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Mathematica Slovaca, Vol. 56 (2006), No. 2, 145--165

Persistent URL: `http://dml.cz/dmlcz/130417`

Math. Slovaca, **56** (2006), No. 2, 145–165

Mathematica Slovaca

Dedicated to Professor Tibor Katriňák

GENERALIZATIONS OF BOOLEAN ALGEBRAS. AN ATTRIBUTE EXPLORATION

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(*Communicated by Sylvia Pulmannová*)

ABSTRACT. This paper gives an overview on generalizations of Boolean algebras. These are lattices enriched by (unary) operations fulfilling some properties of the Boolean negation. The aim is to find the relationship between known extensions of Boolean algebras and the class of weakly dicomplemented lattices, recently introduced for contextual logic purposes, and which generalizes as well Boolean algebras.

1. Introduction and motivation

A *Boolean algebra* is an algebra $(L, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation is a complementation. Boolean algebras were discovered from the investigations of the laws of thought by George Boole [Bo54]. The binary operations $\wedge$ and $\vee$ were used to model the conjunction and the disjunction respectively, while the unary operation models the negation. The constant 1 represents the universe of discourse and 0 encodes "nothing". Boole derived many laws and developed a mathematical theory for the classical logic. Boolean algebras are useful not only in logic and mathematics; they have many applications in computer science and physics (e.g. theory of circuits). They satisfy quite a large number of properties

2000 Mathematics Subject Classification: Primary 03G10; Secondary 03G25, 06B05, 06B10, 06B20.
Keywords: Boolean algebra, concept algebra, Stone algebra, de Morgan algebra, Ockham algebra, Kleene algebra, attribute exploration, triple construction.

The first author was partially supported by GrK 334 of the German Research Foundation (DFG) and Gesellschaft von Freunden und Förderern der TU Dresden.
The second author was supported by GrK 334 of the German Research Foundation (DFG).

and allow easy computations. Which properties really make this class of algebras so nice?

For similar purposes many classes of algebras have been introduced. Of course the class of Boolean algebras is appropriate for the classical propositional logic. A generalization is for example the class of *ortholattices* (lattices with a complementation satisfying the de Morgan laws). They play an important rôle in the logic connected with the foundations of quantum mechanics (quantum logic, [Hi02]). Here the unary operation is interpreted as a negation. The class of *Heyting algebras* is convenient for the intuitionistic logic ([Dz90]). That is the class of algebras $(L, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and for elements a and b in L, the set $\{z \in L : \ z \wedge a \leq b\}$ has a greatest element denoted by $a \rightarrow b$. A unary operation $\neg$ ($\neg x := x \rightarrow 0$) can be derived from the binary operation $\rightarrow$. This encodes the intuitionistic negation. Another example of logic is the 3-valued Łukasiewicz logic. To this logic, the class of *Stone algebras*, a subclass of that of p-algebras, is associated.[1] These algebras are of type $(2, 2, 1, 0, 0)$. *Double p-algebras* are algebras $(L, \wedge, \vee, f, g, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \wedge, \vee, f, 0, 1)$ is a p-algebra and $(L, \wedge, \vee, g, 0, 1)$ a dual p-algebra. In this case a negation is approximated by two unary operations. The class of regular double Stone algebras is convenient for the logic of rough sets ([Du97]).

Going back to the genesis of Boolean algebras, another class has been recently introduced, the class of *concept algebras*. In fact, concepts are considered as basic units of thought. A *concept* is determined by its extent and its intent. The *extent* consists of all items belonging to the concept and the *intent* is the set of all attributes shared by all items of the concept. This notion has been formalized in the eighties by Rudolf Wille and led to the theory of FCA (Formal Concept Analysis. See [Wi82] and [GW99]). The starting point is a *formal context*; this is a triple (G, M, I) of sets such that $I \subseteq G \times M$. The elements of G are called *objects* and that of M *attributes*. *Derivation* operations are defined for $A \subseteq G$ and $B \subseteq M$ by

$$A' := \{m \in M : \ (g, m) \in I \text{ for all } g \in A\}$$

and

$$B' := \{g \in G : \ (g, m) \in I \text{ for all } m \in B\}.$$

A *formal concept* is then a pair (A, B) with $A' = B$ and $B' = A$. The hierarchy of concepts is captured by the relation

$$(A_1, B_1) \leq (A_2, B_2) : \Longleftrightarrow A_1 \subseteq A_2 \quad (\Longleftrightarrow B_1 \supseteq B_2).$$

[1]These classes have been widely investigated by Tibor Katriňák. Their definitions will be given later.

The set $\mathfrak{B}(G,M,I)$ of all formal concepts of the context (G,M,I), ordered by this relation, is a complete lattice, called the *concept lattice* of (G,M,I). Therefore, the conjunction and the disjunction can be respectively encoded by the meet and the join operation of the lattice, while the bottom and the top element will respectively stand for "nothing" and "universe". To develop a Contextual Logic, a negation should also be encoded. An approach, proposed by Rudolf Wille in [Wi00], is to use a *weak negation* $^\triangle$ and a *weak opposition* $^\triangledown$ defined by:

$$(A,B)^\triangle := \big((G\setminus A)'', (G\setminus A)'\big) \qquad \text{and} \qquad (A,B)^\triangledown := \big((M\setminus B)', (M\setminus B)''\big)\,.$$

A concept lattice equipped with these two operations is called a *concept algebra*. They will play a similar rôle in the contextual logic as the power set algebras in the classical logic. An analogue to Boolean algebras for contextual logic should be called *dicomplemented lattices*. These are algebras of type $(2,2,1,1,0,0)$ that generate the quasi-equational theory of concept algebras. Until now no complete axiomatization of the class of dicomplemented lattices is known. Therefore, a super-class is introduced, the class of *weakly dicomplemented lattices*. These are algebras $(L,\wedge,\vee,^\triangle,^\triangledown,0,1)$ such that $(L,\wedge,\vee,0,1)$ is a bounded lattice and the following identities hold for all $x,y\in L$:

$$\begin{array}{llll}
(1) & x^{\triangle\triangle}\le x\,, & (1') & x^{\triangledown\triangledown}\ge x\,,\\
(2) & x\le y \implies x^\triangle \ge y^\triangle\,, & (2') & x\le y \implies x^\triangledown\ge y^\triangledown\,,\\
(3) & (x\wedge y)\vee(x\wedge y^\triangle) = x\,, & (3') & (x\vee y)\wedge(x\vee y^\triangledown)=x\,.
\end{array}$$

These characterize, at least in finite distributive case, concept algebras, [Kw04; Theorem 4.1.7 & Corollary 4.1.8]. The operation $^\triangle$ is called a *weak complementation* and $^\triangledown$ a *dual weak complementation*. They generalize in some sense the complementation of a Boolean algebra. The class of Boolean algebras is exactly the class of (reducts of) weakly dicomplemented lattices for which the unary operations coincide, [Kw04; Corollary 3.3.5]. Therefore, weakly dicomplemented lattices generalize Boolean algebras in a natural way. We are interested in the interrelations with other generalizations.

Note that the main idea in many generalizations is to retain some properties and drop others. Doubtless, the most investigated class of lattices generalizing Boolean algebras is that of distributive lattices. But in this contribution, we focus only on generalizations where a negation comes into play. Here generalizations are divided into two groups. In the first group the negation is captured by a single operation and in the second group it is captured by two operations. For some generalizations of Boolean algebras, the distributivity of the underlying lattice was required. To study these classes together with nondistributive generalizations, we generally drop this assumption. If the initial definition requires distributivity, we write this in brackets. In Figure 1 you can see some properties of unary operations abstracting a negation.

Notation	Definition	Denomination
ane	$x \leq y \Longrightarrow f(x) \geq f(y)$	antitone
mMl	$f(x \wedge y) = f(x) \vee f(y)$	meet de Morgan law
jMl	$f(x \vee y) = f(x) \wedge f(y)$	join de Morgan law
sqex	$x \leq f^2(x)$	square extensive
sqin	$x \geq f^2(x)$	square intensive
inv	$x = f^2(x)$	involution
wcol	$(x \wedge y) \vee \big(x \wedge f(y)\big) = x$	weak complementation's 3rd law
dwcl	$(x \vee y) \wedge \big(x \vee f(y)\big) = x$	dual weak complementation's 3rd law
omol	$x \leq y \Longrightarrow x \vee \big(f(x) \wedge y\big) = y$	orthomodular law

FIGURE 1. Some properties of unary operations abstracting a negation.

In Section 2 more generalizations of Boolean algebras are introduced. Some implications valid between these classes are proved in Section 3. We conclude with some similarities between the class of weakly dicomplemented lattices and that of distributive double p-algebras. The hope is to use the tools developed by Chen, Grätzer, Lakser, Katriňák and others ([CG69], [La71], [Ka72], [Ka73], [KM83], ...) for p-algebras and double p-algebras to characterize weakly dicomplemented lattices.

2. Generalization by unary operations

2.1. Generalization by a single unary operation.

The following section provides definitions of all Boolean algebra generalizations considered here. They are recapitulated in Figure 2. In a Boolean algebra the complementation satisfies the de Morgan laws. If, in a bounded (distributive) lattice, we can define a unary operation that satisfies the de Morgan laws and interchanges 0 and 1, then we obtain what is called an *Ockham algebra*[2]. That

[2]The name Ockham lattices has been introduced by A. Urquhart in [Ur79] with the justification: "the term *Ockham lattice* was chosen because the so-called *de Morgan laws* are due (at least in the case of propositional logic) to William of Ockham" (1290–1349).

is a bounded (distributive) lattice with a unary operation f such that

$$f(x \wedge y) = f(x) \vee f(y)\,, \quad f(x \vee y) = f(x) \wedge f(y)\,, \quad f(0) = 1 \quad \text{and} \quad f(1) = 0\,.$$

The operation f is sometimes called a (de Morgan) "*negation*" although this does not satisfy the law of double negation. The complementation in a Boolean algebra is also a *polarity* (i.e. an antitone involution). If we require that the unary operation of an Ockham algebra should be an involution, we get the so called *de Morgan algebras*[3]. These are algebras $(L, \wedge, \vee, f, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a bounded (distributive) lattice and f a unary operation that satisfies

$$f^2(x) = x\,, \quad f(x \wedge y) = f(x) \vee f(y) \quad \text{and} \quad f(x \vee y) = f(x) \wedge f(y)\,.$$

If in addition f satisfies the inequality $x \wedge f(x) \leq y \vee f(y)$, then $(L, \wedge, \vee, f, 0, 1)$ is called a *Kleene algebra*.

Contrary to the above mentioned generalizations, where less attention is paid to the property of being a complementation, the second approach is more interested in this property. One way of generalizing the notion of complementation is to retain the identity $x \wedge x' = 0$ and drop the dual. The operation obtained is a *semicomplementation*. Of considerable interest are those lattices in which, for any element x, the subset of semicomplements of x has a greatest element (the *pseudocomplement* of x), *pseudocomplemented lattices*. A lattice with a *pseudocomplementation* is called a *p-algebra*. The dual notions are *dual semicomplementation*, *dual pseudocomplementation* and *dual p-algebra*. Of course, if we require that $x \vee x^* = 1$ for every x in a p-algebra L, then x^* becomes a complement of x and, when L is distributive, then L is a *Boolean lattice*[4]. M. H. Stone suggested a restriction of the equation $x \vee x^* = 1$ to those elements x that are pseudocomplements, i.e. that it will be fruitful to consider the equation $x^* \vee x^{**} = 1$ (Stone identity). (Distributive) pseudocomplemented lattices that satisfy this identity are therefore called *Stone lattices*[5]. with an important subclass the regular Stone algebras.

Retaining the properties common to de Morgan algebras and Stone algebras defines the so-called *de Morgan-Stone algebra*, or for short *MS-algebra*. That is an algebra $(L, \wedge, \vee, f, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded (distributive) lattice and the unary operation satisfies the equations

$$f(1) = 0\,, \quad f(x \wedge y) = f(x) \vee f(y) \quad \text{and} \quad x \leq f^2(x)$$

for all x and y in L. Similarly a *dual de Morgan-Stone* algebra is defined.

[3]De Morgan algebras "*arose in the researches on the algebraic treatment of constructive logic with strong negation*" ([BV94]).

[4]The term *Boolean algebra* is used when $x \mapsto x^*$ is considered as a fundamental operation.

[5]When $x \mapsto x^*$ is considered as a fundamental operation, the term *Stone algebra* is used.

The approach omitting distributivity appeared in ortholattices, orthomodular lattices and weakly orthocomplemented lattices. An *orthomodular lattice* is an ortholattice satisfying the condition

$$x \leq y \implies x \vee (f(x) \wedge y) = y \qquad \text{(orthomodular law)}.$$

A *weak orthocomplementation* is a square-extensive antitone semicomplementation. A *weakly orthocomplemented lattice* is a bounded lattice with a weak orthocomplementation. Dually is defined a *dual weak orthocomplementation*. All dual weak complementations are weak orthocomplementation. It is not difficult to see that each weak complementation satisfies the orthomodular law. But, weakly complemented lattices are orthomodular if and only if the weak complementation is also a complementation. In this case we automatically get a Boolean algebra ([Kw04; Lemma 3.3.12 & Corollary 3.3.13]).

There are other generalizations by a single operation worth to be mentioned although this operation is not unary. However, a unary operation can be deduced, as in the case of Heyting algebras. Another class is formed by relatively complemented lattices.

Relatively complemented lattices. Let L be a lattice and $a, b \in L$ with $a < b$. Let $c \in [a, b]$. An element $d \in [a, b]$ is a *relative complement* of c in $[a, b]$ if $c \vee d = b$ and $c \wedge d = a$. A lattice is *relatively complemented* if for every $a, b \in L$ with $a < b$, each element $c \in [a, b]$ has at least one relative complement in $[a, b]$. Of course, if L has 0 and 1 and is relatively complemented, then L is complemented. All complemented modular lattices are relatively complemented. But the converse does not hold.

Heyting algebras. In relation to the intuitionistic logic they play an analogous rôle to that played by Boolean algebras to the classical logic. The binary operation $\rightarrow$ is called *implication*. The deduced unary operation (*intuitionistic negation* or *Heyting negation*, $\neg x := x \rightarrow 0$) is a pseudocomplementation. Dually is defined a *dual Heyting algebra*. Heyting algebras and dual Heyting algebras are completely distributive.

Before we discuss the relations between all these generalizations, we recollect some generalizations where the negation is captured by two operations.

2.2. Generalization by two unary operations.

We have mentioned p-algebras and dual p-algebras as Boolean algebra generalizations. Putting together the two unary operations gives *double p-algebras*. Similarly *double Stone algebras*, weakly dicomplemented lattices (for *double weakly complemented lattices*) and *double Heyting algebras* are defined. For double de Morgan-Stone algebras the two operations are somehow connected.

A *double de Morgan-Stone algebra*, or *DMS-algebra* for short, is an algebra $(L, \wedge, \vee, ^{\perp}, ^{\circ}, 0, 1)$ such that $(L, \wedge, \vee, ^{\circ}, 0, 1)$ is an MS-algebra, $(L, \wedge, \vee, ^{\perp}, 0, 1)$ a dual MS-algebra and the equations $x^{\perp\circ} = x^{\perp\perp}$ and $x^{\circ\perp} = x^{\circ\circ}$ hold for all x in L. Note that in a distributive double Stone algebra $(L, \wedge, \vee, ^{\circ}, ^{\perp}, 0, 1)$ we have

$$x^{\perp\circ} = x^{\perp\perp} \leq x \leq x^{\circ\circ} = x^{\circ\perp}.$$

We know that Boolean algebras are weakly dicomplemented lattices such that the unary operations coincide. The same holds for double p-algebras.

PROPOSITION 2.1. *Boolean algebras are double p-algebras such that the unary operations coincide.*

Proof. Let L be a double p-algebra. If the pseudocomplementation and the dual pseudocomplementation coincide, then the skeleton and dual skeleton are equal to L. But Orrin Frink[6] proved in [Fr62] that skeletons of p-algebras are Boolean algebras. Thus L is a Boolean algebra. □

In contrast to the case of weakly dicomplemented lattices and double p-algebras, requiring the two unary operations to coincide is no longer enough to get Boolean algebras from DMS-algebras. The class obtained is that of de Morgan algebras.

Remark 2.1. A *bi-uniquely complemented lattice* is a bounded lattice in which every element $x \notin \{0,1\}$ has exactly two complements. The lattice $L12$ in Figure 3 is bi-uniquely complemented. The two element Boolean algebra is the unique distributive bi-uniquely complemented lattice. Therefore, we cannot consider bi-uniquely complemented lattices as a generalization of Boolean algebras (with two unary operations).

3. Attribute exploration

3.1. On the methods.

Section 2 shows that there are many ways to generalize Boolean algebras. To apprehend the connection between those negations we use an FCA tool called attribute exploration. This is very useful for finding interrelations between subsets of a given set of attributes. We used the software ConImp developed by Peter Burmeister in Darmstadt. A description can be found in [Bu03]. The attribute exploration generates a finite formal context from a given finite set of attributes. The attributes considered here are generalizations of Boolean algebras. The formal context we are looking for will have as objects algebras

[6]The result was first proved by Valère Glivenko for distributive p-algebras in 1929.

$(L, \wedge, \vee, f, 0, 1)$ of type $(2, 2, 1, 0, 0)$, where $(L, \wedge, \vee, 0, 1)$ is a bounded lattice. The unary operation, denoted by f, is defined on a bounded lattice L and interchanges 0 and 1. Potential properties are in Figure 1. The relation I refers to "is a" in a natural way. We end up with the context in Figure 4. The corresponding concept lattice is depicted in Figure 5. Before we present the results let us see how the attribute exploration works.

Given a finite set of attributes M and a (not well-known) class G of objects susceptible to have some of these attributes. We want to find all valid implications between elements of M. By a valid implication between these attributes we mean a pair (B_1, B_2) of subsets of M such that $B_2 \subseteq B_1''$. (That is, every object that has all attributes of B_1 also has all attributes of B_2.) We denote it by $B_1 \to B_2$. The program systematically generates conjectural implications. We have to decide whether these are true or not. If an implication does not hold, we should provide a counterexample and add it as a new object to the context we are looking for. In this case the program updates his knowledge and conjectures other implications. If we accept an implication, we should provide a proof. (In the next subsection, there are some proofs of valid implications between generalizations of Boolean algebras.) Since M is a finite set, only a finite number of implications can be conjectured. The program does it systematically, so that only non redundant implications (with respect to the Armstrong rules[7], [GW99; Proposition 21]) are conjectured. Therefore, the program terminates. The result is a basis of valid implications between the given attributes. All the valid implications can be deduced by closing this basis under Armstrong rules. In addition the program provides a finite and representative set of counterexamples that refute the false implications. Thus the attribute exploration generates a finite context $(\tilde{G}, M, I)$ where $\tilde{G} \subseteq G$ is the set of counterexamples. The concept lattices $\mathfrak{B}(G, M, I)$ and $\mathfrak{B}(\tilde{G}, M, I)$ are isomorphic. Now, how can we read off the implications from the lattices? Note that concept lattices can be labelled such that each objet (resp. attribute) appears only once, written underneath (resp. above) a single node. Concepts are nodes of the lattices. For each node the corresponding concept has as intent (resp. extent) the attributes (resp. objects) contained in the order filter (resp. ideal) generated by this node. The implications can be read off from the diagram as follows: $B_1 \to B_2$ is a valid implication if and only if the meet of all nodes labelled by elements of B_1 lies below all nodes labelled by elements of B_2. For example the implication

$$\{\text{dsc, MSa}\} \to \{\text{sco, com, wol, Ock}\}$$

[7]For X, Y, Z, W subsets of attributes we have:
a) $X \to X$,
b) $X \to Y \implies X \cup Z \to Y$,
c) $(X \to Y \quad \& \quad Y \cup Z \to W) \implies X \cup Z \to W$.

can be read off from Figure 5, meaning that every dual semicomplemented de Morgan-Stone algebra is already complemented (and therefore semicomplemented), is an Ockham algebra and is a weak ortholattice.

Notation	Definition	Denomination
sco	$x \wedge f(x) = 0$	semicomplementation
dsc	$x \vee f(x) = 1$	dual semicomplementation
com	sco & dsc	complementation
uco	$x \wedge y = 0$ & $x \vee y = 1$ $\iff y = f(x)$	unique complementation
wco	sqin & ane & wcol	weak complementation
dwc	sqex & ane & dwcl	dual weak complementation
Ock	mMl & jMl	Ockham algebra
Mor	Ock & inv	de Morgan algebra
Kle	Mor & $x \wedge f(x) \leq y \vee f(y)$	Kleene algebra
pa	$x \wedge y = 0 \iff y \leq f(x)$	p-algebra
dpa	$x \vee y = 1 \iff y \geq f(x)$	dual p-algebra
Sa	pa & $f(x) \vee f^2(x) = 1$	Stone algebra
dSa	dpa & $f(x) \wedge f^2(x) = 0$	dual Stone algebra
MSa	mMl & sqex	de Morgan-Stone algebra
dMSa	jMl & sqin	dual de Morgan-Stone algebra
ola	ane & com & inv	ortholattice
oml	ola & omol	orthomodular lattice
wol	sqex & ane & sco	weak ortholattice
dwol	sqin & ane & dsc	dual weak ortholattice
Ba	all properties above	Boolean algebra

FIGURE 2. Generalizations of a Boolean algebra complementation by means of a single unary operation.

The subsequent section derives all implications valid in the described context.

3.2. Implications.

The following results present some implications valid between these attributes. The proofs are not so difficult. Nevertheless we provide some of the proofs. Recall that we consider only bounded lattices with a unary operation that interchanges 0 and 1.

PROPOSITION 3.1. *Any unary operation satisfying one of the de Morgan laws is antitone.*

Proof. Let f be a unary operation satisfying the join de Morgan law. Let $x \leq y$. We have $f(y) = f(x \vee y) = f(x) \wedge f(y)$. Thus $f(x) \geq f(y)$. The proof for meet de Morgan law is obtained similarly. □

PROPOSITION 3.2. *Any antitone square-extensive operation satisfies the join de Morgan law. Dually, any antitone square intensive operation satisfies the meet de Morgan law.*

Proof. Let x and y be two elements. Obviously, $f(x \vee y) \leq f(x) \wedge f(y)$ holds. Assuming that $a \leq f(x)$ and $a \leq f(y)$, we get $f(a) \geq f^2(x) \vee f^2(y) \geq x \vee y$. Thus, $a \leq f^2(a) \leq f(x \vee y)$, and $f(x \vee y)$ is the meet of $f(x)$ and $f(y)$. □

Therefore, the de Morgan algebras are exactly bounded lattices with polarities. It also follows that the classes of MS-algebras and of dual MS-algebras are contained in the class of Ockham algebras

PROPOSITION 3.3. *A pseudocomplementation is antitone and square-extensive. Dually, a dual pseudocomplementation is antitone and square-intensive.*

Proof. Let $x \leq y$ and f a pseudocomplementation. From $y \wedge f(y) = 0$ we get $x \wedge f(y) = 0$. Thus, $f(y) \leq f(x)$ and f is antitone. Now $x \wedge f(x) = 0$ implies $x \leq f^2(x)$, and f is square-extensive. □

PROPOSITION 3.4. *Each square-intensive semicomplementation satisfying one of the de Morgan laws is a complementation. Dually each square-extensive dual semicomplementation satisfying one of the de Morgan laws is a complementation.*

Proof. Let f be a semicomplementation satisfying the meet de Morgan law. We have

$$1 = f(0) = f\big(x \wedge f(x)\big) = f(x) \vee f^2(x) \leq f(x) \vee x \implies f(x) \vee x = 1\,.$$

Thus f is a complementation. Now we assume that f is a semicomplementation satisfying the join de Morgan law. We get

$$f\big(f(x) \vee x\big) = f^2(x) \wedge f(x) = 0 \implies 1 = f^2\big(f(x) \vee x\big) \leq f(x) \vee x\,.$$

Thus $f(x) \vee x = 1$ and f is a complementation. The remaining claims follow dually. $\square$

Thus each dually semicomplemented MS-algebra is complemented and each semicomplemented dual MS-algebra complemented.

PROPOSITION 3.5. *Each dual weak complementation satisfying the meet de Morgan law is a pseudocomplementation. Dually, each weak complementation satisfying the join de Morgan law is a dual pseudocomplementation.*

Proof. We shall prove that $x \vee y = 1 \iff x \geq f(y)$, where f is a weak complementation satisfying the join de Morgan law. From the weak complementation's 3rd law (see Figure 1) we have $x \vee f(x) = 1$ for all x. Now we assume that $x \vee y = 1$. We get $f(x) \wedge f(y) = 0$ and

$$f(y) = \big(f(y) \wedge x\big) \vee \big(f(y) \wedge f(x)\big) = f(y) \wedge x\,.$$

Thus $x \geq f(y)$. The rest is proved similarly. $\square$

Another result due to O. Frink (see for example [CG00; Theorem 9.4]) characterizes pseudocomplementation as a square-extensive unary operation f with $f^2(0) = 0$ and $f(x \wedge y) \wedge f\big(x \wedge f(y)\big) = f(x)$. Moreover, by O. Frink's result we obtain:

PROPOSITION 3.6. *Each pseudocomplemented de Morgan algebra is a Boolean algebra. Dually, each dual pseudocomplemented de Morgan algebra is a Boolean algebra.*

Proof. We first prove that the unary operation is a dual weak complementation. It is square-extensive and antitone since it is a pseudocomplementation. Moreover

$$\begin{aligned} f(x) = f(x \wedge y) \wedge f\big(x \wedge f(y)\big) &= \big(f(x) \vee f(y)\big) \wedge \big(f(x) \vee f^2(y)\big) \\ &= \big(f(x) \vee f(y)\big) \wedge \big(f(x) \vee y\big)\,. \end{aligned}$$

We need to prove that $f(x)$ can be replaced by x. This is always possible since f is an involution. Thus

$$x = \big(x \vee f(y)\big) \wedge (x \vee y)$$

and f is a dual weak complementation. Thus f is a Boolean algebra complementation ([Kw04; Lemma 3.3.12 & Corollary 3.3.13]). $\square$

The notions of complementation, unique complementation, Ockham algebra, de Morgan algebra, Kleene algebra, ortholattice and orthomodular lattice ([Be84; Theorem 3.1]) are self dual.

PROPOSITION 3.7. *Each pseudocomplemented dual MS-algebra is a weakly complemented lattice. Dually, each dual pseudocomplemented MS-algebra is dual weakly complemented.*

Proof. From the definition of a dual MS-algebra (see Figure 2) its unary operation f satisfies the join de Morgan law; thus f is antitone. As a pseudocomplementation f is also square-extensive. It remains to check the weak complementation's 3rd law. By the Frink's characterization of pseudocomplementations we have

$$f(x \wedge y) \wedge f\big(x \wedge f(y)\big) = f(x)\,.$$

From the definition of a dual MS-algebra f is square-intensive, and so is an involution. Thus

$$x = f^2(x) = f^2\big[(x \wedge y) \vee \big(x \wedge f(y)\big)\big] = (x \wedge y) \vee \big(x \wedge f(y)\big)\,.$$

The second part follows dually. □

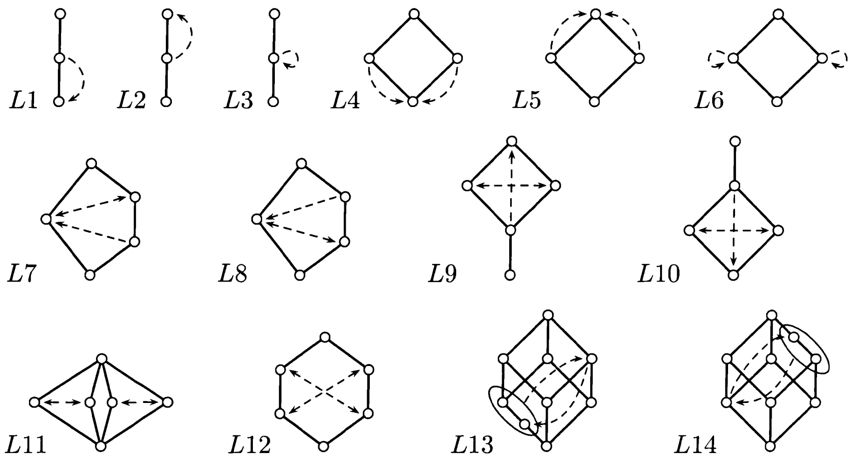

FIGURE 3. Lattice diagrams of algebras used as counter example in the attribute exploration. The dashed lines indicate the image by the unary operation. For the two last cases $L13$ and $L14$ the images of other elements are their (unique) complements.

Remark 3.1. Each antitone unique complementation defines a Boolean algebra ([Sa88; p. 48]). By a result of R. P. Dilworth [Di45] there is a uniquely complemented lattice that is not a Boolean algebra. This lattice is infinite. Its complementation is automatically an involution. It cannot be antitone. A representative of this class is the object $L15$ in the context in Figure 4. This is an infinite uniquely complemented non-distributive lattice and cannot be relatively complemented (see G. Kalmbach in [KAL83]). Though we cannot draw it down, we know exactly to which classes[8] this lattice belongs.

	sco	dsc	com	uco	wco	dwc	Ock	Mor	Kle	pa	dpa	Sa	dSa	MSa	dMSa	ola	oml	wol	dwol	Ba
L1	×					×	×			×		×		×				×		
L2		×			×		×				×		×		×				×	
L3							×	×	×					×	×					
L4	×					×												×		
L5		×			×														×	
L6							×	×						×	×					
L7	×	×	×				×			×		×		×				×		
L8	×	×	×				×				×		×		×				×	
L9		×			×						×								×	
L10	×					×				×								×		
L11	×	×	×				×	×	×					×	×	×	×	×	×	
L12	×	×	×				×	×	×					×	×	×		×	×	
L13	×	×	×								×		×						×	
L14	×	×	×							×		×						×		
L15	×	×	×	×																
L16		×			×						×		×						×	
L17	×					×				×		×						×		

FIGURE 4. Context of generalizations of the Boolean algebra complementation by a unary operation. For $L15$, $L16$ and $L17$ see Remarks 3.1 and 3.2 below.

Remark 3.2. One implication is still open: $\{\text{wco}, \text{dSa}\} \to \{\text{Ock}\}$. i.e.:

Is any weakly complemented dual Stone algebra an Ockham algebra?

[8] among the classes considered in this investigation

Note that the meet de Morgan law follows from the properties of weakly complemented lattices. Thus to get this implication we should prove the join de Morgan law. In the case of distributive lattices, the dual Stone identity is equivalent to the join de Morgan law ([Fr62; Theorem 3]). Up to now no counter example is found. We believe that the implication might be true. However, if the conjecture happens to be false, a representative of this class will be the object $L16$ in the second part of the context[9] in Figure 4. The dual will be $L17$. Contrary to $L15$, which really exists, $L16$ and $L17$ are fictitious. If they existed, we would know to which classes they would belong. The context in Figure 4 would be considered in whole. Its concept lattice would be the lattice in Figure 5, read without the dash lines around $L2$ and $L16$ on the one hand, and around $L1$ and $L17$ on the other hand. If they do not exist, the corresponding context is just the first part of the context in Figure 4. Its concept lattice is obtained from Figure 5 by removing the markings $L16$ and $L17$, and their corresponding concepts; i.e. the dash lines collapse the two pairs of concepts into two concepts: the lower ones.

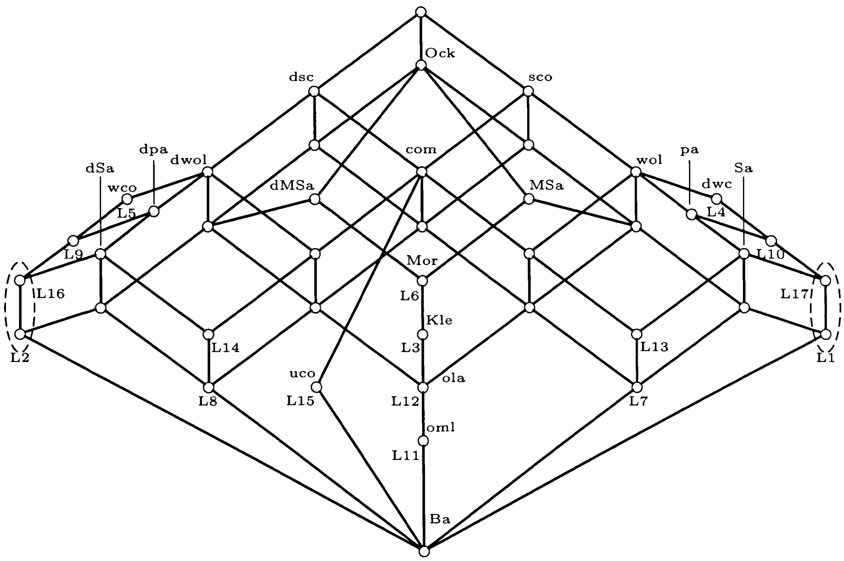

FIGURE 5. Lattice of extensions of the Boolean complementation by a unary operation.

Via the attribute exploration we established the context shown in Figure 4. A list of algebras used as counter examples in this exploration is given in Figure 3.

[9]This figure presents in fact two contexts: one with 15 objects and the second with 2 more objects. Their concept lattices are slightly different. This difference is materialized by the dash lines on Figure 5.

The corresponding concept lattice is shown in Figure 5. This diagram and the diagram in Figure 6 have been drawn using ANACONDA, a preparator software for TOSCANA. TOSCANA is a Management System for Conceptual Information Systems.

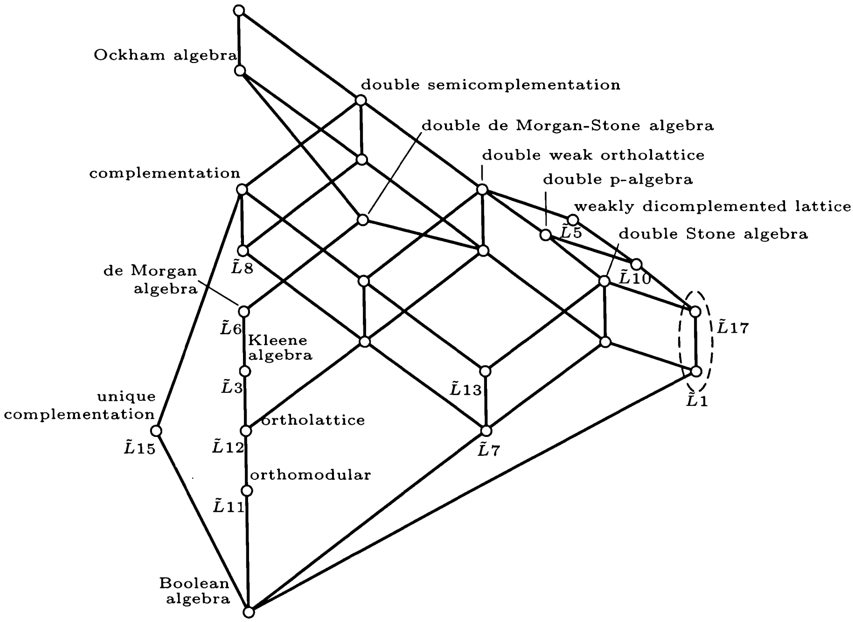

FIGURE 6. Lattice of extensions of the Boolean negation by two unary operations.

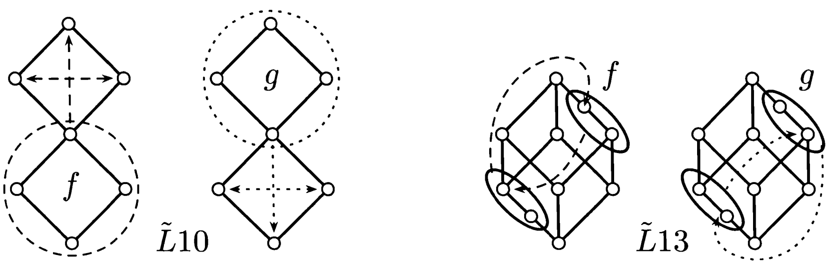

FIGURE 7. Lattice diagrams of algebras used as counter example. The images of other elements are their (unique) complements.

Remark 3.3. In [Da00] the author explored the elementary properties of unary mappings used to define most of the important types of complementations. Neither the weak complementation's 3rd law nor its dual were considered. He considered only finite lattices. In this case one cannot distinguish between Boolean algebras and uniquely complemented lattices.

Remark 3.4. The consideration of algebras with two unary operations extending the Boolean negation leads to the concept lattice shown in Figure 6. The algebras $\tilde{L} := (L, \wedge, \vee, f, g, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ named as representatives of classes can be derived from those in Figure 3 in the following way. For the algebras $\tilde{L}3$, $\tilde{L}6$, $\tilde{L}8$, $\tilde{L}11$ and $\tilde{L}12$ the two unary operations coincide with the ones of $L3$, $L6$, $L8$, $L11$ and $L12$ respectively.

Differently from this, $\tilde{L}7$ is formed by the unary operations of $L7$ and $L8$, $\tilde{L}1$ by that of $L1$ and $L2$, and $\tilde{L}5$ by that of $L4$ and $L5$. The remaining ones are either not known yet ($\tilde{L}15$, $\tilde{L}17$) or shown in Figure 7 ($\tilde{L}10$, $\tilde{L}13$).

4. Triple characterization

Our analysis in Section 3 has shown that double p-algebras and weakly dicomplemented lattices are closely related. Both classes are contained in the class of double weak ortholattices. Furthermore, distributive double p-algebras are weakly dicomplemented lattices as well; p-algebras are characterized by means of triples. The aim is to simulate such characterization to better understand the structure of weakly dicomplemented lattices. Of course this is a work in progress. We present the problems faced as open, and by then, motivate further research.

4.1. Triple characterization for p-algebras.

To every p-algebra L, a Boolean algebra $S(L)$ (its skeleton) and a lattice filter $D(L)$ (dense elements) can be assigned. Nemitz, Chen and Grätzer observed that a third bit of information is needed in order to characterize L. That is the structure map $\phi(L)$ between $S(L)$ and $D(L)$.

For a distributive Stone algebra, the identity

$$x = x^{**} \wedge (x \vee x^*) \qquad \text{with} \quad (x^{**}, x \vee x^*) \in S(L) \times D(L)$$

holds. The structures of $S(L)$ and $D(L)$ together with the relationship between their elements describe L. The relationship here is expressed by the homomorphism $\phi(L)\colon S(L) \to \mathcal{D}\big(D(L)\big)$ defined by

$$\phi(L)(x) := \big\{y \in D(L) :\ y \geq x^*\big\}\,,$$

where $\mathcal{D}\big(D(L)\big)$ is the set of lattice filters of $D(L)$. Chen and Grätzer proved that the triple $\big\langle S(L), D(L), \phi(L)\big\rangle$ characterizes L up to isomorphism. To characterize a much larger class, Katriňák and Mederly [KM83] called a p-algebra *decomposable* if for every $x \in L$ there is $d \in D(L)$ such that $x = x^{**} \wedge d$. For a decomposable p-algebra a binary relation $\phi_L(a)$ on $D(L)$ can be assigned to each element a of $S(L)$ as follows:

$$d\,\phi_L(a)\,e :\Longleftrightarrow\ a^* \wedge d = a^* \wedge e\,.$$

ϕ_L is a $(0,1)$-isotone map from $S(L)$ into the lattice of $\wedge$-compatible equivalence relations on $D(L)$, denoted by $Eq_\wedge(D)$. Note that $\phi(L)(a)$ is the equivalence class $[1]_{\phi_L(a)}$. They further called (S,D,ϕ) an *abstract triple* if S is a Boolean algebra, D a lattice with 1, and $\phi\colon S \to Eq_\wedge(D)$ a $(0,1)$-isotone map such that for any $d,e \in D$ and $a,b \in S$ there exists $t \in D$ such that

$$[[d]_{\phi(a)}) \cap [[e]_{\phi(b)}) = [[t]_{\phi(a\wedge b)})\,.$$

They defined an isomorphism between the triples (S_1,D_1,ϕ_1) and (S_2,D_2,ϕ_2) to be a pair (f,g), where $f\colon B_1 \cong B_2$, $g\colon D_1 \cong D_2$ (inducing an isomorphism $\bar{g}\colon Eq_\wedge(D_1) \cong Eq_\wedge(D_2)$), and $\bar{g}\circ\phi_1 = \phi_2\circ f$. Then, they proved these two results, respectively called *triple characterization* and *triple construction* of p-algebras.

(i) Two decomposable p-algebras are isomorphic if and only if their associated triples are isomorphic.
(ii) For any triple (S,D,ϕ) there is a decomposable p-algebra L such that the associated triple $(S(L),D(L),\phi_L)$ is isomorphic to (S,D,ϕ).

4.2. Characterization for weakly dicomplemented lattices.

Now L denotes a weakly dicomplemented lattice. Its skeleton

$$S(L) := \{x \in L:\ x^{\triangledown\triangledown} = x\}$$

and dual skeleton $\bar{S}(L)$ are ortholattices ([Kw04; Proposition 1.3.4]). Its Boolean part

$$B(L) := \{x \in L:\ x^{\vartriangle} = x^{\triangledown}\}$$

is the largest Boolean algebra, subalgebra of the skeletons and sublattice of L. The set

$$D(L) := \{x \in L:\ x^{\triangledown} = 0\}$$

of dense elements forms an order filter of L. Dually $\bar{D}(L)$ is an order ideal containing $x\wedge x^{\vartriangle}$ for all $x \in L$. It is not in general a lattice ideal.

For every $x \in L$, the element $x \vee x^{\triangledown}$ is in $D(L)$ and $x^{\triangledown\triangledown}$ is in $S(L)$. Moreover, the equality $x = x^{\triangledown\triangledown}\wedge(x\vee x^{\triangledown})$ holds. This identity can be interpreted, as in the case of Stone algebras, that every $x \in L$ can be represented by a pair $(y,z) \in S(L)\times D(L)$. Such an interpretation suggests that the structures of $S(L)$ and $D(L)$ together with the relationship between their elements may characterize L. How does this relationship look like?

A map $\phi(L)$ is defined from $S(L)$ to the power set of $D(L)$ by

$$\phi(L)(a) := \{x \in D(L):\ x \geq a^{\triangledown}\}\,.$$

We have the equalities $\phi(L)(0) = \{1\}$ and $\phi(L)(1) = D(L)$. In addition, $\phi(L)(a)$ is an order filter of $D(L)$. All $\phi(L)(a)$'s are lattice filters iff $D(L)$ is a lattice filter. Let $a \leq b$ in $S(L)$. From $a^{\triangledown} \geq b^{\triangledown}$ we get $\phi(L)(a) \subseteq \phi(L)(b)$. Thus, $\phi(L)$ is a $(0,1)$-isotone map from $S(L)$ into the lattice of order filters of $D(L)$.

For $a \in S(L)$ we set $F_a := \{x \in L : x^{\triangledown\triangledown} = a\}$. F_a is a $\vee$-semilattice[10] with a greatest element a. The family $(F_a)_{a \in S(L)}$ forms a partition of L. We get $F_0 = \{0\}$ and $F_1 = D(L)$. The mapping $\psi_a \colon x \mapsto x \vee a^{\triangledown}$ maps F_a into $D(L)$. In fact $x \in F_a$ implies

$$(x \vee a^{\triangledown})^{\triangledown} = x^{\triangledown} \wedge a = a^{\triangledown} \wedge a = 0\,.$$[11]

If $\psi_a(x) = \psi_a(y)$ for x and y in F_a, then we have

$$x = (x \vee x^{\triangledown}) \wedge x^{\triangledown\triangledown} = (x \vee a^{\triangledown}) \wedge a = (y \vee a^{\triangledown}) \wedge a = (y \vee y^{\triangledown}) \wedge y^{\triangledown\triangledown} = y\,.$$

Thus, ψ_a is a bijection from F_a onto $\psi_a(F_a)$. Moreover,

$$\psi_a(x \vee y) = \psi_a(x) \vee \psi_a(y)\,.$$[12]

Each element $x \in F_a$ is completely determined by $a \in S(L)$ and $x \vee a^{\triangledown} \in \psi_a(L)$, that is, by a pair (a, z) with $a \in S(L)$ and $z \in \psi_a(F_a)$. Every such pair determines one and only one element of L, namely $a \wedge z \in F_a$. We want to show that the partial ordering of L can be determined by such pairs. Let x and y be in L. If they belong to F_a, then

$$x \leq y \iff (a, x \vee a^{\triangledown}) \leq (a, y \vee a^{\triangledown})$$

since $x^{\triangledown\triangledown} = a$ and $a \wedge (x \vee a^{\triangledown}) = x$. Without loss of generality we assume $x \in F_a$ and $y \in F_b$. From $x \leq y$ we get $a = x^{\triangledown\triangledown} \leq y^{\triangledown\triangledown} = b$. In addition

$$x \leq y \implies x \vee a \leq y \vee a \qquad \text{and} \qquad x \vee a^{\triangledown} \leq y \vee a^{\triangledown}\,.$$

The converse holds if L is distributive. In this case we have $x \leq y$ is equivalent to $x \vee a^{\triangledown} \leq y \vee a^{\triangledown}$ since $a \vee x = a \leq a \vee y$ is obvious.

Construction problem. To each weakly complemented lattice L can be associated an order filter $D(L)$, a Boolean algebra $B(L)$ and an ortholattice $S(L)$. Of course, if D is an order filter of a lattice L (a $\vee$-semilattice with 1), then the lattice $L_D := \{0\} \oplus D$ endowed with the trivial weak complementation has D as the set of dense elements. It is also evident that each Boolean algebra L can be considered as a weakly complemented lattice with $B(L) = L$. The first problem to consider is to find if each ortholattice is the skeleton of some weakly dicomplemented lattice. The construction problem can be formulated as follows:

Given a Boolean algebra B, an ortholattice S and a $\vee$-semilattice D with 1. Is there any weakly complemented lattice L such that

$$B(L) = B\,, \qquad S(L) = S\,, \qquad \text{and} \qquad D(L) = D?$$

---

[10] Is F_a a $\wedge$-semilattice?

[11] $x^{\triangledown\triangledown} = a \iff x^{\triangledown} = a^{\triangledown}$ for all $a \in S(L)$.

[12] If L is distributive, then $\psi_a(x \wedge y) = \psi_a(x) \wedge \psi_a(y)$.

Of course, these structures should be somehow connected. Finding these conditions is an interesting problem to be considered in future works.

Characterization problem. If L_1 and L_2 are isomorphic weakly dicomplemented lattices, then their skeletons are isomorphic ortholattices, their Boolean parts are isomorphic Boolean algebras, their dense elements form two isomorphic $\vee$-semilattices and the dual dense elements two isomorphic $\wedge$-semilattices. The corresponding isomorphisms are the restrictions of the isomorphism between L_1 and L_2 to the corresponding subsets.

> Now, if L_1 and L_2 are two weakly dicomplemented lattices such that $S(L_1) \cong S(L_2)$, $D(L_1) \cong D(L_2)$ and $B(L_1) \cong B(L_2)$ as well as for their dual, under which conditions is L_1 isomorphic to L_2?

We know that for each $x \in L$ the equalities

$$x^{\bigtriangledown\bigtriangledown} \wedge (x \vee x^{\bigtriangledown}) = x = x^{\vartriangle\vartriangle} \vee (x \wedge x^{\vartriangle})$$

hold. If the isomorphisms $f\colon S(L_1) \cong S(L_2)$, $g\colon D(L_1) \cong D(L_2)$ and $u\colon B(L_1) \cong B(L_2)$ as well as the dual $\bar{f}\colon \bar{S}(L_1) \cong \bar{S}(L_2)$ and $\bar{g}\colon \bar{D}(L_1) \cong \bar{D}(L_2)$ have a common extension $h\colon L \cong L$, then necessarily

$$f(x^{\bigtriangledown\bigtriangledown}) \wedge g(x \vee x^{\bigtriangledown}) = h(x) = \bar{f}(x^{\vartriangle\vartriangle}) \vee \bar{g}(x \wedge x^{\vartriangle}).$$

This can be set as a definition of a candidate map h. As the Boolean part is a subalgebra of the skeleton, the map u should be the restriction of f and $\bar{f}$ on $B(L_1)$. The problem now is to find conditions under which h is an isomorphism. Like the construction problem, it is still open and will be considered in future works.

5. Conclusion

In this contribution, our aim was to find the "logic" of the classes extending the class of Boolean algebras. We focussed only on those where the Boolean complementation is abstracted by a unary operation. This is up to one implication now completely determined (see Remark 3.2). This suggests that other classes should be investigated, for example the class of complemented Ockham algebras or complemented p-algebras, or their intersection. Some analogies between the class of weakly dicomplemented lattices and double p-algebras suggest that the techniques developed for double p-algebras might be useful to understand the structure of weakly dicomplemented lattices. A preliminary work should be to find a "good" description of the structure of skeletons and dual skeletons.

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Received June 21, 2004
Revised October 28, 2005

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