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# ON PROJECTION MEASURABILITY OF FUNCTIONS AND MULTIFUNCTIONS

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ABSTRACT. The notion of projection measurable multifunction is defined and studied. The article generalizes some results concerning projection measurable functions, some projection measurability-preserving convergencies are studied.

In paper [1] one of us introduced the notion of a projection measurable function and established some conditions for the projection measurability to imply or be implied by the usual measurability. In the present paper, the notion of a projection measurable multifunction is defined and studied.

We begin with recalling the original definition from [1].

**DEFINITION 1.** *Let $X, Y$ be nonempty sets, $\mathcal{X}$ any family of subsets of $X$ and $\mathcal{V}$ any family of subsets of $X \times Y$. We say that a function $g\colon X \to Y$ is $(\mathcal{X}, \mathcal{V})$-projection measurable if and only if the set $\{x\colon\ (x, g(x)) \in V\}$ is in $\mathcal{X}$ for each $V \in \mathcal{V}$.*

As the name suggests a function $f\colon X \to Y$ is projection measurable if and only if the projection into $X$ of the intersection of the graph of $f$ with any "measurable" (i.e. belonging to $\mathcal{V}$) set in $X \times Y$ is measurable (i.e. in $\mathcal{X}$). This property can easily be formulated for multifunctions in the following definition. By a multifunction $F\colon X \to Y$ we mean a mapping of $X$ into $2^Y$, which here will denote the family of all nonempty subsets of $Y$.

**DEFINITION 2.** *Let $X, Y$ be nonempty sets, $\mathcal{X}$ any family of subsets of $X$ and $\mathcal{V}$ any family of subsets of $X \times Y$. A multifunction $F\colon X \to Y$ is said to be $(X, \mathcal{V})$-projection measurable if and only if the set $\{x\colon\ \{x\} \times F(x) \cap V \neq \emptyset\}$ is in $\mathcal{X}$ for every $V \in \mathcal{V}$.*

The following theorem is analogous to Theorem 1 in [1].

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**THEOREM 1.** *Let $X, Y, Z$ be nonempty sets and let $\mathcal{X} \subset 2^X$ $\mathcal{V} \subset 2^{X \times Y}$ and $\mathcal{Z} \subset 2^Z$. Let $F\colon X \times Y \to Z$ be a $(\mathcal{V}, \mathcal{Z})$-measurable multifunction, that is, $\forall B \in \mathcal{Z}$ $F^{-1}(B) \in \mathcal{V}$, where $F^{-1}(B) = \{(x, y)\colon\ F(x, y) \cap B \neq \emptyset\}$. Let $G\colon X \to Y$ be an $(X, \mathcal{V})$-projection measurable multifunction. Then the multifunction $H\colon X \to Z$ defined by $H(x) = F(x, G(x)) = \bigcup\limits_{y \in G(x)} F(x, y)$ for all $x \in X$, is $(\mathcal{X}, \mathcal{Z})$-measurable.*

P r o o f . If $P \in \mathcal{Z}$, then the set

$$H^{-1}(P) = \{x\colon\ F(x, G(x)) \cap P \neq \emptyset\} = \{x\colon\ \{x\} \times G(x) \cap F^{-1}(P) \neq \emptyset\}$$

is in $\mathcal{X}$, because $F^{-1}(P) \in \mathcal{V}$ and $G$ is an $(\mathcal{X}, \mathcal{V})$-projection measurable multifunction. □

It is well known that pointwise limits of sequence of (real valued) measurable functions are measurable. An answer to the natural question whether a limit function of a sequence of projection measurable functions or multifunctions is itself projection measurable will be given in Theorem 2. To simplify its formulation and proof let us first introduce some notation.

**DEFINITION 3.** *Let $X, Y$ be nonempty sets. We shall use the notation $D^x$ for an $x$-section of a set $D \subset X \times Y$, i.e., for a given $x \in X$, $D^x = \{y \in Y\colon (x, y) \in D\}$. If $Y$ is a topological space and $\mathcal{V} \subset X \times Y$, we shall denote by $\widetilde{\mathcal{V}}$ the set $\bigcup\limits_{x \in X} (\{x\} \times \overline{V^x})$, where the bar denotes closure in $Y$.*

In order to produce our results we must specify in what sense a sequence of multifunctions converges to a function. Here is one of the most frequently used topologies in which the convergence may be considered.

**DEFINITION 4.** *Let $Y$ be a topological space, denote $\mathcal{P}(Y) = \{C \subset Y\colon\ C = \overline{C} \neq \emptyset\}$. The topology whose base is formed by all sets of the form*

$$(U, U_1, \dots, U_n) = \{C \in \mathcal{P}(Y)\colon\ C \subset U \text{ and } C \cap U_i \neq \emptyset \text{ for all } i = 1, 2, \dots, n\},$$

*where $n$ is a positive integer and $U, U_1, \dots, U_n$ are open subsets of $Y$, is called the Vietoris topology on $\mathcal{P}(Y)$.*

**THEOREM 2.** *Let $\mathcal{X} \subset 2^X$ be a $\sigma$-ring, let $Y$ be a $T_1$ topological space. Assume that $\mathcal{V}$ is a $\sigma$-ring in $X \times Y$ generated by a family $\mathcal{V}_0$ such that for any $V \in \mathcal{V}_0$ there is a sequence of sets $V_k \in \mathcal{V}$ satisfying*

(i) $\forall x \in X$ $\{y \in Y\colon\ (x, y) \in V_k\}$ *is open in* $Y$

*and*

(ii) $V = \bigcup_{k=1}^{\infty} V_k = \bigcup_{k=1}^{\infty} \widetilde{\mathcal{V}}_k$.

*If a sequence of $(X, \mathcal{V})$-projection measurable multifunctions $F_n\colon X \to Y$ with closed nonempty values converges pointwise in the Vietoris topology to a function $f\colon X \to Y$, then $f$ is $(\mathcal{X}, \mathcal{V})$-projection measurable.*

Proof. It is sufficient to prove that, for each $V \in \mathcal{V}_0$, the set $f^{-1}(V) = \{x\colon (x, f(x)) \in V\}$ is in $X$. Let $V \in \mathcal{V}_0$ and let $\{V_k\}_{k=1}^{\infty}$ be such a sequence that the sets $V, V_k$ satisfy the assumptions (i) and (ii). Put $A = f^{-1}(V)$ and

$$B = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \{x\colon \{x\} \times F_n(x) \cap V_k \neq \emptyset\}.$$

Evidently, $B \in \mathcal{X}$. We are going to show that $A = B$.

It is easy to verify that

$$A = \bigcup_{k=1}^{\infty} \{x\colon f(x) \in V_k^x\} = \{x\colon f(x) \in \overline{V_k^x}\}$$

and

$$B = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \{x\colon F_n(x) \cap V_k^x \neq \emptyset\}.$$

If $x \in A$, then for some positive integer $k$ we have $f(x) \in V_k^x$, that is, $\{f(x)\} \cap V_k^x \neq \emptyset$. Since $\{C\colon C \cap V_k^x \neq \emptyset\}$ is a neighbourhood of $\{f(x)\}$ in the Vietoris topology, we infer that there is a positive integer $i$ such that for every $n \geq i$ we have $F_n(x) \cap V_k^x \neq \emptyset$. Hence $x \in B$.

Conversely, let $x \in B$. Then

($*$) there are positive integers $i, l$ such that $F_n(x) \cap V_l^x \neq \emptyset$ for all $n \geq i$.

Let $k$ be arbitrary and assume that $f(x) \in Y - \overline{V_k^x}$. Then $\{C\colon C \cap \overline{V_k^x} = \emptyset\}$ is a neighbourhood of $\{f(x)\}$ and since $F_n(x)$ converge to $\{f(x)\}$, we obtain, for some positive integer $j$ and every $n \geq j$, the inclusion $F_n(x) \subset Y - \overline{V_k^x}$, which contradicts ($*$). Therefore $f(x) \in \overline{V_k^x}$ for some $k$, and hence $x$ belongs to $A$. □

Remark. The assumptions of Theorem 2 sound naturally especially for any regular topological space.

**COROLLARY.** *If $X$ and $Y$ satisfy the assumptions of Theorem 2, then every function which is a pointwise limit of a sequence of $(X, \mathcal{V})$-projection measurable functions $f_n\colon X \to Y$ is $(X, \mathcal{V})$-projection measurable.*

The following example shows that if the hypotheses of Theorem 2 are not met, a sequence of projection measurable functions may have a limit that is not projection measurable.

Example 1. Put $X = \mathbb{R}$, $\mathcal{X} = \{E \subset X\colon\ E$ is countable or $X - E$ is countable$\}$. Let $Y = \mathbb{R}$ be endowed with the cofinite topology $\mathcal{S} = \{G \subset Y\colon Y - G$ is finite$\}$ and let $\mathcal{V}$ be the $\sigma$-ring generated by $\{E \times G\colon\ E \in \mathcal{X}$ and $G \in \mathcal{S}\}$. As usual, let the function $\operatorname{sign}(x)$ be defined by

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

Then evidently the functions $f_n(x) = |x|^{\frac{1}{n}} \operatorname{sign}(x)$ are bijections of $X$ onto $Y$ and hence they are $(\mathcal{X}, \mathcal{V})$-projection measurable, while the sign function is not. But $\lim\limits_{n\to\infty} f_n(x) = \operatorname{sign}(x)$ also in the cofinite topology. □

It is an open problem whether "function $f$" may be replaced by "multifunction $F$" in the statement of Theorem 2. The authors conjecture that the answer is negative.

## REFERENCES

[1] DRAVECKÝ, J.: *On measurability of superpositions*, Acta Math. Univ. Comenian. **44–45** (1984), 181–183.

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