dual distributivity law. Indeed in the notation of Example 1 we have

$$\frac{1}{x} \in (A_1 \vee A_2) \cap (A_1 \vee A_3) \quad \text{but} \quad \frac{1}{x} \notin A_1 = A_1 \vee L = A_1 \vee (A_3 \cap A_2).$$

A final remark. Let A_1, A_2, A_3 be subrings of the field K such that A_2, A_3 are valuation rings. Then

$$(A_1 \vee A_2) \cap (A_1 \vee A_3) = A_1 \vee (A_2 \cap A_3). \tag{3}$$

To show this we need the following Lemma P.

Lemma P. *Every overring B of the intersection $A = A_3 \cap A_2$ is an intersection of two valuation overrings B_1, B_2 of the rings A_3, A_2, respectively.*

To see this we shall use the following well — known facts about valuation rings have not been quoted above.

First of all we recall one of the many possible equivalent definitions of a Prüfer ring.

The subring R of the field K is a Prüfer ring if and only if any ring S between R and K is integrally closed. ([2], Theorem (11.10) (ii))

Next we have:

Every integrally closed ring is an intersection of valuation rings. ([2], Corollary (10.9))

Every finite intersection of valuation rings is a Prüfer ring. ([2] Theorem (11.12))

Every valuation ring B which contains a finite intersection $\bigcap_{i=1} A_i$ of valuation rings contains some of the valuation rings A_i. ([1], Chapter E, Corollary 2c)

Now we are ready to prove the Lemma P. Indeed if we use the above facts we have gradually:

$A_3 \cap A_2 = A$ is a Prüfer ring. B is integrally closed. B is an intersection $\bigcap_{i \in I} C_i$ of valuation rings. Every C_i is an overring of A_3 or A_2.

$$B = \Big(\bigcap_{C_k \supset A_3} C_k\Big) \cap \Big(\bigcap_{C_j \supset A_2} C_j\Big).$$

Hence, indeed, B is equal to the intersection of two valuation overrings of the rings A_3, A_2, respectively.

Now we prove (3). We have

$$(A_1 \vee A_2) \cap (A_1 \vee A_3) \subset [A_2 \vee (A_1 \vee (A_2 \cap A_3))] \cap [A_3 \vee (A_1 \vee (A_2 \cap A_3))].$$

But according to (P) there exist rings A_4, A_5 such that $A_4 \supset A_2$, $A_5 \supset A_3$ and $A_4 \cap A_5 = A_1 \vee (A_2 \cap A_3)$. From this and from the distributive law for every triple of valuation rings we have

$$\begin{aligned} &[A_2 \vee (A_1 \vee (A_2 \cap A_3))] \cap [A_3 \vee (A_1 \vee (A_2 \cap A_3))] \\ &= [A_2 \vee (A_4 \cap A_5)] \cap [A_3 \vee (A_4 \cap A_5)] \\ &= (A_2 \vee A_4) \cap (A_2 \vee A_5) \cap (A_3 \vee A_4) \cap (A_3 \vee A_5) \\ &= A_4 \cap (A_3 \vee A_4) \cap A_5 \cap (A_2 \vee A_5) \\ &= A_4 \cap A_5, \end{aligned}$$

and so $(A_1 \vee A_2) \cap (A_1 \vee A_3) \subset A_1 \vee (A_2 \cap A_3)$.

Since the converse is true in every lattice, the relation (3) is proved.

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ЗАМЕТКА О СВОЙСТВЕ ДИСТРИБУТИВНОСТИ В КОЛЬЦАХ НОРМИРОВАНИЯ

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Резюме

В статье показывается, что все семейства, состоящие из трех колец нормирования, удовлетворяют обом дистрибутивным тождествам, но это не всегда верно, если только два элемента из этого семейства являются кольцами нормирования.