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ON TOTALLY BOUNDED GAMES

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The aim of this paper is to establish a closer connection between totally bounded games and compact-continuous ones (cf. the definitions below). It is known that each compact-continuous game is totally bounded. Here we show that each zero-sum two-person game of strategy with bounded payoff function is a dense subgame of a complete game, and thus, in particular, each totally bounded game is a dense subgame of a compact-continuous game.

For the background in game theory the reader is refered to [7], and in topology to [1] or [3].

Let (X, Y, P) be a game of strategy ([7], p. 114), i.e., X and Y are the sets of strategies of Player I and Player II, respectively, and $P\colon X\times Y\to R$ is the real-valued payoff function. Player I and Player II independently choose x in X and y in Y, respectively. If $P(x, y)\geqq 0$, then Player I receives from Player II the amount $P(x, y)$; if $P(x, y)<0$, then Player II receives from Player I the amount $|P(x, y)|$. Player I (Player II) tries to maximize (minimize, resp.) the value of $P(x, y)$. The game (X, Y, P) is said to be determined if

$$\sup_x \inf_y P(x, y) = \inf_y \sup_x P(x, y).$$

In the sequel we assume that the payoff function P is bounded.

The natural (intrinsic) pseudometric p_X for X and p_Y for Y is defined by the formula

$$p_X(x_1, x_2) = \sup_y |P(x_1, y) - P(x_2, y)|$$

and

$$p_Y(y_1, y_2) = \sup_x |P(x, y_1) - P(x, y_2)|$$

respectively [2, 7, 9, 10]. Let us notice that one can convert the pseudometrics into metrics by identifying points which are not distinguished by the corresponding pseudometrics; the conversion has no influence on strategic properties of the game.

It is easy to check that

$$|P(x_1, y_1) - P(x_2, y_2)| \leqq p_X(x_1, x_2) + p_Y(y_1, y_2).$$

Hence it follows that P is uniformly continuous with respect to the product pseudometric $p_{X\times Y}$ defined by the formula

$$p_{X\times Y}((x_1, y_1), (x_2, y_2)) = p_X(x_1, x_2) + p_Y(y_1, y_2).$$

A game (A, B, Q) is said to be *a subgame* of a game (X, Y, P) if $A \subset X$, $B \subset Y$ and $Q = P|(A \times B)$.

From the definition of the natural pseudometrics it follows immediately that for any subgame (A, B, Q) of (X, Y, P) we have

$$p_A \leqq p_X|(A \times A) \quad \text{and} \quad p_B \leqq p_Y|(B \times B).$$

A game (A, B, Q) is said to be *a dense subgame* of a game (X, Y, P) if A and B are dense subsets of the pseudometric spaces (X, p_X) and (Y, p_Y) respectively, and $Q = P|(A \times B)$.

Let us notice that for any dense subgame (A, B, Q) of (X, Y, P) we have

$$p_A = p_X|(A \times A) \quad \text{and} \quad p_B = p_Y|(B \times B).$$

Furthermore, one can easily prove that

$$\sup_a \inf_b Q(a, b) = \sup_x \inf_y P(x, y)$$

and

$$\inf_b \sup_a Q(a, b) = \inf_y \sup_x P(x, y).$$

We say that a game (X, Y, P) is *complete* if the pseudometric spaces (X, p_X) and (Y, p_Y) are complete (cf. [3], Chapter 6).

Theorem 1. *Each game (X, Y, P) is a dense subgame of a complete game $(\bar{X}, \bar{Y}, \bar{P})$, i.e., each game has a completion.*

Proof. Let $(\bar{X}, \bar{p}_X)$ and $(\bar{Y}, \bar{p}_Y)$ be the completions of (X, p_X) and (Y, p_Y) respectively. Then setting

$$\bar{p}_{X\times Y}((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)) = \bar{p}_X(\bar{x}_1, \bar{x}_2) + \bar{p}_Y(\bar{y}_1, \bar{y}_2)$$

we get a complete pseudometric for $\bar{X} \times \bar{Y}$ so that

$$\bar{p}_{X\times Y}|(X \times Y \times X \times Y) = p_{X\times Y}$$

and $X \times Y$ is dense in $(\bar{X} \times \bar{Y}, \bar{p}_{X\times Y})$. Since P is uniformly continuous with respect to $p_{X\times Y}$, there is a function $\bar{P}$: $\bar{X} \times \bar{Y} \to R$ such that $\bar{P}|(X \times Y) = P$ and $\bar{P}$ is uniformly continuous with respect to $\bar{p}_{X\times Y}$. Since (X, Y, P) is a subgame of $(\bar{X}, \bar{Y}, \bar{P})$, it suffices to prove that $\bar{p}_X = p_{\bar{X}}$ and $\bar{p}_Y = p_{\bar{Y}}$. We shall show that $\bar{p}_X = p_X$ only, because the other equality can be shown similarly. So, let $\bar{x}$, $\bar{z} \in \bar{X}$. To prove that $p_{\bar{X}}(\bar{x}, \bar{z}) \leqq \bar{p}_X(\bar{x}, \bar{z})$, let us take any $\varepsilon > 0$ and Cauchy sequences $(x_1, x_2, \ldots)$ and

$(z_1, z_2, \dots)$ in (X, p_X) so that $\bar{p}_X(\bar{x}, x_n) \to 0$ and $\bar{p}_X(\bar{z}, z_n) \to 0$ as $n \to \infty$. Then we have

$$p_{\bar{X}}(\bar{x}, \bar{z}) = \sup_{\bar{y}} |\bar{P}(\bar{x}, \bar{y}) - \bar{P}(\bar{z}, \bar{y})| = \sup_{y} |\bar{P}(\bar{x}, y) - \bar{P}(\bar{z}, y)|,$$

because Y is dense in $(\bar{Y}, \bar{p}_Y)$ and $|\bar{P}(\bar{x}, \cdot) - \bar{P}(\bar{z}, \cdot)|$ is continuous on $(\bar{Y}, \bar{p}_Y)$. Hence

$$p_{\bar{X}}(\bar{x}, \bar{z}) < |\bar{P}(\bar{x}, y) - P(\bar{z}, y)| + \varepsilon$$

for some $y \in Y$. However,

$$|P(\bar{x}, y) - \bar{P}(\bar{z}, y)| \leqq$$
$$\leqq |\bar{P}(\bar{x}, y) - P(x_n, y)| + |P(x_n, y) - P(z_n, y)| + |P(z_n, y) - \bar{P}(\bar{z}, y)|,$$

where $|\bar{P}(\bar{x}, y) - P(x_n, y)| \to 0$ and $|P(z_n, y) - \bar{P}(\bar{z}, y)| \to 0$ as $n \to \infty$, because $\bar{P}(\cdot, y)$ is continuous on $(\bar{X}, \bar{p}_X)$. Hence there is an $m \in N$ such that for each $n \geqq m$ we have

$$p_{\bar{X}}(\bar{x}, \bar{z}) < |P(x_n, y) - P(z_n, y)| + 3\varepsilon \leqq p_X(x_n, z_n) + 3\varepsilon.$$

However,

$$p_X(x_n, z_n) = \bar{p}_X(x_n, z_n) \leqq \bar{p}_X(x_n, \bar{x}) + \bar{p}_X(\bar{x}, \bar{z}) + \bar{p}_X(\bar{z}, z_n)$$

and

$$\bar{p}_X(x_k, \bar{x}) < \varepsilon \quad \text{and} \quad \bar{p}_X(z_k, \bar{z}) < \varepsilon$$

for some $k \geqq m$. Therefore

$$p_{\bar{X}}(\bar{x}, \bar{z}) < \bar{p}_X(\bar{x}, \bar{z}) + 5\varepsilon.$$

Finally, we prove the inequality $\bar{p}_X(\bar{x}, \bar{z}) \leqq p_{\bar{X}}(\bar{x}, \bar{z})$. Clearly

$$\bar{p}_X(\bar{x}, \bar{z}) \leqq \bar{p}_X(\bar{x}, x_n) + \bar{p}_X(x_n, z_n) + \bar{p}_X(z_n, \bar{z})$$

and hence

$$\bar{p}_X(\bar{x}, \bar{z}) < \bar{p}_X(x_n, z_n) + 2\varepsilon$$

for some $n \in N$. However

$$\bar{p}_X(x_n, z_n) = p_X(x_n, z_n) \leqq p_{\bar{X}}(x_n, z_n) \leqq p_{\bar{X}}(x_n, \bar{x}) +$$
$$+ p_{\bar{X}}(\bar{x}, \bar{z}) + p_{\bar{X}}(\bar{z}, z_n) \leqq \bar{p}_X(x_n, \bar{x}) + p_{\bar{X}}(\bar{x}, \bar{z}) + \bar{p}_X(\bar{z}, z_n) < p_{\bar{X}}(\bar{x}, \bar{z}) + 2\varepsilon$$

and hence

$$\bar{p}_X(\bar{x}, \bar{z}) < p_{\bar{X}}(\bar{x}, \bar{z}) + 4\varepsilon.$$

The proof is complete.

Remark 1. The families $\{P(\cdot, y): y \in Y\}$ and $\{P(x, \cdot): x \in X\}$ of functions induce the natural uniformities $\mathscr{U}_X$ and $\mathscr{U}_Y$ on X and Y respectively. Then obviously P is separately uniformly continuous on the product of the uniformities $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$. The following result, analogous to Theorem 1, was obtained by N. J. Young ([11], Theorem 7): P admits a separately uniformly continuous extension $\bar{P}$ onto the product of completions $(\bar{X}, \bar{\mathcal{U}}_X)$ and $(\bar{Y}, \bar{\mathcal{U}}_Y)$ iff P satisfies the repeated limit condition, i.e.,

$$\lim_m \lim_n P(x_m, y_n) = \lim_n \lim_m P(x_m, y_n)$$

provided that both iterated limits exist.

We say that a game (X, Y, P) is *totally bounded* if the pseudometric spaces (X, p_X) and (Y, p_Y) are totally bounded (cf. [1], p. 332).

Let us notice that a game (X, Y, P) is totally bounded iff at least one of the spaces (X, p_X) and (Y, p_Y) is totally bounded (A. Wald [10], 2.1.3). Furthermore, each subgame of a totally bounded game is also totally bounded, because each subspace of a totally bounded pseudometric space is totally bounded.

Totally bounded games constitute a natural generalization of matrix games because for each $\varepsilon > 0$ there is a finite subgame which is ε-close to the given game (cf. [10], Theorem 2.3).

A game (X, Y, P) is said to be *compact-continuous* (J. E. Fenstad [2]) if X and Y are compact and P is continuous on $X \times Y$.

Since the completion of a totally bounded pseudometric space is totally bounded, and each totally bounded complete pseudometric space is compact, by Theorem 1 we get

Corollary. *Each totally bounded game (X, Y, P) is a dense subgame of the compact-continuous game $(\bar{X}, \bar{Y}, \bar{P})$.*

Remark 2. It is well known that the mixed extension of a compact-continuous game constituted by countably additive probability Borel measures is again a compact-continuous game, where both players have optimal strategies. On the other hand, there are several types of mixed extensions of totally bounded games that are determined (cf. J. Kindler [4, 5]), and moreover, the players have optimal mixed strategies which are finitely additive probability Borel measures (J. E. Fenstad [2]). However, countably additive optimal mixed strategies need not exist in general (cf. [2]). Now, according to Corollary, if (X, Y, P) is totally bounded, then the players have optimal strategies which are countably additive probability measures defined on Borel σ-fields of the completions $\bar{X}$ and $\bar{Y}$. (Note that the values of the mixed extensions of (X, Y, P) and $(\bar{X}, \bar{Y}, \bar{P})$ coincide.)

It is easy to show that each compact-continuous game is totally bounded and complete. Hence, by Corollary, we get

Theorem 2. *A game (X, Y, P) is totally bounded iff it is a (dense) subgame of a compact-continuous game.*

Remark 3. Theorem 2 reaches beyond the scope of the game theory. E.g., its topological version reads as follows: a function $P: X \times Y \to R$, where X and Y are sets, can be extended to a continuous function $P_1: X_1 \times Y_1 \to R$, where X_1 and Y_1 are compact spaces, iff P is bounded and the pseudometric space (X, p_X) is totally bounded. In view of that interpretation, Theorem 2 is analogous to the following result of V. Pták [8]: a function $P: X \times Y \to R$, where X and Y are sets, can be extended to a separately continuous function $P_2: X_2 \times Y_2 \to R$, where X_2 and Y_2 are compact spaces, iff P is separately bounded and satisfies the repeated limit condition (cf. Remark 1 above).

Remark 4. Denote by (X^*, Y^*, P^*) the mixed extension of (X, Y, P) consisting of finite mixtures of pure strategies (i.e., X^* is the set of all $x^* \in [0, 1]^X$ such that $x^*(x) = 0$ for all but finitely many $x \in X$, $x^*(x) \geqq 0$ for all $x \in X$, $\sum_x x^*(x) = 1$, etc.). If (X, Y, P) is a totally bounded game, then

(∗) for each subgame (A, B, Q) of (X, Y, P) the games (A^*, B^*, Q^*) and $(A^*, B^*, -Q^*)$ are determined.

However, (∗) holds iff P satisfies the repeated limit condition (J. Kindler [4, 5, 6] and N. J. Young [11]). Next, the repeated limit condition is equivalent to the following:

$$\int_X \int_Y P \, d\eta \, d\xi = \int_Y \int_X P \, d\xi \, d\eta$$

for each pair of finitely additive probability measures ξ and η defined for all subsets of X and Y respectively (cf. [5] and [12]). The assumption of the total boundedness of (X, Y, P) is therefore too strong, even for getting (∗); it incorporates, however, a reasonable condition ensuring the equality of repeated integrals of P with respect to countably additive probability measures on X and Y. On the other hand, so far there is no topological characterization of those functions $P: X \times Y \to R$ for whose the analogue of (∗) holds with the mixed extensions constituted by the countably additive probability measures on X and Y.

Remark 5. The above definitions and theorems can be extended to non-cooperative games $(X_1, \ldots, X_n, P_1, \ldots, P_n)$ of n players, where all $P_i: x_1 \times \ldots \times X_n \to R$ are bounded. To be more specific, let

$$p_i^j(x_i, x_i') = \sup |P_j(x_1, \ldots, x_n) - P_j(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)|$$

where the supremum is taken over all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, and

$$p_i(x_i, x_i') = \sum_{j=1}^n p_i^j(x_i, x_i').$$

Then p_i is a pseudometric on X_i and it is easy to check that for each $k \leqq n$

$$|P_k(x_1, \ldots, x_n) - P_k(x_1', \ldots, x_n')| \leqq \sum_{i=1}^n p_i(x_i, x_i').$$

Thus each P_k is uniformly continuous with respect to the product pseudometric

$$p((x_1, \ldots, x_n), (x_1', \ldots, x_n')) = \sum_{i=1}^{n} p_i(x_i, x_i').$$

Assuming that all (X_i, p_i^j) are totally bounded it is easy to show that also all (X_i, p_i) are totally bounded. Hence the completions $(\bar{X}_i, \bar{p}_i)$ of (X_i, p_i) are compact and the extensions $\bar{P}_k$ of P_k are continuous on $\left(\bar{X}_1 \times \ldots \times \bar{X}_n, \sum_{i=1}^{n} \bar{p}_i\right)$. Let us note, moreover, that in the case of $n = 2$, it is sufficient to assume that (X_1, p_1^1) and (X_2, p_2^2) are totally bounded, since then (X_1, p_1^2) and (X_2, p_2^1) are totally bounded by a theorem of A. Wald ([9], 2.1.3). For $n \geqq 3$, however, the assumption of the total boundedness of all (X_i, p_i^j) cannot be weakened in general, as the following example shows: $X_1 = X_2 = X_3 = \{1, 2, \ldots, n, \ldots\}$, $P_1(x_1, x_2, x_3) = \operatorname{sgn}(x_2 - x_3)$, and $P_2 = P_3 = 0$.

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О ВПОЛНЕ ОГРАНИЧЕННЫХ ИГРАХ

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Резюме

В работе доказывается, что каждая стратегическая игра с ограниченной функцией выигрыша является плотной подигрой полной игры. Отсюда получается, что каждая вполне ограниченная игра является плотной подигрой компактно-непрерывной игры.