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Mathematica Slovaca, Vol. 46 (1996), No. 2-3, 269--277

Persistent URL: http://dml.cz/dmlcz/130925

Math. Slovaca, **46** (1996), No. 2–3, 269–277

Mathematica Slovaca
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Slovak Academy of Sciences

Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

INTERMEDIATE VALUE PROPERTY AND FUNCTIONAL CONNECTEDNESS OF MULTIVALUED MAPS[1]

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(*Communicated by Ľubica Holá*)

ABSTRACT. Unions, intersections, limits, algebraic sums and products of multivalued maps which either both are functionally connected, or have the intermediate value property are considered. It is shown that the composition of a continuous and a functionally connected function need not be functionally connected. Conditions equivalent to the intermediate value property of multivalued maps and some conditions under which upper continuity and the intermediate value property of multivalued maps are equivalent, are given.

1. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, I any interval contained in $\mathbb{R}$. If $A \subset I$, we let $\mathrm{cl}(A)$ denote the closure of the set A in I, and $A^c = I \setminus A$. For a non-empty set $A \subset \mathbb{R}$ and a number $\varepsilon > 0$ we denote

$$K_\varepsilon(A) = \{x \in \mathbb{R};\ \text{there exists } y \in A \text{ such that } |x - y| < \varepsilon\}$$

By the algebraic sum (product) of sets $A, B \subset \mathbb{R}$, we mean the set $A + B = \{a + b;\ a \in A,\ b \in B\}$ ($AB = \{ab;\ a \in A,\ b \in B\}$). For a set $A \subset \mathbb{R}$ and a number $b \in \mathbb{R}$ we denote $A - b = \{a - b;\ a \in A\}$. For $M \subset X \times Y$, where $X, Y \subset \mathbb{R}$, we put $\pi(M) = \{x \in X;\ \text{there exist } y \in Y \text{ such that } (x, y) \in M\}$ and $M_x = \{y \in Y;\ (x, y) \in M\}$.

AMS Subject Classification (1991): Primary 54C60.
Key words: functional connectedness, functional connectedness point, Darboux property.

[1] This work was supported by the University of Gdańsk, grant nr BW 5100–5–0151–4.

In this paper, $\Phi\colon I \to \mathbb{R}$ will denote a multivalued map Φ which to each point $x \in I$ assigns a non-empty subset $\Phi(x) \subset \mathbb{R}$. By the graph of Φ, we mean the following set $\Gamma_\Phi = \bigcup\{(x,y);\ y \in \Phi(x)\}$. For any sets $A \subset I$, $B \subset \mathbb{R}$ and any point $y \in \mathbb{R}$ let $\Phi(A) = \bigcup\{\Phi(x);\ x \in A\}$, $\Phi^+(B) = \{x \in I;\ \Phi(x) \subset B\}$, $\Phi^-(B) = \{x \in I;\ \Phi(x) \cap B \neq \emptyset\}$ and $\Phi^-(y) = \{x \in I;\ y \in \Phi(x)\}$. Φ is *lower* (*upper*) *semicontinuous* if for any open set $V \subset \mathbb{R}$ the set $\Phi^-(V)$ ($\Phi^+(V)$) is open in I. Φ is *lower* (*upper*) *first class* if for any open set $V \subset \mathbb{R}$, $\Phi^-(V)$ ($\Phi^+(V)$) is an F_σ-set.

We say that a sequence of multivalued maps $\{\Phi_n\}_{n\in\mathbb{N}}$ *converges to* Φ if for any x and any number $\varepsilon > 0$ there exist $k \in \mathbb{N}$, such that for any $n > k$, $\Phi_n(x) \subset K_\varepsilon(\Phi(x))$ and $\Phi(x) \subset K_\varepsilon(\Phi_n(x))$. If $f\colon A \to \mathbb{R}$, $A \subset I$, is any function, then by a *multivalued extension of* f we mean any multivalued map $\Phi\colon I \to \mathbb{R}$ such that $\Phi|_A = f$.

2. Unions, intersections, limits, algebraic sums and products

In papers [2] and [3], the following definitions are given:

Definition 1. A multivalued map $\Phi\colon I \to \mathbb{R}$ has the *intermediate value property* if for any distinct points $x_1, x_2 \in I$ and every $y_1 \in \Phi(x_1)$ there exists $y_2 \in \Phi(x_2)$ such that $(y_1, y_2) \subset \Phi((x_1, x_2))$.

Definition 2. A multivalued map $\Phi\colon I \to \mathbb{R}$ is *functionally connected* if for any distinct points $x_1, x_2 \in I$ and every $y_1 \in \Phi(x_1)$ there exists $y_2 \in \Phi(x_2)$ such that for any continuous function $h\colon [x_1, x_2] \to \mathbb{R}$ such that $h(x_1) > y_1$ and $h(x_2) < y_2$ or $h(x_1) < y_1$ and $h(x_2) > y_2$, there exists $x \in (x_1, x_2)$ for which $h(x) \in \Phi(x)$ (i.e., $\Gamma_\Phi|_{(x_1,x_2)} \cap \Gamma_h \neq \emptyset$).

It is not difficult to show that the union of functionally connected multivalued maps (or maps which have the intermediate value property) has this property too, but it is not true for non-empty intersections. In order to realise that, one may consider the intersection $(\Psi \cap \Phi)(x) = \Psi(x) \cap \Phi(x)$, where

$$\Psi(x) = \begin{cases} [0,1]; & x \in \mathbb{Q}, \\ \{0\}; & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

$$\Phi(x) = \begin{cases} \{1\}; & x \in \mathbb{Q}, \\ [0,1]; & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and $\mathbb{Q}$ denotes the set of rational numbers.

LEMMA 1. *Let $M \subset I$ be c-dense in I. Then there exists a function $f\colon M \to \mathbb{R}$ such that any multivalued extension $\Phi\colon I \to \mathbb{R}$ of f is functionally connected.*

Proof. Since M is c-dense in I, then $M = \bigcup_{t\in T} M_t$, where T has continuum cardinality, M_t is dense in I for each $t \in T$, and $M_{t_1} \cap M_{t_2} = \emptyset$ if $t_1 \neq t_2$. Let $\mathcal{G} = \{g\colon [a,b] \to \mathbb{R};$ where g is a continuous function on $[a,b] \subset I,\ a < b\}$. Since the cardinality of $\mathcal{G}$ is continuum, there exists a one-to-one function $p\colon T \to \mathcal{G}$, and let $g_t = p(t)$, $g_t\colon [a_t, b_t] \to \mathbb{R}$ for $t \in T$. Let us define the function $f\colon M \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} g_t(x); & x \in M_t \cap [a_t, b_t], \\ 0; & x \in M \setminus \bigcup\limits_{t\in T} M_t \cap [a_t, b_t]. \end{cases}$$

Since the graph of f intersects the graph of any continuous function $g\colon [a,b] \to \mathbb{R}$, $[a,b] \subset I$, then the same holds for any multivalued extension $\Phi\colon I \to \mathbb{R}$ of f. Therefore any multivalued extension is functionally connected. □

COROLLARY 1. *For any multivalued map $\Phi\colon I \to \mathbb{R}$ there exists a functionally connected multivalued map $\Psi\colon I \to \mathbb{R}$ such that the set $\{x \in I;\ \Phi(x) \neq \Psi(x)\}$ is first category and Lebesgue measure zero.*

Proof. Let $M \subset I$ be c-dense in I, first category, and Lebesgue measure zero. By Lemma 1, it is enough to take:

$$\Psi(x) = \begin{cases} \{f(x)\}; & x \in M, \\ \Phi(x); & x \in I \setminus M. \end{cases}$$

□

PROPOSITION 1. *Each multivalued map $\Phi\colon I \to \mathbb{R}$ is an algebraic sum of two functionally connected multivalued maps.*

Proof. Let $M_1, M_2 \subset I$ be two disjoint sets such that $M_1 \cup M_2 = I$, and each of them is c-dense in I. By Lemma 1, there exist functions $f_i\colon M_i \to \mathbb{R}$, $i = 1,2$, for which any multivalued extension on I is functionally connected. Let us define $\Phi_1, \Phi_2\colon I \to \mathbb{R}$ as follows:

$$\Phi_1(x) = \begin{cases} \{f_1(x)\}; & x \in M_1, \\ \Phi(x) - f_2(x); & x \in M_2, \end{cases}$$

$$\Phi_2(x) = \begin{cases} \Phi(x) - f_1(x); & x \in M_1, \\ \{f_2(x)\}; & x \in M_2. \end{cases}$$

Then Φ_1, Φ_2 are functionally connected, and $\Phi(x) = \Phi_1(x) + \Phi_2(x)$ for every $x \in I$. □

The product of functionally connected functions need not have even the intermediate value property. In order to realize that, one may consider the following functions $f, g\colon I \to \mathbb{R}$:

$$f(x) = \begin{cases} f_1(x); & x \in M_1, \\ 1; & x \in M_2, \\ 0; & x \in M_3, \end{cases}$$

$$g(x) = \begin{cases} 0; & x \in M_1, \\ 1; & x \in M_2, \\ f_2(x); & x \in M_3, \end{cases}$$

where $M_1, M_2, M_3 \subset I$ are c-dense in I, each two of them do not intersect, $M_1 \cup M_2 \cup M_3 = I$, and $f_i\colon M_i \to \mathbb{R}$, $i = 1, 3$, are functions from Lemma 1. However, the "product version" of Proposition 1 does not hold since there exist functions which are not products even of Darboux functions.

Proposition 2. *Each multivalued map $\Phi\colon I \to \mathbb{R}$ is a limit of a sequence of functionally connected multivalued maps.*

P r o o f . Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of sets such that each of the set is c-dense in I, every two of them do not intersect, and $\bigcup_{n=1}^{\infty} M_n = I$. By Lemma 1, there exist functions $f_i\colon M_i \to \mathbb{R}$, $i \in \mathbb{N}$, for which any multivalued extension on I is functionally connected. Let us define the sequence of functionally connected multivalued maps as follows:

$$\Phi_n(x) = \begin{cases} \Phi(x); & x \in \bigcup_{i=1}^{n} M_i, \\ \{f_i(x)\}; & x \in M_i, \ i > n. \end{cases}$$

It is easy to see that $\{\Phi_n\}_{n=1}^{\infty}$ converges to Φ. □

3. Compositions

Theorem 1. *Let $\Psi\colon I \to J$, where $J \subset \mathbb{R}$ is an interval, $\Phi\colon J \to \mathbb{R}$ be multivalued maps such that $\Psi(I) = J$, and Ψ, Φ have the intermediate value property. Then the composition $\Phi \circ \Psi$ has the intermediate value property, too.*

P r o o f . Let $x_1, x_2 \in I$ be two different points, and let $y_1 \in \Phi \circ \Psi(x_1)$. We have to show that there exists $y_2 \in \Phi\circ\Psi(x_2)$ such that $(y_1, y_2) \subset \Phi\circ\Psi\big((x_1, x_2)\big)$. There exists $z_1 \in \Psi(x_1)$ such that $y_1 \in \Phi(z_1)$. Since Ψ has the intermediate value property, there exists $z_2 \in \Psi(x_2)$ such that $(z_1, z_2) \subset \Psi\big((x_1, x_2)\big)$. Let us assume that $z_2 \neq z_1$. Since Φ has the intermediate value property, there exists $y_2 \in \Phi(z_2)$ such that $(y_1, y_2) \subset \Phi((z_1, z_2))$. Then, since $\Phi((z_1, z_2)) \subset \Phi(\Psi((x_1, x_2))) = \Phi \circ \Psi((x_1, x_2))$, we have $(y_1, y_2) \subset \Phi \circ \Psi((x_1, x_2))$. In the case when $z_2 = z_1$, it is enough to take $y_2 = y_1$. □

THEOREM 2. *Let $\varphi\colon I \to J$, $J \subset \mathbb{R}$ is an interval, be a continuous function such that $\varphi(I) = J$, and let $\Phi\colon J \to \mathbb{R}$ be a multivalued map. Then Φ has the intermediate value property if and only if the composition $\Phi \circ \varphi$ has the intermediate value property.*

Proof. Assume that $\Phi \circ \varphi$ has the intermediate value property. Let $x_1, x_2 \in J$ be two different points, and let $y_1 \in \Phi(x_1)$. Let us take $a, b \in I$ such that $\varphi(a) = x_1$, $\varphi(b) = x_2$, and $\varphi((a,b)) = (\varphi(a), \varphi(b)) = (x_1, x_2)$. Since $a \neq b$ and $\Phi \circ \varphi$ has the intermediate value property, for $y_1 \in \Phi(x_1) = \Phi \circ \varphi(a)$, there exists $y_2 \in \Phi \circ \varphi(b) = \Phi(x_2)$ such that $(y_1, y_2) \subset \Phi \circ \varphi((a,b)) = \Phi((x_1, x_2))$. Therefore Φ has the intermediate value property. Since φ has the intermediate value property as a continuous function, the converse implication is an obvious corollary from Theorem 1. □

THEOREM 3. *Let $\varphi\colon I \to J$, where $J \subset \mathbb{R}$ is an interval, be a continuous function such that $\varphi(I) = J$, and let $\Phi\colon J \to \mathbb{R}$ be a multivalued map. If $\Phi \circ \varphi$ is functionally connected, then Φ is functionally connected.*

Since the proof of Theorem 3 is similar to the proof of Theorem 2, it is omitted. Theorem 4 shows that the converse of Theorem 3 does not hold.

In Theorems 2 and 3, we can not replace "continuous function" by "continuous multivalued map". To see this, let $\Psi(x) = [-x, x]$, $x \in \mathbb{R}$, and

$$\Phi(x) = \begin{cases} [0,2]; & x \le 0, \\ [0,1]; & x > 0. \end{cases}$$

Then $\Phi \circ \Psi(x) = [0,2]$ is functionally connected, thus has the intermediate value property, but Φ does not have any of these properties.

LEMMA 2. *Let $M \subset [0,1]^2$ be a continuum such that $\pi(M) = [0,1]$, $M_x \neq [0,1]$ for any $x \in [0,1]$, and $\Gamma_g \not\subset M$ for each continuous function $g\colon [a,b] \to \mathbb{R}$, $[a,b] \subset [0,1]$, $a < b$. Then there exists a functionally connected function $\varphi\colon [0,1] \to [0,1]$ such that $M \cap \Gamma_\varphi = \emptyset$.*

Proof. Let $\mathcal{G} = \{g\colon [a,b] \to [0,1];$ where g is continuous function on $[a,b] \subset [0,1]$, $a < b\}$. For any $g \in \mathcal{G}$ the set $X_g = \{x \in [a,b];\ (x, g(x)) \in M\}$ is nowhere dense. Let $[0,1] = \bigcup_{t \in T} M_t$ be a disjoint union where M_t is dense in I for any $t \in T$, and T has continuum cardinality. Since $\mathcal{G}$ has continuum cardinality, then there exists a one-to-one function $p\colon T \to \mathcal{G}$, and let $g_t = p(t)$, $g_t\colon [a_t, b_t] \to \mathbb{R}$, for $t \in T$. Let us define the function $\varphi\colon [0,1] \to [0,1]$ as follows:

$$\varphi(x) = \begin{cases} g_t\,; & x \in [a_t, b_t] \cap M_t \setminus X_{g_t}\,, \\ y_x\,; & \text{for the remaining } x \in [0,1]\,, \\ & \text{where } y_x \text{ is any point in } [0,1] \setminus M_x\,. \end{cases}$$

Then from the construction it is easy to see that φ is functionally connected and $\Gamma_\varphi \cap M = \emptyset$. □

For real functions the following inclusions $\mathcal{C} \subset \mathcal{C}_f \subset \mathcal{D}$ hold (where $\mathcal{C}$, $\mathcal{C}_f$ and $\mathcal{D}$ denote functions with connected graph, functionally connected and Darboux functions respectively). It is known that compositions of functions from $\mathcal{C}$ or $\mathcal{D}$ with continuous functions fall into $\mathcal{C}$ or $\mathcal{D}$ respectively. It seems to be interesting that it is not true for the class $\mathcal{C}_f$, which is the assertion of the following theorem.

THEOREM 4. *There exist a continuous function $g\colon [0,1] \to [0,1]$ and functionally connected function $\varphi\colon [0,1] \to [0,1]$ such that the compositions $g \circ \varphi$ and $\varphi \circ g$ are not functionally connected.*

Proof. Let $g\colon [0,1] \to [0,1]$ be the Cantor function, and let $M = \{(g(x), x);\ x \in [0,1]\}$. Then M is a continuum such that $\pi(M) = [0,1]$, $M_x \neq [0,1]$ for each $x \in [0,1]$, and there is no continuous function on any interval $[a,b] \subset [0,1]$, $a < b$, with the graph contained in M. By Lemma 2, there exists a functionally connected function $\varphi\colon [0,1] \to [0,1]$ such that $M \cap \Gamma_\varphi = \emptyset$.

To show that $g \circ \varphi$ is not functionally connected, let us take $h(x) = x$ for $x \in [0,1]$. Since $\varphi(0) > 0$ and $\varphi(1) < 1$, then $g \circ \varphi(0) > h(0)$ and $g \circ \varphi(1) < h(1)$. We will show that $\Gamma_h \cap \Gamma_{g\circ\varphi} = \emptyset$. Suppose on the contrary that there exists $x \in (0,1)$ for which $h(x) = g \circ \varphi(x)$. Consequently, $x = g \circ \varphi(x)$ and $(x, \varphi(x)) = (g \circ \varphi(x), \varphi(x)) \in M$, but this contradicts that $M \cap \Gamma_\varphi = \emptyset$.

To show that $\varphi \circ g$ is not functionally connected, let again $h(x) = x$ for $x \in [0,1]$. As before, $h(0) < \varphi \circ g(0)$ and $h(1) > \varphi \circ g(1)$. Suppose that there exists $x \in (0,1)$ such that $h(x) = \varphi \circ g(x)$. Then $(g(x), x) \in \Gamma_\varphi$ which means that $M \cap \Gamma_\varphi \neq \emptyset$, a contradiction. □

4. Generalization of Zahorski theorem

DEFINITION 3. A set $P \subset I$ is *dense-in-itself* (*c-dense-in-itself*) provided if $x \in P$ and $U \subset I$ is an open set with $x \in \mathrm{cl}(U)$, then $P \cap U$ contains a point other than x ($P \cap U$ has continuum cardinality).

THEOREM 5. *If $\Phi\colon I \to \mathbb{R}$ has the intermediate value property, then for any open set $V \subset \mathbb{R}$ the counter image $\Phi^-(V)$ is dense-in-itself.*

P r o o f . Let $V \subset \mathbb{R}$ be an open set, and let $x_0 \in \Phi^-(V)$. Let us take an open set $U \subset I$ for which $x_0 \in \mathrm{cl}(U)$. Assume on the contrary that $U \cap \Phi^-(V) = \emptyset$ or $U \cap \Phi^-(V) = \{x_0\}$. Choose $x \in U$ such that $x \neq x_0$ and $(x, x_0) \subset U$. Let $y_0 \in \Phi(x_0) \cap V$, then, for any $y \in \Phi(x)$, $(y_0, y) \not\subset \Phi\big((x, x_0)\big)$. This contradicts that Φ has the intermediate value property. □

THEOREM 6. *Suppose $\Phi\colon I \to \mathbb{R}$ is upper first class. If Φ has the intermediate value property, then for any open set $V \subset \mathbb{R}$ the counter image $\Phi^-(V)$ is c-dense-in-itself.*

P r o o f . Let $V \subset \mathbb{R}$ be an open set, and let $x_0 \in \Phi^-(V)$. Let $U \subset I$ be an open set for which $x_0 \in \mathrm{cl}(U)$. Since Φ has the intermediate value property, by Theorem 5, there exists $x_1 \in U$ such that $\Phi(x_1) \cap V \neq \emptyset$. Let $y_1 \in \Phi(x_1) \cap V$, and let V_1 be an open set such that $y_1 \in V_1$ and $\mathrm{cl}(V_1) \subset V$. Let us take an open set W for which $x_1 \in W$ and $\mathrm{cl}(W) \subset U$, and define $D = W \cap \Phi^-(V_1)$. Since $\Phi^-(V_1)$ is dense-in-itself, the set D is dense-in-itself, too, and hence $\mathrm{cl}(D)$ is a perfect set. Since Φ is upper first class, the set $C_u\Big(\Phi|_{\mathrm{cl}(D)}\Big)$ of upper continuity points of $\Phi|_{\mathrm{cl}(D)}$ is residual in $\mathrm{cl}(D)$. Thus, it has continuum cardinality, and inclusions $C_u\Big(\Phi|_{\mathrm{cl}(D)}\Big) \subset \mathrm{cl}(W) \cap \Phi^-\big(\mathrm{cl}(V_1)\big) \subset U \cap \Phi^-(V)$ imply that $U \cap \Phi^-(V)$ has continuum cardinality. □

LEMMA 3. (Z a h o r s k i [9; Lemma 7]) *An open interval I cannot be the union of two non-empty, disjoint F_σ-sets A and B such that both of the sets are dense-in-itself.*

THEOREM 7. *Suppose $\Phi\colon I \to \mathbb{R}$, $\Phi(x)$ is closed for each $x \in I$, and Φ is both lower and upper first class. If for any open set $V \subset \mathbb{R}$ the counter images $\Phi^+(V)$ and $\Phi^-(V)$ are dense-in-itself, then Φ has the intermediate value property.*

P r o o f . Suppose, on the contrary, that Φ does not have the intermediate value property. Then, since Φ has closed values, numbers $x_1, x_2 \in I$, $x_1 \neq x_2$, and α_1, α_2, $\alpha_1 < \alpha_2$ (we admit $\alpha_1 = -\infty$ or $\alpha_2 = +\infty$), exist such that $\Phi(x_1) \cap (\alpha_1, \alpha_2) \neq \emptyset$, $\Phi(x_2) \subset [\alpha_1, \alpha_2]^c$ and $\Phi^-\big(\{\alpha_1, \alpha_2\}\big) \cap (x_1, x_2) = \emptyset$. Let us consider the following sets:

$$A^- = \Phi^-\big((\alpha_1, \alpha_2)\big) \cap [x_1, x_2]\,, \qquad A^+ = \Phi^+\big((\alpha_1, \alpha_2)^c\big) \cap [x_1, x_2]\,.$$

By the selection of the numbers α_1 and α_2, we have $A^- \cap A^+ = \emptyset$ and $A^- \cup A^+ = [x_1, x_2]$. By the assumption of the theorem, the sets A^- and A^+ are dense-in-itself F_σ-sets, contrary to Lemma 3. □

COROLLARY 2. (Zahorski [1]) *Let $f\colon I \to \mathbb{R}$ be first Baire's class. Then the following conditions are equivalent:*

(a) *f is a Darboux function.*
(b) *The set $f^{-1}(V)$ is dense-in-itself for any open set $V \subset \mathbb{R}$.*
(c) *The set $f^{-1}(V)$ is c-dense-in-itself for any open set $V \subset \mathbb{R}$.*

5. Upper continuity and the intermediate value property

THEOREM 8. *Suppose $\Phi\colon I \to \mathbb{R}$, $\Phi(x)$ is a compact set for any $x \in I$, and Φ has the intermediate value property. If there exists a dense subset Y of $\mathbb{R}$ such that for each $y \in Y$ the set $\Phi^-(y)$ is closed, then Φ is upper semicontinuous.*

Proof. Let $x_0 \in I$ and $U \subset \mathbb{R}$ be an open set such that $\Phi(x_0) \subset U$. Since $\Phi(x_0)$ is a compact set, there exist $a_i, b_i \in Y$, $1 \le i \le n$, such that $\Phi(x_0) \subset \bigcup_{i=1}^{n} (a_i, b_i)$, $\bigcup_{i=1}^{n} (a_i, b_i) \subset U$ and (a_i, b_i) are pairwise disjoint for $1 \le i \le n$. Since $\Phi^-\Big(\bigcup_{i=1}^{n} \{a_i, b_i\}\Big) = \bigcup_{i=1}^{n} \Phi^-(a_i) \cup \Phi^-(b_i)$ and each of the sets $\Phi^-(a_i)$, $\Phi^-(b_i)$, $1 \le i \le n$, is closed, then the set $\Phi^-\Big(\bigcup_{i=1}^{n} \{a_i, b_i\}\Big)$ is closed. Obviously $x_0 \notin \Phi^-\Big(\bigcup_{i=1}^{n} \{a_i, b_i\}\Big)$, therefore there exists an open set G such that $x_0 \in G$ and

$$G \subset I \setminus \Phi^-\Big(\bigcup_{i=1}^{n} \{a_i, b_i\}\Big). \tag{1}$$

We will show that $\Phi(G) \subset \bigcup_{i=1}^{n} (a_i, b_i)$. Let as assume that it is not true. Then there exist $x \in G$ and $y_* \in \Phi(x)$ such that $y_* \notin \bigcup_{i=1}^{n} (a_i, b_i)$ and, taking account of (1), $y_* \notin \bigcup_{i=1}^{n} [a_i, b_i]$. Therefore, for any $y \in \Phi(x_0)$, $(y_*, y) \not\subset \Phi\big((x, x_0)\big)$, and it means that Φ does not have the intermediate value property, a contradiction. □

COROLLARY 3. (Lipiński [7]) *Let $f\colon I \to \mathbb{R}$ be a Darboux function. If there exists a dense subset Y of $\mathbb{R}$ such that for each $y \in Y$ the set $f^{-1}(y)$ is closed, then f is continuous.*

Remark 1. If $\Phi\colon I \to \mathbb{R}$ has the intermediate value property, then $\Phi\big(\mathrm{cl}(E)\big) \subset \mathrm{cl}\big(\Phi(E)\big)$ for any connected set $E \subset I$.

P r o o f . Let us assume that $\Phi(\mathrm{cl}(E)) \not\subset \mathrm{cl}(\Phi(E))$ for some interval $E \subset I$ joining points a and b. Suppose that there exists $y_* \in \Phi(a) \setminus \mathrm{cl}(\Phi(E))$. Let $x \in E$, then, for any $y \in \Phi(x)$, $(y_*, y) \not\subset \Phi((a,x))$. It means that Φ does not have the intermediate value property. □

THEOREM 9. *If $\Phi\colon I \to \mathbb{R}$, $\Phi(x)$ is a closed set for any $x \in I$, Φ is upper semicontinuous and $\Phi(\mathrm{cl}(E)) \subset \mathrm{cl}(\Phi(E))$ for any connected set $E \subset I$, then Φ has the intermediate value property.*

P r o o f . Let us assume that Φ does not have the intermediate value property. Since Φ has closed values, numbers $x_1, x_2 \in I$, $x_1 \neq x_2$, and α_1, α_2, $\alpha_1 < \alpha_2$ (we admit $\alpha_1 = -\infty$ or $\alpha_2 = +\infty$), exist such that $\Phi(x_1) \cap (\alpha_1, \alpha_2) \neq \emptyset$, $\Phi(x_2) \subset [\alpha_1, \alpha_2]^c$ and $\Phi^-(\{\alpha_1, \alpha_2\}) \cap (x_1, x_2) = \emptyset$. Let us consider the following sets

$$A^- = \Phi^-((\alpha_1, \alpha_2)) \cap [x_1, x_2], \qquad A^+ = \Phi^+([\alpha_1, \alpha_2]^c) \cap [x_1, x_2].$$

Obviously, $A^- \cap A^+ = \emptyset$ and $A^- \cup A^+ = [x_1, x_2]$. Since Φ is upper semicontinuous, then A^+ is an open set in $[x_1, x_2]$. Let E be a component of A^+. Then $\Phi(\mathrm{cl}(E)) \not\subset \mathrm{cl}(\Phi(E))$, a contradiction. □

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Received July 8, 1994

Revised June 6, 1995

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