Mathematica Slovaca

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Mathematica Slovaca, Vol. 55 (2005), No. 2, 169--181

Persistent URL: http://dml.cz/dmlcz/130333

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WEAK SOLUTION FOR FRACTIONAL ORDER INTEGRAL EQUATIONS IN REFLEXIVE BANACH SPACES

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(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper, we define the fractional order Pettis-integral operator in reflexive Banach spaces and we investigate the properties of such operator. A fixed point theorem is used to establish an existence result for the nonlinear Pettis-fractional order integral equation of the following type

$$x(t) = g(t) + \lambda I^{\alpha} f(t, x(t)) , \qquad t \in [0,1] , \ \ 0 < \alpha < 1 .$$

Moreover, the existence of a solutions for the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, \mathrm{D}^{\beta} x(t)), \qquad t \in [0,1] , \ \ 0 < \beta < 1 , \ \ x(0) = x_0 ,$$

is proved.

1. Introduction and preliminaries

Let $L^1(I)$ be the space of Lebesgue integrable functions on the interval $I = [0,1]$, let C_0 be the space of all null sequences endowed with the maximum norm. Unless otherwise stated, E is a reflexive Banach space with norm $\|\cdot\|$ and dual E^*. We will denote by E_w the space E endowed with the weak topology $\sigma(E, E^*)$ and denote by $C[I,E]$ the Banach space of strongly continuous functions $x \colon I \to E$ with sup-norm $\|\cdot\|_0$.

We recall that the *fractional integral operator of order* $\alpha > 0$ with left-hand point a is defined by

$$I_a^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \, \mathrm{d}s .$$

2000 Mathematics Subject Classification: Primary 26A33, 34A12, 47G10.
Keywords: fractional calculus, pseudo solution, Cauchy problem.

Using the known relations between the Beta- and Gamma-function, a well-known calculation with the Fubini-Tonelli theorem shows that $I_a^{\alpha+\beta}x = I_a^{\alpha} I_a^{\beta} x$ for each $x \in L^1([a,b])$ and each $\alpha, \beta > 0$. In particular, I_a^n is the nth iterate of the usual integral operator, and so I_a^{α} may indeed be considered as a corresponding fractional integral. When $a = 0$, we can write $I_0^{\alpha} x(t) = x(t) * \eta_{\alpha}(t)$ where $\eta_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, $\eta_{\alpha}(t) = 0$ for $t \le 0$, and $\eta_{\alpha} \to \delta(t)$ as $\alpha \to 0$ where δ is the delta function.

The following lemma is folklore:

LEMMA 1.1. *Let $\alpha, \beta \in \mathbb{R}^+$, and $n = 1, 2, 3, \dots$. Then we have $\lim\limits_{\alpha \to n} I_a^{\alpha} x(t) = I_a^n x(t)$ uniformly, where $I_a^1 x(t) = \int\limits_a^t x(s)\, \mathrm{d}s$. Moreover $I_a^{\alpha}\colon L^1([a,b]) \to L^1([a,b])$ is a continuous operator.*

We recall the following definitions. Let E be a Banach space and let $x\colon I \to E$. Then

1. $x(\cdot)$ is said to be *weakly continuous* (*measurable*) at $t_0 \in I$ if for every $\varphi \in E^*$, $\varphi(x(\cdot))$ is continuous (measurable) at t_0.
2. A function $h\colon E \to E$ is said to be *weakly sequentially continuous* if h takes weakly convergent sequences in E to weakly convergent sequences in E.
3. $x(\cdot)$ is said to be *strongly measurable* if it is the limit (in the norm topology in E) of a sequence of step functions a.e. on I.
4. If the function $\varphi(x(\cdot))$ is differentiable for every $\varphi \in E^*$ and if there is a function $y\colon I \to E$ such that $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi(x(t))) = \varphi(y(t))$ for every $\varphi \in E^*$ and every $t \in I$, then x is weakly differentiable and we write $x'(t) = y(t)$, where $x'(t)$ denotes the weak derivative of the function x.

If E is weakly complete and $\varphi(x(\cdot))$ is differentiable for every $\varphi \in E^*$, then x is weakly differentiable. If x is weakly continuous on I, then x is strongly measurable ([16; p. 73]), hence weakly measurable. Also it can be easily proved, that weak differentiability implies weak continuity. Note that in reflexive Banach spaces weakly measurable functions are Pettis integrable (see [7], [16] and [22] for the definition) if and only if $\varphi(x(\cdot))$ is Lebesgue integrable on I for every $\varphi \in E^*$ ([16; p. 78]).

Consider the problem (modelled off a first order differential equation)

$$x(t) = x_0 + \int\limits_0^t f(s, x(s))\, \mathrm{d}s\,, \qquad t \in I\,, \tag{1}$$

where $f\colon I \times E \to E$, $x_0 \in E$ and the integral understood to be the Pettis integral. The existence of weak solutions of this problem was proved for example by Cichoń [4], Cramer, Lakshmikantham and Mitchell [6], Kinght [17] Kubiaczyk [18], Mitchell and Smith [20] and more recently by O'Regan [21]. Here we deal with a general case of equation (1), namely, we consider the nonlinear fractional order integral equation

$$x(t) = g(t) + \lambda I^{\alpha} f\big(t, x(t)\big), \qquad t \in I, \ \ 0 < \alpha < 1. \tag{2}$$

Definition 1.1. By a *weak solution* to (2) we mean a function $x \in C[I,E]$ such that for all $\varphi \in E^*$

$$\varphi\big(x(t)\big) = \varphi\big(g(t)\big) + \lambda I^{\alpha} \varphi\big(f\big(t, x(t)\big)\big), \qquad t \in [0,1], \ \ 0 < \alpha < 1.$$

Some interesting existence results for (2), in the case $E = \mathbb{R}^n$, may be found in [1], [3], [5], [10], [11], [12], [15] and [26]. In [1], [3], [15], [26], the real-valued function f is assumed to be continuous while in [5], [10] it is assumed that f is a Carathéodory function, and in [11], [12] the case of a Carathéodory function with monotonicity condition respectively a function of bounded variation was studied. In case that E is an ideal space (see [29] for the definition), equations (2) and generalization have provoked some interest in the literature [27]. In comparison with earlier results, we drop the requirement that f is real-valued and we consider the general case of a vector-valued continuous function f.

The plan of this paper is as follows. In Section 2, we prove an existence result for the fractional integral operator in the sense of Pettis, which will be our main tool. After recalling some properties of such operators, we prove an existence result in Section 3. Moreover, we will discuss the existence of pseudo-solution (for the definitions and basic results, see [4] or [17]) for the Cauchy problem

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f\big(t, x(t)\big), \qquad t \in I, \\ x(0) &= x_0. \end{aligned} \tag{3}$$

Also, we discuss the existence of solutions for the Cauchy problem

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f\big(t, \mathrm{D}^{\beta} x(t)\big), \qquad t \in I, \ \ 0 < \beta < 1, \\ x(0) &= x_0, \end{aligned} \tag{4}$$

with x taking values in E. Now, we present some auxiliary results that will be needed in this paper. First, we state a fixed point result (proved in [21]), which was motivated by ideas in [2].

Theorem 1.1. *Let E be a Banach space and let Q be nonempty, bounded, closed and convex subset of $C[I,E]$. Suppose, that $T\colon Q \to Q$ is weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0,1]$. Then the operator T has a fixed point in Q.*

Proposition 1.1. ([30]) *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 1.2. ([20]) *Let Q be a weakly compact subset of $C[I,E]$. Then $Q(t)$ is weakly compact subset of E for each $t \in I$.*

Finally, we state some results which directly follow from the Hahn-Banach theorem.

Proposition 1.3. *Let E be a normed space with $x_0 \neq 0$. Then there exits a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

Proposition 1.4. *If $x_0 \in E$ is such that $\varphi(x_0) = 0$ for every $\varphi \in E^*$, then $x_0 = 0$.*

2. Fractional order integrals in reflexive Banach spaces

Here, we define the fractional order integral operator in reflexive Banach spaces. Definition given below is an extension of such a notion for real-valued functions.

Definition 2.1. Let $x\colon I \to E$ be a weakly measurable function such that $\varphi(x(\cdot)) \in L^1(I)$, and let $\alpha > 0$. Then the *fractional* (arbitrary order) *Pettis-integral* $I^\alpha x(t)$ is defined by

$$I^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, \mathrm{d}s\,. \tag{5}$$

In the above definition the sign $\int$ denotes the Pettis-integral.

Such an integral is well-defined:

Theorem 2.1. *Let $x\colon I \to E$ be a weakly measurable function, $\varphi(x(\cdot)) \in L^1(I)$ and let $\alpha > 0$. The fractional (arbitrary order) Pettis-integral*

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, \mathrm{d}s$$

exists for almost every $t \in I$ as a function from I into E and $\varphi(I^\alpha x(t)) = I^\alpha \varphi(x(t))$.

Proof. Since $x\colon I \to E$ is weakly measurable and $\varphi(x(\cdot)) \in L^1(I)$, x is a Pettis integrable function. From the definition of the integral of fractional order we have

$$I^\alpha \varphi(x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, \mathrm{d}s = \int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) \mathrm{d}s$$

exists for almost every $t \in I$ and is an element from $L^1(I)$, that is, for almost every $t \in I$, $s \in (0,t)$ the measurable function $\varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))$ is Lebesgue integrable, hence the function $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)$ is Pettis integrable on I. From the definition of the Pettis-integral there exists a function denoted by $I^\alpha x(\cdot)$ from I into E that satisfies

$$\varphi(I^\alpha x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, \mathrm{d}s = I^\alpha \varphi(x(t))$$

for all $\varphi \in E^*$ and for a.e. $t \in I$.

By definition,

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, \mathrm{d}s \qquad \text{for} \quad \text{a.e. } t \in I. \tag{6}$$

$\square$

Remarks.

1. It is clear that if x is weakly continuous (hence weakly measurable) and $\varphi(x(\cdot)) \in L^1(I)$, then $I^\alpha x(t)$ exists for every $t \in I$ as a weakly continuous function.
2. We cannot define $I^\alpha x(t)$ for arbitrary bounded weakly measurable function x since from the Phillip example (see [23]) it follows that a bounded weakly measurable function need not be Pettis integrable function.
3. We can define the above integral for the spaces E which contain no isomorphic copy of C_0 (cf. [7; p. 54, Theorem 7]).

Now, for the properties of the integrals of fractional orders in reflexive spaces we have the following Lemma.

Lemma 2.1. *Let $x\colon I \to E$ be weakly measurable and $\varphi(x(\cdot)) \in L^1(I)$. If $\alpha, \beta \in (0,1)$, we have:*

1. $I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t)$ *for a.e.* $t \in I$.
2. $\lim_{\alpha \to 1} I^\alpha x(t) = I^1 x(t)$ *weakly uniformly on I if only these integrals exist on I.*

3. $\lim\limits_{\alpha\to 0} I^\alpha x(t) = x(t)$ *weakly in* E *for a.e.* $t \in I$.

4. *If, for a fixed* $t \in I$, $\varphi(x(t))$ *is bounded for each* $\varphi \in E^*$, *then* $\lim\limits_{t\to 0} I^\alpha x(t) = 0$.

Remarks.

1. $\{x_n(t)\}_{n=1}^{\infty}$ converges weakly uniformly on I to a function $x(t)$ if for almost every $t \in I$, $\varepsilon > 0$ and $\varphi \in E^*$, there exists an integer N so that $n > N$ implies $|\varphi(x_n(t) - x(t))| < \varepsilon$.

2. If we assume that $x\colon I \to E$ is weakly continuous, then all the statements of Lemma 2.1 hold for every $t \in I$.

Proof of Lemma 2.1. For any $\varphi \in E^*$, from the properties of the fractional integral operators in the Banach space $L^1(I)$ (see e.g. [25], [27] and [28]), we have:

1. $\varphi(I^\alpha I^\beta x(t)) = I^\alpha \varphi(I^\beta x(t)) = I^\alpha I^\beta \varphi(x(t)) = I^{\alpha+\beta}\varphi(x(t)) = \varphi(I^{\alpha+\beta}x(t))$ implies $\varphi(I^\alpha I^\beta x(t) - I^{\alpha+\beta}x(t)) = 0$, hence according to Proposition 1.4, we obtain

$$I^\alpha I^\beta x(t) - I^{\alpha+\beta}x(t) = 0 \qquad (\text{for a.e. } t \in I).$$

2. $|\varphi(I^\alpha x(t)) - \varphi(I^1 x(t))| = |\varphi(I^\alpha x(t) - I^1 x(t))| = |I^\alpha \varphi(x(t)) - I^1 \varphi(x(t))|$ since $\varphi(x(\cdot)) \in L^1(I)$. Then we have $\varphi(I^\alpha x(t)) \to \varphi(I^1 x(t))$ uniformly on I, so $I^\alpha x(t) \to I^1 x(t)$ weakly uniformly on I.

3. $\lim\limits_{\alpha\to 0} \varphi(I^\alpha x(t)) = \lim\limits_{\alpha\to 0} I^\alpha \varphi(x(t)) = \lim\limits_{\alpha\to 0} \varphi(x(t)) * \eta_\alpha(t) = \varphi(x(t)) * \delta(t) = \varphi(x(t))$.

4. The proof of this part follows immediately from Proposition 1.4 and the inequality

$$|\varphi(I^\alpha x(t))| = |I^\alpha \varphi(x(t))| \le K \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \text{where} \quad K = \sup_{t\in I} |\varphi(x(t))|.$$

□

Example. Let $x\colon I \to L_\infty$ be defined by $x(t) = \chi_{[0,t]}$. This function is weakly measurable and for each $\varphi \in L_\infty^*$, we have $\varphi x \in L^1$ (each φx is a function of bounded variation) (cf. [8], [9], [14]). Thus, according to Theorem 2.1, $I^\alpha x$ exists. To calculate the fractional order Pettis-integral of x, let $\rho \in L^1$ and let φ be the element in L_∞^* corresponding to ρ. For any $t > 0$, we have

$$\begin{aligned}
&\int_0^t \varphi\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\right) \mathrm{d}s \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi\big(x(s)\big) \mathrm{d}s = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \rho(\theta)\chi_{[0,s]}(\theta) \mathrm{d}\theta \,\mathrm{d}s \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \rho(\theta) \mathrm{d}\theta \,\mathrm{d}s = \int_0^t \int_0^s \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(\theta) \mathrm{d}\theta \,\mathrm{d}s \\
&= \int_0^t \int_\theta^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(\theta) \mathrm{d}s \,\mathrm{d}\theta = \int_0^t \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)} \rho(\theta) \mathrm{d}\theta \\
&= \int_0^1 \frac{(t-\theta)^{\alpha}}{\Gamma(1+\alpha)} \rho(\theta)\chi_{[0,t]}(\theta) \mathrm{d}\theta = \varphi\left(\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \chi_{[0,t]}(s)\right).
\end{aligned}$$

Hence

$$I^{\alpha}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \mathrm{d}s = \frac{(t-\cdot)^{\alpha}}{\Gamma(1+\alpha)} \chi_{[0,t]}(\cdot) \in L^{\infty}.$$

Remark. As $\alpha \to 1$, we get the same result due to R. Geitz (see [14]).

Now, we give the definition of the weak derivative of fractional order.

DEFINITION 2.2. Let $x\colon I \to E$ be a weakly differentiable function and x' is weakly continuous. We define the *weak derivative* of x of order $\beta \in (0,1]$ by

$$\mathrm{D}^{\beta}x(t) := I^{1-\beta}\,\mathrm{D}x(t), \qquad \mathrm{D} \text{ denote the weakly differential operator.}$$

3. Main results

In this section we present our main result by proving the existence of solutions of the equation (2) in $C[I,E]$.

Let E be a reflexive Banach space. For a fixed $g \in C[I,E]$ let

$$E_r = \big\{x \in C[I,E] : \ \|x\|_0 < \|g\|_0 + r\big\} \qquad (r > 0),$$

where $\|\cdot\|_0$ is the sup-norm. We will consider the set

$$B_r = \big\{x(t) \in E : \ x \in E_r, \ t \in I\big\}.$$

Let us state the following assumptions:

1. Assume, that $g \in C[I,E]$.
2. Let $f\colon I \times B_r \to E$ satisfies the following conditions:
   (a) For each $t \in I$, $f_t = f(t,\cdot)$ is weakly sequentially continuous.
   (b) For each $x \in E_r$, $f(\cdot,x(\cdot))$ is weakly measurable on I.
   (c) For any $r > 0$, the weak closure of the range of $f(I \times B_r)$ is weakly compact in E (or equivalently: there exists an M such that $\|f(t,x)\| \le M$ for all $(t,x) \in I \times B_r$).

Now, we are in a position to formulate and prove our main result.

THEOREM 3.1. *Let the conditions* (1) *and* (2) *be satisfied in addition to the following inequality*

$$\frac{M|\lambda|}{\Gamma(1+\alpha)} < r\,.$$

Then the equation (2) *has at least one weak solution* $x \in C[I,E]$.

P r o o f . Let us define the operator T as

$$(Tx)(t) = g(t) + \lambda I^\alpha f\big(t,x(t)\big) \qquad t \in I\,, \quad 0 < \alpha < 1\,. \tag{7}$$

We will solve equation (2) by finding a fixed point of the operator T.

We claim

$$T\colon C[I,E] \to C[I,E]\,.$$

To prove our claim, first note that assumption (b) implies that for each $x \in C[I,E]$, $f(\cdot,x(\cdot))$ is weakly measurable on I. The fact that f has weakly compact range means that $\varphi\big(f(\cdot,x(\cdot))\big)$ is Lebesgue integrable on I for every $\varphi \in E^*$ and thus the operator T is well-defined. Now, we show that if $x \in C[I,E]$, then $Tx \in C[I,E]$. Note, that there exists $r > 0$ with $\|x\|_0 = \sup_{t\in I}\|x(t)\| < \|g\|_0 + r$. Now assumption (c) implies that

$$\big\|f\big(t,x(t)\big)\big\| \le M \qquad \text{for} \quad t \in [0,1]\,.$$

Let $t,s \in [0,1]$ with $t > s$. Without loss of generality, assume $Tx(t) - Tx(s) \ne 0$. Then there exists (a consequence of Proposition 1.3) $\varphi \in E^*$ with $\|\varphi\| = 1$ and

$$\|Tx(t) - Tx(s)\| = \varphi\big(Tx(t) - Tx(s)\big)\,.$$

Thus

$$\begin{aligned}
&\|Tx(t) - Tx(s)\| \\
&\leq \big|\varphi\big(g(t) - g(s)\big)\big| + |\lambda| \bigg| \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi\big(f\big(\theta, x(\theta)\big)\big)\ \mathrm{d}\theta \\
&\qquad\qquad - \int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi\big(f\big(\theta, x(\theta)\big)\big)\ \mathrm{d}\theta \bigg| \\
&\leq \|g(t) - g(s)\| + |\lambda| \bigg| \int_0^s \left(\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)} \right) \varphi\big(f\big(\theta, x(\theta)\big)\big)\ \mathrm{d}\theta \bigg| \\
&\qquad + |\lambda| \bigg| \int_s^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi\big(f\big(\theta, x(\theta)\big)\big)\ \mathrm{d}\theta \bigg| \\
&\leq \|g(t) - g(s)\| + \frac{|\lambda| M}{\Gamma(1+\alpha)} \big\{ |t^\alpha - s^\alpha| + 2(t-s)^\alpha \big\}\,.
\end{aligned}$$

Hence

$$\|Tx(t) - Tx(s)\| \leq \|g(t) - g(s)\| + \frac{|\lambda| M}{\Gamma(1+\alpha)} \big\{ |t^\alpha - s^\alpha| + 2(t-s)^\alpha \big\} \tag{8}$$

and so $Tx \in C[I,E]$.

Now, let

$$\begin{aligned}
Q = \Big\{ x \in E_r :\ &(\forall t \in I)(\forall s \in I)\Big(\|x(t) - x(s)\| \\
&\leq \|g(t) - g(s)\| + \tfrac{|\lambda| M}{\Gamma(1+\alpha)} \big\{ |t^\alpha - s^\alpha| + 2(t-s)^\alpha \big\} \Big) \Big\}\,,
\end{aligned}$$

note that Q is nonempty, closed, bounded, convex and equicontinuous subset of $C[I,E]$. Now, we claim that $T\colon Q \to Q$ and is weakly sequentially continuous. If this is true then according to Proposition 1.1, TQ is bounded in $C[I,E]$ (hence, Proposition 1.2, implies $TQ(t)$ is weakly relatively compact in E for each $t \in I$) and the result follows immediately from Theorem 1.1. It remains to prove our claim. First, we show that T maps Q into Q. To see this, note that the inequality (8) shows that TQ is norm continuous. Now, take $x \in Q$; without loss of generality, we may assume that $I^\alpha f\big(t, x(t)\big) \neq 0$, then, by Proposition 1.3, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\big\|I^\alpha f\big(t, x(t)\big)\big\| = \varphi\big(I^\alpha f\big(t, x(t)\big)\big)$. Thus

$$\begin{aligned}
\|Tx(t)\| &\leq \|g(t)\| + \big\|\lambda I^\alpha f\big(t, x(t)\big)\big\| = \|g(t)\| + \varphi\big(\lambda I^\alpha f\big(t, x(t)\big)\big) \\
&\leq \|g(t)\| + \lambda I^\alpha \varphi\big(f\big(t, x(t)\big)\big) \leq \|g(t)\| + |\lambda| M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ \mathrm{d}s \\
&\leq \|g(t)\| + \frac{|\lambda| M}{\Gamma(1+\alpha)} < \|g(t)\| + r\,,
\end{aligned}$$

therefore

$$\|Tx\|_0 < \|g\|_0 + r\,. \tag{9}$$

Thus $T\colon Q \to Q$. Finally, we will show that T is weakly sequentially continuous. To see this, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in Q and let $x_n(t) \to x(t)$ in E_w for each $t \in [0,1]$. Recall ([20]) that a sequence $\{x_n\}_{n=1}^{\infty}$ is weakly convergent in $C[I,E]$ if and only if it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t,\cdot)$, the Lebesgue dominated convergence theorem (see assumption (c)) for the Pettis-integral ([13; Corollary 4]) implies for each $\varphi \in E^*$ that $\varphi(Tx_n(t)) \to \varphi(Tx(t))$ a.e. on I, $Tx_n(t) \to Tx(t)$ in E_w. So $T\colon Q \to Q$ is weakly sequentially continuous. The proof is now completed. □

Remark. As in the proof of [17; Theorem 2] we can generalize the assumption (c) in Theorem 3.1 to the form

(c*) There exists a null subset N of I such that the weak closure of $f\big((I/N) \times B_r\big)$ is weakly compact in E.

In the remaining part of this paper, we will consider the Cauchy problems (3) and (4) as a special cases.

DEFINITION. A function $x\colon I \to E$ is said to be a *pseudo-solution* of (3) (cf. [4]) if

(a) $x(\cdot)$ is absolutely continuous,
(b) $x(0) = x_0$,
(c) for each $\varphi \in E^*$ there exists a null set $N(\varphi)$ (i.e. N is depending on φ and $\mathrm{mes}\big(N(\varphi)\big) = 0$) such that for each $t \notin N(\varphi)$

$$(\varphi x)'(t) = \varphi\big(f\big(t, x(t)\big)\big) \tag{10}$$

where x' denote the pseudo-derivative (see Pettis [4] or [22]).

In other words by a pseudo-solution of (3) we will understand an absolutely continuous function such that $x(0) = x_0$, and for each $\varphi \in E^*$, $x(\cdot)$ satisfies (10) a.e. on I. A strong (weak) solution of (3) is an absolutely (weakly) continuous function with strong (weak) derivative satisfying (3) a.e. on I. Each strong (or weak) solution is also a pseudo-solution. The converse is not (in general) true (cf. [4]).

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *the Cauchy problem* (3) *has a pseudo-solution. In fact this solution is a strong solution.*

Proof. Putting $\alpha \to 1$, $\lambda = 1$ and $g(t) = x_0$ in equation (2), we get

$$x(t) = x_0 + \int_0^t f\big(s, x(s)\big)\,\mathrm{d}s\,. \tag{11}$$

From the definition of Q it follows that (as $\alpha \to 1$) every solution $x \in Q$ will be absolutely continuous. Since f is Pettis integrable, then the existence of a pseudo-solution of (11) is equivalent to the existence of the solution of (3) (cf. [22; Sec. 8]). Moreover, since E is reflexive, E has the Radon-Nikodym property (cf. [7]) and [17; Corollary 6] implies that each pseudo-solution is a strong solution. □

THEOREM 3.2. *In Theorem* 3.1 *replace assumption* (b) *by the following:*

(b1) *For each* $x \in E_r$, $f(\cdot, x(\cdot))$ *is weakly continuous on* I.

Then Cauchy problem (4) *has at least one solution* $x \in C[I, E]$.

DEFINITION. A function $x\colon I \to E$ is called a *solution* of (4) if

(a) x is weakly differentiable and x' is weakly continuous,
(b) $x(0) = x_0$,
(c) $\frac{\mathrm{d}x}{\mathrm{d}t} = f\big(t, \mathrm{D}^\beta x(t)\big)$, $t \in I$.

Proof. Putting $\alpha = 1 - \beta$, $g(t) \equiv 0$ and $\lambda = 1$ in the equation (2) and considering $y\colon I \to B_r$ be a solution, then $y(\cdot)$ satisfies

$$y(t) = I^{1-\beta} f\big(t, y(t)\big) . \tag{12}$$

Operating by I^β on both sides we obtain

$$I^\beta y(t) = I^1 f\big(t, y(t)\big) , \qquad t \in I ,$$

but, f is weakly continuous in t on I (assumption (b1)) and it is well known that the integral of weakly continuous function is weakly differentiable with respect to the right endpoint of the integration interval and its derivative equals the integrand at that point (cf. [20]), therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} I^\beta y(t) = f\big(t, y(t)\big) .$$

Now, set

$$x(t) = x_0 + I^\beta y(t) = x_0 + I^1 f\big(t, y(t)\big) , \tag{13}$$

then $x(\cdot)$ is weakly differentiable and

$$x(0) = x_0 , \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = f\big(t, y(t)\big) ,$$

since f is weakly continuous in t on I, $I^{1-\beta} \frac{\mathrm{d}x}{\mathrm{d}t}$ exists. Moreover, we have

$$\mathrm{D}^\beta x(t) = I^{1-\beta} \frac{\mathrm{d}x}{\mathrm{d}t} = I^{1-\beta} f\big(t, y(t)\big) = y(t) .$$

Then any solution to equation (12) will be a solution to the Cauchy problem (4), this solution is given by equation (13). This completes the proof. □

Acknowledgments

The authors would like to acknowledge a useful discussion with Professor M. Cichoń.

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Received October 30, 2002
Revised March 8, 2004

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