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Ivan Chajda

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SOME MODIFICATIONS OF THE CONGRUENCE EXTENSION PROPERTY

IVAN CHAJDA

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ABSTRACT. Hamiltonian algebras satisfying the Congruence Extension Property are characterized by the strong Congruence Extension Property. A characterization of classes of algebras satisfying the Unique Congruence Extension Property is given.

The concept of congruence extension property was firstly investigated by A. Day [3]: An algebra A satisfies the *Congruence Extension Property* (briefly CEP) if for every subalgebra B of A and each $\theta \in \operatorname{Con} B$ there exists $\phi \in \operatorname{Con} A$ such that $\phi|_A = \theta$; the congruence ϕ is called an *extension of* θ. A variety $\mathscr{V}$ of algebras satisfies CEP if each $A \in \mathscr{V}$ has this property. Varieties satisfying CEP are in connection with the co called Hamiltonian Varieties, see [5]: an algebra A is *Hamiltonian* if every its subalgebra is a class of some congruence on A. A variety $\mathscr{V}$ is *Hamiltonian* if each $A \in \mathscr{V}$ has this property.

It was proven by E. W. Kiss [4] that every Hamiltonian variety satisfies CEP. The converse implication does not hold, see e.g.:

Example 1. Let $L = \{0, a, 1\}$ be a three element lattice, i.e., $0 < a < 1$. Thus L is distributive and, by [3], L satisfies CEP. On the other hand, $S = \{0, 1\}$ is a sublattice of L but it cannot be a class of any $\theta \in \operatorname{Con} L$ since S is not a convex subset of L. Thus L is not Hamiltonian, and hence also the variety of all distributive lattices (which satisfies CEP) is not Hamiltonian.

Moreover, the theorem of E. W. Kiss does not hold for a single algebra:

Example 2. Let $\mathbf{A} = \{a, b, c, d\}$, and $A = (\mathbf{A}, \cdot, c)$ be an algebra of the type $(2, 0)$, i.e., c is the nullary operation of A and $\cdot$ is given as follows:

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$\cdot$	a	b	c	d
a	b	b	c	b
b	b	a	c	c
c	c	c	c	a
d	a	b	c	d

Evidently, A has exactly three subalgebras, namely $\{c\}$, $S = \{a,b,c\}$ and the whole A. Then $\{c\}$ is the class of identity congruence and A is the class of $A \times A$. Moreover, S is a class of $\theta \in \operatorname{Con} A$ determined by its classes $\{a,b,c\}$, $\{d\}$. Hence A is Hamiltonian. On the contrary, we show that A does not satisfy CEP. The equivalence ϕ on $\{a,b,c\}$ determined by its classes $\{a,b\}$, $\{c\}$ is clearly a congruence on S. Suppose the existence of $\psi \in \operatorname{Con} A$ such that $\psi|_S = \phi$. Since $\langle a,b\rangle \in \phi$, also $\langle a,b\rangle \in \psi$. Further $\langle d,d\rangle \in \psi$, thus also

$$\langle b,c\rangle = \langle a\cdot d,\, b\cdot d\rangle \in \psi .$$

Since $b,c \in S$, we obtain $\langle b,c\rangle \in \psi|_S$, but $\langle b,c\rangle \notin \phi$, which is a contradiction.

For the sake of characterizing Hamiltonian algebras satisfying CEP, let us firstly recall from [2]: an algebra A satisfies the *Strong Congruence Extension Property* (briefly SCEP) if for every subalgebra B of A and each $\theta \in \operatorname{Con} B$ there exists $\phi \in \operatorname{Con} A$ with $[b]_\theta = [b]_\phi$ for each $b \in B$. A variety $\mathscr{V}$ satisfies SCEP if every A of $\mathscr{V}$ has this property. It was proven in [2] that a variety $\mathscr{V}$ satisfies SCEP if and only if $\mathscr{V}$ is Hamiltonian. Since SCEP implies CEP, and the algebra A in Example 2 is Hamiltonian, but it does not satisfy CEP, and thus it also does not satisfy SCEP, the analogous theorem is not valid for a single algebra. However, we can prove:

THEOREM 1. *For an algebra A, the following conditions are equivalent:*

(1) *A satisfies SCEP;*
(2) *A is Hamiltonian and satisfies CEP.*

Proof.

(1) $\Longrightarrow$ (2): Let B be a subalgebra of A. Put $\theta = B \times B \in \operatorname{Con} B$. By SCEP, there exists $\phi \in \operatorname{Con} A$ with $B = [b]_\theta = [b]_\phi$ for each $b \in B$, i.e., the subalgebra B is a congruence class of ϕ, thus A is Hamiltonian. Since A satisfies SCEP, it also satisfies CEP.

(2) $\Longrightarrow$ (1): Let A be Hamiltonian and satisfies CEP. Let B be a subalgebra of A and $\theta \in \operatorname{Con} B$. By CEP, there exists $\phi \in \operatorname{Con} A$ with $\phi|_B = \theta$. Since A is Hamiltonian, B is a block of some $\psi \in \operatorname{Con} A$. Put $\theta^* = \phi \cap \psi$. Then $[b]_{\theta^*} = [b]_\phi \cap [b]_\psi = [b]_\phi \cap B = [b]_\theta$ for each $b \in B$, thus A satisfies SCEP. $\square$

We can introduce another modification of CEP: an algebra A satisfies the *Unique Congruence Extension Property* (briefly UCEP) if for every subalgebra B of A and each $\theta \in \operatorname{Con} B$ there exists a unique $\phi \in \operatorname{Con} A$ such that $\phi|_B = \theta$. A class $\mathscr{C}$ of algebras satisfies UCEP if each $A \in \mathscr{C}$ has this property.

Let A be an algebra. Denote by ω_A the identity relation on A, i.e., $\langle x, y\rangle \in \omega_A$ if and only if $x = y$.

LEMMA. *Let an algebra A has a one-element subalgebra. A satisfies UCEP if and only if A is a one-element algebra.*

P r o o f. Let A satisfies UCEP and E be a one element subalgebra of A. Then

$$\omega_E = \omega_A|_E = A \times A|_E$$

implies $\omega_A = A \times A$, which gives $\operatorname{card} A = 1$. The converse implication is trivial. □

E x a m p l e 3. Let $A = (\{a, b, c\}, \cdot)$, where the binary operation $\cdot$ is given as follows:

$\cdot$	a	b	c
a	b	a	c
b	a	a	a
c	c	c	b

Evidently, A has the only one subalgebra different from A, namely $B = \{a, b\}$. Since B has exactly two elements, $\operatorname{Con} B = \{\omega_B,\ B \times B\}$. Prove that A satisfies UCEP. Clearly $\omega_B = \omega_A|_B$ and $B \times B = A \times A|_B$. It is easy to verify that $\operatorname{Con} A = \{\omega_A,\ A \times A\}$, hence UCEP is evident.

THEOREM 2. *Let $\mathscr{C}$ be a class of algebras closed under subalgebras and homomorphic images. The following conditions are equivalent:*

(1) *$\mathscr{C}$ satisfies UCEP;*
(2) *$\mathscr{C}$ has CEP and for each $A \in \mathscr{C}$ and every subalgebra B of A, the congruence ω_B has the unique extension $\omega_A \in \operatorname{Con} A$.*

P r o o f. (1) $\Longrightarrow$ (2) is trivial. Prove (2) $\Longrightarrow$ (1):

Let $A \in \mathscr{C}$, B be a subalgebra of A and $\theta \in \operatorname{Con} B$. Since $\mathscr{C}$ satisfies CEP, there exists an extension of θ onto the whole A. Suppose $\phi, \psi \in \operatorname{Con} A$ with

$$\phi|_B = \theta = \psi|_B\,.$$

Clearly also $(\phi \cap \psi)|_B = \theta$. Hence, we can suppose $\phi \subseteq \psi$ without loss of generality. Since $\mathscr{C}$ is closed under homomorphic images, also $A/\phi \in \mathscr{C}$. Consider the factor congruence ψ/ϕ on the factor algebra $D = A/\phi$. Clearly $C = B/\phi|_B$ is a subalgebra of A/ϕ and $\psi/\phi|_C = \omega_C$. By (2), it implies $\psi/\phi = \omega_D$, thus $\psi = \phi$ proving UCEP. □

REFERENCES

[1] CHAJDA, I.: *Characterizations of Hamiltonian algebras*, Czechoslovak Math. J. **42** (1992), 487–489.

[2] CHAJDA, I.: *Algebras and varieties satisfying the congruence extension property*, Acta Sci. Math. (Szeged) **56** (1992), 19–21.

[3] DAY, A.: *A note on the congruence extension property*, Algebra Universalis **1** (1971), 234–235.

[4] KISS, E. W.: *Each Hamiltonian variety has the congruence extension property*, Algebra Universalis **12** (1981), 395–398.

[5] KLUKOVITS, L.: *Hamiltonian varieties of universal algebra*, Acta Sci. Math. (Szeged) **37** (1975), 11–15.

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Department of Algebra
and Geometry
UP Olomouc
Tomkova 40
CZ 779 00 Olomouc
CZECH REPUBLIC