Mathematica Slovaca

Jozef Džurina

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Mathematica Slovaca, Vol. 42 (1992), No. 3, 299--315

Persistent URL: http://dml.cz/dmlcz/131008

Mathematica Slovaca
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Slovak Academy of Sciences

Math. Slovaca, 42 (1992), No. 3, 299–315

COMPARISON THEOREMS FOR NONLINEAR ODEs

JOZEF DŽURINA

ABSTRACT. In this paper the asymptotic properties of the solutions of the nonlinear differential equation

$$L_n u(t) + p(t) f\big(u[g(t)]\big) = 0$$

are deduced from those of the differential equation

$$M_n z(t) + q(t) h\big(z[\tau(t)]\big) = 0 .$$

As application of this comparison principle the sufficient conditions for the linear differential equation

$$L_n u(t) + p(t) u\big(g(t)\big) = 0$$

to have certain asymptotic behaviour are presented.

1. Introduction

On the basis of suitable comparison theorems this paper presents results concerning oscillatory and asymptotic properties of solutions of linear differential equations of the form

$$L_n u(t) + p(t) u(t) = 0 , \tag{1}$$

and

$$L_n u(t) + p(t) u\big(g(t)\big) = 0 , \tag{2}$$

where $n \geq 2$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{\mathrm{d}}{\mathrm{d}t} r_{n-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} r_{n-2}(t) \dots \frac{\mathrm{d}}{\mathrm{d}t} r_1(t) \frac{\mathrm{d}}{\mathrm{d}t} .$$

It is assumed that:

(3) $r_i(t)$ are continuous and positive on $[t_0, \infty)$, and

$$\int_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)} \to \infty \qquad \text{as} \quad t \to \infty , \quad 1 \leq i \leq n-1 .$$

AMS Subject Classification (1991): Primary 34C10.
Key words: Comparison, Deviating argument.

(4) $p(t)$ is continuous and with constant sign on $[t_0,\infty)$,

(5) $g(t)$ is continuous on $[t_0,\infty)$ and $g(t)\to\infty$ as $t\to\infty$.

In the sequel we will restrict our attention to those solutions of equations considered which exist on some ray $[t_0,\infty)$ and are nontrivial in any neighbourhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise. Considered equations are said to be oscillatory if all their solutions are oscillatory.

Throughout the paper we will use the following functions:

$$r_0(t)=r_n(t)\equiv 1 \qquad \text{on} \quad [t_0,\infty).$$

The following notation is employed:

$$\mathrm{D}_0(u;\, r_0)(t)=u(t)$$
$$\mathrm{D}_i(u;\, r_0,\dots,r_i)(t)=r_i(t)\big[\mathrm{D}_{i-1}(u;\, r_0,\dots,r_{i-1})(t)\big]', \qquad 1\le i\le n.$$

Equation (2) can then be rewritten as

$$\mathrm{D}_n(u;\, r_0,\dots,r_n)(t)+p(t)u\big(g(t)\big)=0.$$

If $u(t)$ is a nonoscillatory solution of (2), then according to a generalization of a lemma of K i g u r a d z e (Lemma 1 in [7]), there is an integer ℓ, $0\le\ell\le n$ such that $\ell\equiv n-1 \pmod 2$ if $p(t)>0$, and $\ell\equiv n \pmod 2$ if $p(t)<0$, and

$$\begin{aligned} u(t)\,\mathrm{D}_i(u;\, r_0,\dots,r_i)(t)&>0, & 0\le i\le\ell,\\ (-1)^{i-\ell}u(t)\,\mathrm{D}_i(u;\, r_0,\dots,r_i)(t)&>0, & \ell\le i\le n, \end{aligned} \tag{6}$$

for all sufficiently large t. A function $u(t)$ satisfying (6) is said to be a function of degree ℓ.

DEFINITION 1. *Equation (2) is said to have property (A) if for n even equation (2) is oscillatory and for n odd, every nonoscillatory solution $u(t)$ of (2) satisfies*

$$\mathrm{D}_i(u;\, r_0,\dots,r_i)(t)\to 0 \qquad \textit{as} \quad t\to\infty, \quad 0\le i\le n-1. \tag{P$_0$}$$

DEFINITION 2. *Equation (2) is said to have property (B) if, for n even, every nonoscillatory solution $u(t)$ of equation (2) satisfies either (P_0) or*

$$\big|\mathrm{D}_i(u;\, r_0,\dots,r_i)(t)\big|\to\infty \qquad \textit{as} \quad t\to\infty, \qquad 0\le i\le n-1, \tag{P$_n$}$$

and, for n odd, every nonoscillatory solution $u(t)$ of (2) *satisfies* (P_n).

We shall transfer some asymptotic properties of solutions of the Euler equation

$$t^n y^{(n)}(t) + \alpha y(t) = 0 \tag{7}$$

to the equation (2).

It is known that if b is a real root of the algebraic equation

$$-k(k-1)\dots(k-n+1) = \alpha\,, \tag{8}$$

then $y(t) = t^b$ is a nonoscillatory solution of (7) and if $b + \mathrm{i}\,c$ is a complex root of (8), than $y_1(t) = t^b \cos(c \ln t)$ and $y_2(t) = t^b \sin(c \ln t)$ are a couple of oscillatory solutions of (7).

In the next three lemmas we introduce some properties of the polynomial

$$P_n(k) = -k(k-1)\dots(k-n+1)$$

to be able to solve the equation (8) (and also (7)) graphically.

Lemma 1. *The polynomial $P_n(k)$*

(i) *has, for n even, $n/2$ local maxima and $n/2 - 1$ local minima,*
(ii) *has, for n odd, $(n-1)/2$ local maxima and $(n-1)/2$ local minima.*

Lemma 2. *The graph of $P_n(k)$*

(i) *is, for n even, symmetrical with respect to the line $x = (n-1)/2$,*
(ii) *is, for n odd, symmetrical with respect to the point $[(n-1)/2,\, 0]$.*

Lemma 3. *Let V_i be the local extreme of $P_n(k)$ on the interval $[i, i+1]$, for $i = 0, 1, \dots, n-2$. Then the sequence $(|V_i|)_{i=0,1,\dots,n-2}$*

(i) *is, for n even, decreasing for $i = 0, 1, \dots, n/2 - 1$ and increasing for $i = n/2 - 1,\, n/2, \dots, n-2$,*
(ii) *is, for n even, decreasing for $i = 0, 1, \dots, (n-3)/2$ and increasing for $i = (n-3)/2,\, (n-1)/2, \dots, n-2$.*

The proofs of the preceeding lemmas are simple and can be omitted. Now it is easy to see that the Euler equation (7) has for certain a value of parameter α, in addition to the solutions $u(t)$ satisfying either condition (P_0) or (P_n) also the solutions $u(t)$ satisfying for some $k \in \{1, 2, \dots, n-1\}$ the inequalities

$$\begin{aligned} \left|\mathrm{D}_i(u;\, r_0, \dots, r_i)(t)\right| &\to \infty \quad \text{as} \quad t \to \infty \quad \text{for} \quad 0 \le i \le k-1\,, \\ \mathrm{D}_i(u;\, r_0, \dots, r_i)(t) &\to 0 \quad \text{as} \quad t \to \infty \quad \text{for} \quad k \le i \le n-1\,, \end{aligned} \tag{P_k}$$

which for the equation (7) gain the form

$$\begin{aligned} \left|u^{(i)}(t)\right| &\to \infty \qquad \text{as} \quad t \to \infty \quad \text{for} \quad 0 \le i \le k-1\,, \\ u^{(i)}(t) &\to 0 \qquad \text{as} \quad t \to \infty \quad \text{for} \quad k \le i \le n-1\,. \end{aligned}$$

R e m a r k 1. Conditions $(\mathrm{P}_0),(\mathrm{P}_n)$ and (P_k) are equivalent to the conditions $\mathrm{D}_0(u;\,r_0)(t) \to 0$ as $t \to \infty$, $\left|\mathrm{D}_{n-1}(u;\,r_0,\dots,r_{n-1})(t)\right| \to \infty$ as $t \to \infty$ and

$$\begin{aligned} \left|\mathrm{D}_{k-1}(u;\,r_0,\dots,r_{k-1})(t)\right| &\to \infty \qquad \text{as} \quad t \to \infty\,, \\ \mathrm{D}_k(u;\,r_0,\dots,r_k)(t) &\to 0 \qquad \text{as} \quad t \to \infty\,, \end{aligned}$$

respectively.

R e m a r k 2. If a function $u(t)$ satisfies condition (P_k) for some $k \in \{0,1,\dots,n\}$, then $u(t)$ is a function of degree $\ell = k$.

From the above it is obvious that we can characterize the solutions of the Euler equation (7) more exactly than the above mentioned properties (A) and (B) permit. We define next properties of the equation (2) and investigate when (2) has those new properties.

Let m be a positive integer number, let $k_1, k_2, \dots, k_m$ be all mutually different integer numbers such that $1 \le k_i \le n-1$ and $k_i \equiv n-1 \pmod 2$ if $p(t) > 0$, $k_i \equiv n \pmod 2$ if $p(t) < 0$, for $1 \le i \le m$.

DEFINITION 3. *Equation* (2) *is said to have property* $\mathrm{A}_{k_1,k_2,\dots,k_m}$ *if for* n *even, every nonoscillatory solution* $u(t)$ *of* (2) *satisfies some kind of conditions* (P_{k_i}), *where* $1 \le i \le m$, *and for* n *odd, every nonoscillatory solution* $u(t)$ *of* (2) *satisfies either* (P_0) *or some kind of conditions* (P_{k_i}), *where* $1 \le i \le m$.

DEFINITION 4. *Equation* (2) *is said to have property* $\mathrm{B}_{k_1,k_2,\dots,k_m}$ *if for* n *even, every nonoscillatory solution* $u(t)$ *of* (2) *satisfies either* (P_0) *or* (P_n) *or some kind of conditions* (P_{k_i}), *where* $1 \le i \le m$ *and for* n *odd, every nonoscillatory solution* $u(t)$ *of* (2) *satisfies either* (P_n) *or some kind of conditions* (P_{k_i}), *where* $1 \le i \le m$.

It is clear from the above definitions that properties (A) and (B) are a special case of properties $\mathrm{A}_{k_1,k_2,\dots,k_m}$ and $\mathrm{B}_{k_1,k_2,\dots,k_m}$. Those new properties admit for the equation (2) to have special solutions of degree $\ell = k_i$, where $i \in \{1,2,\dots,m\}$.

We have defined properties (A), (B), $\mathrm{A}_{k_1,k_2,\dots,k_m}$ and $\mathrm{B}_{k_1,k_2,\dots,k_m}$ only for the linear differential equation (2), but we can use those definitions also for all equations and inequalities considered in the sequel (c.f. (12), (14), (15) etc.).

2. Preliminaries

The following lemma is elementary but quite useful in the sequel.

LEMMA 4. *Suppose that $U(t)$ is a continuous, positive and nondecreasing function on $[t_0, \infty)$. Let $p(t) \geq 0$ and $q(t) \geq 0$ and*

$$\int_t^\infty p(s)\,\mathrm{d}s \geq \int_t^\infty q(s)\,\mathrm{d}s\,, \qquad \text{for} \quad t \geq t_0\,.$$

Then

$$\int_t^\infty p(s)U(s)\,\mathrm{d}s \geq \int_t^\infty q(s)U(s)\,\mathrm{d}s\,, \qquad \text{for} \quad t \geq t_0\,.$$

To recall a result from [6] we need the functions $r(t)$ and $R(t)$ satisfying the following conditions:

(9) $r(t)$ is continuous and positive on $[t_0, \infty)$ and

$$R(t) = \int_{t_0}^t \frac{\mathrm{d}s}{r(s)} \to \infty \qquad \text{as} \quad t \to \infty\,.$$

THEOREM 1. *Let* (9) *hold. Then $y(t)$ is a solution of the equation*

$$\big(r(t)\ldots\big(r(t)y'(t)\big)'\ldots\big)' + p(t)y(t) = 0 \tag{10}$$

if and only if the function $v(t) = y\big(R^{-1}(t)\big)$ is a solution of the equation

$$v^{(n)}(t) + p\big(R^{-1}(t)\big)r\big(R^{-1}(t)\big)v(t) = 0\,. \tag{11}$$

Moreover there holds

$$\mathrm{D}_i(y;\, r, \ldots, r)(t) = v^{(i)}\big(R(t)\big)\,, \qquad 0 \leq i \leq n-1\,.$$

For the proof see Theorem 1.3 in [6].

In the sequel we will need a modification of the well-known result of T. Kusano and M. Naito (see Theorem 2 and Corollary 1 in [3]).

Let us consider the more general differential equation

$$L_n u(t) + p(t) f\big(u(g(t))\big) = 0\,, \tag{12}$$

for which (3) – (5) are assumed to hold and we suppose that

(13) f is continuous on $(-\infty, \infty)$ and $uf(u) > 0$ for $u \neq 0$.

THEOREM 2. *Suppose that f is nondecreasing.*

(i) *Let $p(t) > 0$.*
Then the equation (12) *has the property $\mathrm{A}_{k_1,k_2,\dots,k_m}$ if and only if the inequality*

$$\{L_n u(t) + p(t) f(u(g(t)))\} \operatorname{sgn} u(g(t)) \le 0 \tag{14}$$

has such a property.

(ii) *Let $p(t) < 0$.*
Then the equation (12) *has the property $\mathrm{B}_{k_1,k_2,\dots,k_m}$ if and only if the inequality*

$$\{L_n u(t) + p(t) f(u(g(t)))\} \operatorname{sgn} u(g(t)) \ge 0 \tag{15}$$

has such a property.

Proof.

(i) Suppose that (12) has the property $\mathrm{A}_{k_1,k_2,\dots,k_m}$. Let $u(t)$ be a nonoscillatory solution of (14). Without loss of generality we may suppose that $u(t)$ is eventually positive. According to a modification of the Kiguradze lemma there exist a t_1 and an integer $\ell \in \{0, 1, \dots, n-1\}$ such that $\ell \equiv n-1 \pmod 2$ and the inequalities (6) hold.

Let $\ell \in \{1, 2, \dots, n-1\}$. An integration of (14), with the aid of (6) yields

$$\begin{aligned} u(t) \ge u(t_1) + c_\ell \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \, \mathrm{d}s_\ell \dots \mathrm{d}s_1 \\ + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \int_{s_\ell}^{\infty} \frac{1}{r_{\ell+1}(s_{\ell+1})} \dots \int_{s_{n-1}}^{\infty} p(s_n) f(u(g(s_n))) \, \mathrm{d}s_n \dots \mathrm{d}s_1, \end{aligned} \tag{16}$$

where $c_\ell = \lim\limits_{t\to\infty} \mathrm{D}_\ell(u; r_0, \dots, r_\ell)(t)$. Then proceeding as in the proof of Theorem 2 in [3] we obtain that the integral equation

$$\begin{aligned} z(t) = u(t_1) + c_\ell \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \, \mathrm{d}s_\ell \dots \mathrm{d}s_1 \\ + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \int_{s_\ell}^{\infty} \frac{1}{r_{\ell+1}(s_{\ell+1})} \dots \int_{s_{n-1}}^{\infty} p(s_n) f(z(g(s_n))) \, \mathrm{d}s_n \dots \mathrm{d}s_1, \end{aligned} \tag{17}$$

has a solution $z(t)$ satisfying

$$u(t_1) \le z(t) \le u(t) \qquad \text{for} \quad t \in [t_1, \infty). \tag{18}$$

It is easy to verify that $z(t)$ is a solution of (12), is a function of degree ℓ and by integration of (14) and differentiation of (17) with the help of (18) we can see that

$$\begin{aligned} \mathrm{D}_i(z;\, r_0,\dots,r_i)(t) \le \mathrm{D}_i(u;\, r_0,\dots,r_i)(t) \qquad &\text{for} \quad 0 \le i \le \ell-1\,, \\ \mathrm{D}_\ell(z;\, r_0,\dots,r_\ell)(t) \ge c_\ell\,. & \end{aligned} \tag{19}$$

As $z(t)$ is a function of degree $\ell \ge 1$, then $z(t)$ must satisfy condition (P_{k_i}) for some $i \in \{1,2,\dots,m\}$ and so $\ell = k_i$. From (19) if follows that $u(t)$ satisfies (P_{k_i}) too.

Next let $\ell = 0$. Note that it is possible only when n is odd. We shall show that $u(t)$ satisfies (P_0). Let us admit that $u(t) \to c > 0$ as $t \to \infty$. Then repeated integration of (14) provides

$$u(t) \ge c + \int_t^\infty \frac{1}{r_1(s_1)} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty p(s_n) f\big(u(g(s_n))\big)\, \mathrm{d}s_n \dots \mathrm{d}s_1\,,$$

for $t \ge t_1$. Arguing as in the proof of the Theorem 2 in [3] we get that the integral equation

$$z(t) = c + \int_t^\infty \frac{1}{r_1(s_1)} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty p(s_n) f\big(z(g(s_n))\big)\, \mathrm{d}s_n \dots \mathrm{d}s_1$$

has a solution $z(t)$ with the property

$$c/2 \le z(t) \le u(t) \qquad \text{for} \quad t \in [t_1,\, \infty)\,.$$

Since $z(t)$ is a function of degree $\ell = 0$ and is a solution of (12), the last inequality contradicts the hypothesis.

We have verified that the inequality (14) has the property $\mathrm{A}_{k_1,k_2,\dots,k_m}$. The converse implication is obvious.

(ii) Let $u(t)$ be an eventually positive solution of (15). By a modification of the Kiguradze lemma we can find a t_1 and an integer $\ell \in \{0,1,\dots,n\}$ such that $\ell \equiv n \pmod 2$ and the inequalities (6) hold. If $\ell \in \{1,2,\dots,n-2\}$ or $\ell = 0$, then exactly as in Case (i) (we only replace the function $p(t)$ in all formulas by $|p(t)|$) it can be shown that $u(t)$ satisfies either (P_{k_i}) for some $i \in \{1,2,\dots,m\}$ or (P_0).

Let $\ell = n$. We shall verify that $u(t)$ satisfies (P_n). Suppose that $\mathrm{D}_{n-1}(u;\, r_0,\dots,r_{n-1})(t)$ is bounded. Integrating an inequality (15), we have

$$u(t) \ge u(t_1) + \int_{t_1}^t \frac{1}{r_1(s_1)} \cdots \int_{t_1}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_1}^{s_{n-1}} |p(s_n)| f\big(u(g(s_n))\big)\, \mathrm{d}s_n \dots \mathrm{d}s_1\,.$$

Taking the proof of the Theorem 2 from [3] into account we see that the integral equation

$$z(t) = u(t_1) + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \cdots \int_{t_1}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_1}^{s_{n-1}} |p(s_n)| f\big(z(g(s_n))\big)\, \mathrm{d}s_n \ldots \mathrm{d}s_1$$

has a solution $z(t)$, such that

$$u(t_1) \le z(t) \le u(t) \qquad \text{for} \quad t \in [t_1, \infty).$$

It is obvious that $z(t)$ is a solution of (12), and is a function of degree $\ell = n$. On the other hand, since $\mathrm{D}_{n-1}(u; r_0, \ldots, r_{n-1})(t)$ is bounded, integrating the inequality (15), we have

$$\int^{\infty} |p(t)| f\big(u(g(t))\big)\, \mathrm{d}t < \infty .$$

Noting that f is nondecreasing, the last inequality implies

$$\int^{\infty} |p(t)| f\big(z(g(t))\big)\, \mathrm{d}t < \infty . \tag{20}$$

An integration of (12) with $u = z$ yields

$$\mathrm{D}_{n-1}(z; r_0, \ldots, r_{n-1})(t) - \mathrm{D}_{n-1}(z; r_0, \ldots, r_{n-1})(t_1) = \int_{t_1}^{t} |p(s)| f\big(z(g(s))\big)\, \mathrm{d}s ,$$

which with the aid of (20) implies that $\mathrm{D}_{n-1}(z; r_0, \ldots, r_{n-1})(t)$ is bounded. It contradicts the hypothesis. The proof is complete.

Remark 3. An analogous assertion concerning the properties (A) and (B) can be found in [3].

THEOREM 3. (*Theorem 1 in* [2]). *Let* $p(t) < 0$ *and* f *be nondecreasing. Then the equation* (12) *has a nonoscillatory solution* $u(t)$ *satisfying*

$$\lim_{t \to \infty} \big|\mathrm{D}_{n-1}(u; r_0, \ldots, r_{n-1})(t)\big| = \text{const} > 0$$

if and only if

$$\left| \int^{\infty} p(t) f\Big(c\big(J(g(t))\big)\Big)\, \mathrm{d}t \right| < \infty , \tag{21}$$

for some $c \ne 0$, *where* $J(t) = \int_{t_0}^{t} \frac{1}{r_1(s_1)} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})}\, \mathrm{d}s_{n-1} \ldots \mathrm{d}s_1$.

3. Main results

The following comparison theorem enables to extend some properties of the Euler equation (7) to equation (1).

We are interested in comparing the oscillatory and asymptotic properties of the equation (12) with those of the equation

$$M_n z(t) + q(t)h\big(z(\tau(t))\big) = 0\,, \tag{22}$$

where

$$M_n = \frac{\mathrm{d}}{\mathrm{d}t} v_{n-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} v_{n-2}(t) \dots \frac{\mathrm{d}}{\mathrm{d}t} v_1(t) \frac{\mathrm{d}}{\mathrm{d}t}\,,$$

and the following conditions are assumed to hold:

(23) $v_i(t)$ are continuous and positive on $[t_0, \infty)$, and

$$\int_{t_0}^{t} \frac{\mathrm{d}s}{v_i(s)} \to \infty \qquad \text{as} \quad t \to \infty, \quad 1 \le i \le n-1\,,$$

(24) $q(t)$ is continuous and with constant sign on $[t_0, \infty)$,
(25) $\tau(t)$ is continuous on $[t_0, \infty)$ and $\tau(t) \to \infty$ as $t \to \infty$,
(26) h is continuous on $(-\infty, \infty)$ and $uh(u) > 0$ for $u \ne 0$,

We put formally $v_0(t) \equiv 1$ on $[t_0, \infty)$.

THEOREM 4. *Suppose that the following conditions are satisfied:*

(27) $r_i(t) \le v_i(t)$ *for* $t \ge t_0$, $1 \le i \le n-1$,
(28) $g(t) \ge \tau(t)$ *for* $t \ge t_0$,
(29) h, τ *are nondecreasing,*
(30) $\int_t^\infty |p(s)|\,\mathrm{d}s \ge \int_t^\infty |q(s)|\,\mathrm{d}s$, *for* $t \ge t_0$,
(31) $f(u)\operatorname{sgn} u \ge h(u)\operatorname{sgn} u$ *for* $u \in (-\infty, \infty)$.

(i) *Let* $p(t) > 0$ *and* $q(t) > 0$.
Then the equation (12) *has the property* $\mathrm{A}_{k_1,k_2,\dots,k_m}$ *if the equation* (22) *has this property.*

(ii) *Let* $p(t) < 0$, $q(t) < 0$ *and* f *be nondecreasing.*
Then the equation (12) *has the property* $\mathrm{B}_{k_1,k_2,\dots,k_m}$ *if the equation* (22) *has this property.*

Proof.

Part (i). Let $u(t)$ be a nonoscillatory solution of (12). Without loss of generality we may suppose that $u(t) > 0$ on $[t_0, \infty)$. There is an integer $\ell \in \{0, 1, \dots, n-1\}$, $\ell \equiv n-1 \pmod 2$ and a number $t_1 \ge t_0$ such that (6) holds for $t \ge t_1$. Assume that $\ell \ge 1$. Successive integrating (12) we have, for $t \ge t_1$,

$$u(t) \ge c_\ell \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \, \mathrm{d}s_\ell \dots \mathrm{d}s_1 \tag{32}$$
$$+ \int_{t_1}^{t} \frac{1}{r_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \int_{s_\ell}^{\infty} \frac{1}{r_{\ell+1}(s_{\ell+1})} \dots \int_{s_{n-1}}^{\infty} p(s_n) f\big(u(g(s_n))\big) \, \mathrm{d}s_n \dots \mathrm{d}s_1,$$

where $c_\ell = \lim\limits_{t\to\infty} \mathrm{D}_\ell(u; r_0, \dots, r_\ell)(t)$. Since $u(t)$ is increasing ($\ell \ge 1$), from (31), (29) and (28) we conclude that

$$\int_t^{\infty} p(s) f\big(u(g(s))\big) \, \mathrm{d}s \ge \int_t^{\infty} p(s) h\big(u(\tau(s))\big) \, \mathrm{d}s \,.$$

As composite function $h\big(u(\tau)\big)$ is nondecreasing, Lemma 4 and (30) imply

$$\int_t^{\infty} p(s) f\big(u(g(s))\big) \, \mathrm{d}s \ge \int_t^{\infty} q(s) h\big(u(\tau(s))\big) \, \mathrm{d}s \,, \qquad t \ge t_1 \,. \tag{33}$$

Combining (33) and (27) with (32) we see that

$$u(t) \ge c_\ell \int_{t_1}^{t} \frac{1}{v_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{v_\ell(s_\ell)} \, \mathrm{d}s_\ell \dots \mathrm{d}s_1 \tag{34}$$
$$+ \int_{t_1}^{t} \frac{1}{v_1(s_1)} \dots \int_{t_1}^{s_{\ell-1}} \frac{1}{v_\ell(s_\ell)} \int_{s_\ell}^{\infty} \frac{1}{v_{\ell+1}(s_{\ell+1})} \dots \int_{s_{n-1}}^{\infty} q(s_n) h\big(u(\tau(s_n))\big) \, \mathrm{d}s_n \dots \mathrm{d}s_1 .$$

Let us denote the right-hand side of (34) by $y(t)$. Repeated differentiation of $y(t)$ gives us

$$M_n y(t) + q(t) h\big(u(\tau(t))\big) = 0 \,, \qquad t \ge t_1 \,,$$

and we see that $y(t)$ is a function of degree ℓ. Since $u(\tau(t)) \geq y(\tau(t))$, for $t \geq t_2$, provided t_2 is large enough, we obtain

$$\{M_n y(t) + q(t)h(y(\tau(t)))\}\operatorname{sgn} y(\tau(t)) \leq 0, \qquad t \geq t_2. \tag{35}$$

It follows from Theorem 2 that the inequality (35) has the property $\mathrm{A}_{k_1,k_2,\ldots,k_m}$. We supposed that $\ell \geq 1$ and thus $y(t)$ satisfies (P_{k_i}) for some $i \in \{1,2,\ldots,m\}$ and so $\ell = k_i$. Repeated differentiation of $y(t)$ and integration of (12) leads to

$$\begin{aligned} \mathrm{D}_{\ell-1}(y;\, v_0,\ldots,v_{\ell-1})(t) &\leq \mathrm{D}_{\ell-1}(u;\, r_0,\ldots,r_{\ell-1})(t), \qquad t \geq t_2, \\ \mathrm{D}_{\ell}(y;\, v_0,\ldots,v_{\ell})(t) &\geq c_\ell, \qquad t \geq t_2, \end{aligned}$$

where we have used (33) and (27). From the above inequalities we see that $u(t)$ obeys condition (P_{k_i}).

Let $\ell = 0$. Our aim is to verify that (P_0) holds. If $u(t) \to c > 0$ as $t \to \infty$, then there exists a $t_2 \geq t_1$ such that

$$3c/2 \geq u(g(t)) \geq c, \qquad \text{for} \quad t \geq t_2. \tag{36}$$

Integrating (12), we see that

$$u(t) \geq c + \int_t^\infty \frac{1}{r_1(s_1)} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty p(s_n) f(u(g(s_n)))\,\mathrm{d}s_n \ldots \mathrm{d}s_1.$$

Noting

$$\begin{aligned} \int_t^\infty p(s) f(u(g(s)))\,\mathrm{d}s &\geq \int_t^\infty p(s) h(u(g(s)))\,\mathrm{d}s \\ &\geq \int_t^\infty p(s) h(c)\,\mathrm{d}s \geq \int_t^\infty q(s) h(c)\,\mathrm{d}s, \end{aligned} \tag{37}$$

and using (27) with (36) we can write for $t \geq t_2$

$$c \geq c/2 + \int_t^\infty \frac{1}{v_1(s_1)} \cdots \int_{s_{n-2}}^\infty \frac{1}{v_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty q(s_n) h(c)\,\mathrm{d}s_n \ldots \mathrm{d}s_1. \tag{38}$$

Denote the right-hand side of (38) by $y(t)$. Then

$$M_n y(t) + q(t)h(c) = 0, \qquad t \geq t_2.$$

We may suppose that $y(\tau(t)) \le c$ and then we can see that $y(t)$ is a solution of the inequality (35) and moreover $\lim_{t\to\infty} y(t) = c/2$, which contradicts the hypothesis as Theorem 2 insures that the inequality (35) has the property $\mathrm{A}_{k_1,k_2,\dots,k_m}$.

Part (ii). Let $u(t)$ be a nonoscillatory solution of (12) which is eventually positive. Then $u(t)$ is a function of degree $\ell \in \{0,1,\dots,n\}$, $\ell \equiv n \pmod 2$ and (6) holds for all large t, say $t \ge t_1$. If $\ell \in \{1,2,\dots,n-2\}$ or $\ell = 0$, then exactly as in Part (i) it can be shown that $u(t)$ satisfies either (P_{k_i}) for some $i \in \{1,2,\dots,m\}$ or (P_0). We only replace the function $p(t)$ in all formulas by $|p(t)|$ and use the inequality

$$\big\{M_n y(t) + q(t)h\big(y(\tau(t))\big)\big\}\operatorname{sgn} y(\tau(t)) \ge 0\,, \qquad t \ge t_2\,.$$

instead of (35).

Assume $\ell = n$. We will show that $u(t)$ satisfies (P_n). Let us suppose that $\lim_{t\to\infty} \mathrm{D}_{n-1}(u; r_0,\dots,r_{n-1})(t) = \text{const} > 0$. Applying Theorem 3 we see that (21) holds for some $c \ne 0$. It is clear that

$$J\big(g(t)\big) \ge J\big(\tau(t)\big) \ge I\big(\tau(t)\big)\,, \qquad t \ge t_1\,,$$

where $I(t) = \int\limits_{t_0}^{t} \frac{1}{v_1(s_1)} \cdots \int\limits_{t_0}^{s_{n-2}} \frac{1}{v_{n-1}(s_{n-1})}\,\mathrm{d}s_{n-1}\dots\mathrm{d}s_1$, $t \ge t_1$.
That is why

$$\big|f\big(cJ(g(t))\big)\big| \ge \big|h\big(cJ(g(t))\big)\big| \ge \big|h\big(cI(\tau(t))\big)\big|\,, \qquad t \ge t_1\,.$$

Consequently

$$\infty > \int\limits_{t_1}^{\infty} \big|p(s)f\big(cJ(g(s))\big)\big|\,\mathrm{d}s \ge \int\limits_{t_1}^{\infty} \big|p(s)h\big(cI(\tau(t))\big)\big|\,\mathrm{d}s \ge \int\limits_{t_1}^{\infty} \big|q(s)h\big(cI(\tau(s))\big)\big|\,\mathrm{d}s\,.$$

We have also used Lemma 4 to obtain the last inequality, which is according to Theorem 3 sufficient for the equation (22) to have a nonoscillatory solution $y(t)$ of degree $\ell = n$ such that $\lim_{t\to\infty} \mathrm{D}_{n-1}(y; v_0,\dots,v_{n-1})(t) = \text{const} > 0$. This is a contradiction with the fact that the equation (22) has the property $\mathrm{B}_{k_1,k_2,\dots,k_m}$. The proof of Theorem 4 is complete.

COROLLARY 1. *The above comparison theorem remains valid also for the properties* (A) *and* (B).

The above comparison theorem provides the following results for the equation (1).

Let $(M_i)_{i=1}^{\sigma}$ be the decreasing sequence of all mutually different local maxima of the polynomial $P_n(k) = -k(k-1)\dots(k-n+1)$. Let $(N_i)_{i=1}^{\lambda}$ be the increasing sequence of all mutually different local minima of the polynomial $P_n(k)$. Let us suppose that the functions $r(t)$ and $R(t)$ are given in (9) and

$$r(t) \geq \max\{r_i(t) \mid \ i = 1, 2, \dots, n-1\}. \tag{39}$$

Denote

$$\beta = (n-1)\liminf_{t\to\infty} R^{n-1}(t) \int_t^{\infty} p(s)\,\mathrm{d}s\,, \quad \gamma = (n-1)\limsup_{t\to\infty} R^{n-1}(t) \int_t^{\infty} p(s)\,\mathrm{d}s\,,$$

then the following theorems hold:

THEOREM 5. *Let* (39) *hold.*

(i) *If* $\beta > M_1$, *then the equation* (1) *has the property* (A).
(ii) *If* $\gamma < N_1$ *and* $n \geq 3$, *then the equation* (1) *has the property* (B).

THEOREM 6. *Let* (39) *hold. Let* n *be even.*

(i) *If* $M_i > \beta > M_{i+1}$, *for some* $i \in \{1, 2, \dots, \sigma-1\}$, *then the equation* (1) *has the property* $\mathrm{A}_{1,3,\dots,2i-1,\,n-2i+1,\dots,n-1}$.
(ii) *If* $N_i < \gamma < N_{i+1}$, *for some* $i \in \{1, 2, \dots, \lambda-1\}$, *then the equation* (1) *has the property* $\mathrm{B}_{2,4,\dots,2i,\,n-2i,\dots,n-2}$.

THEOREM 7. *Let* (39) *hold. Let* n *be odd.*

(i) *If* $M_1 > \beta > M_2$, *then the equation* (1) *has the property* A_{n-1}.
(ii) *If* $N_1 < \gamma < N_2$, *then the equation* (1) *has the property* B_1.
(iii) *If* $M_i > \beta > M_{i+1}$, *for some* $i \in \{2, 3, \dots, \sigma-1\}$ *and* i *is even (odd), then equation* (1) *has the property* $\mathrm{A}_{2,4,\dots,i,\,n-i+1,\dots,n-1}$ (*the property* $\mathrm{A}_{2,4,\dots,i-1,\,n-i,\dots,n-1}$).
(iv) *If* $N_i < \gamma < N_{i+1}$, *for some* $i \in \{2, 3, \dots, \lambda-1\}$ *and* i *is even (odd), then equation* (1) *has the property* $\mathrm{B}_{1,3,\dots,i-1,\,n-i,\dots,n-2}$ (*the property* $\mathrm{B}_{1,3,\dots,i,\,n-i+1,\dots,n-2}$).

THEOREM 8. *Let* (39) *hold.*

(i) *If* $M_\sigma > \beta > 0$ *and* n *is even (odd), then equation* (1) *has the property* $\mathrm{A}_{1,3,\dots,n-1}$ (*the property* $\mathrm{A}_{2,4,\dots,n-1}$).

(ii) *If* $N_\lambda < \gamma < 0$ *and* n *is even (odd), then equation* (1) *has the property* $\mathrm{B}_{2,4,\dots,n-2}$ (*the property* $\mathrm{B}_{1,3,\dots,n-2}$).

We give the outline of the proof of Theorem 5. For details the reader is referred, e.g., to paper [6]. Taking Lemmas 1 – 3 into account it is not difficult to verify that the Euler equation (7) has the property (A) if $\alpha \in (M_1, \beta)$. Noting that

$$\liminf_{t\to\infty} R^{n-1}(t)\int_t^\infty p(s)\,\mathrm{d}s = \liminf_{t\to\infty} t^{n-1}\int_t^\infty p\big(R^{-1}(s)\big)r\big(R^{-1}(s)\big)\,\mathrm{d}s\,,$$

Theorem 4 permits to transfer the property (A) from the equation (7) onto equation (11) under condition $\beta > M_1$. Theorem 1 insures that the equation (10) also has the property (A). Applying Theorem 4 to the equations (10) and (1) we can transfer the property (A) onto equation (1) as (39) holds.

Theorems 5 (i), (6), (7) and (8) are proved similarly.

We mention that Theorem 5 extends a result of J. Ohriska (see [6]), concerning (1) for $n = 3$ and 4 and part (i) of the Theorem 5 covers a result of T. A. Chanturija (see [1]), which is known for equation $y^{(n)} + p(t)y = 0$. Moreover Theorems 6 – 8 provide an information about the asymptotic behaviour of the solutions of (1) even if (1) has not the property (A) nor (B).

The next comparison theorem will serve to transfer the above mentioned properties from equation (1) onto equation (2).

THEOREM 9. *Let* $g(t) \in C^1([t_0, \infty))$, $g'(t) > 0$ *and* $g(t) \le t$. *Suppose that* f *is nondecreasing.*

(i) *Let* $p(t) > 0$.
Then the equation (12) *has the property* $\mathrm{A}_{k_1,k_2,\dots,k_m}$ *if equation*

$$L_n u(t) + \frac{p\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)} f\big(u(t)\big) = 0\,, \tag{40}$$

has.

(ii) *Let* $p(t) < 0$.
Then the equation (12) *has the property* $\mathrm{B}_{k_1,k_2,\dots,k_m}$ *if the equation* (40) *has.*

Proof.

Part (i). We can proceed similarly as in the proof of Theorem 4. Let $u(t)$ be a positive solution of (12) on $[t_0, \infty)$, which is a function of degree $\ell \geq 1$. Then $u(t)$ satisfies (32). Since $g'(t) > 0$ and $g(t) \leq t$ we have for all large t

$$\int_t^\infty p(s) f\big(u(g(s))\big)\,\mathrm{d}s \geq \int_{g(t)}^\infty \frac{p(g^{-1}(x))}{g'\big(g^{-1}(x)\big)} f\big(u(x)\big)\,\mathrm{d}x \geq \int_t^\infty \frac{p(g^{-1}(s))}{g'\big(g^{-1}(s)\big)} f\big(u(s)\big)\,\mathrm{d}s\,. \tag{41}$$

Putting (41) into (32) we conclude

$$u(t) \geq c_\ell \int_{t_1}^{t} \frac{1}{r_1(s_1)} \cdots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)}\,\mathrm{d}s_\ell \ldots \mathrm{d}s_1$$

$$+ \int_{t_1}^{t} \frac{1}{r_1(s_1)} \cdots \int_{t_1}^{s_{\ell-1}} \frac{1}{r_\ell(s_\ell)} \int_{s_\ell}^{\infty} \frac{1}{r_{\ell+1}(s_{\ell+1})} \cdots \int_{s_{n-1}}^{\infty} \frac{p(g^{-1}(s_n))}{g'\big(g^{-1}(s_n)\big)} f\big(u(s_n)\big)\,\mathrm{d}s_n \ldots \mathrm{d}s_1. \tag{42}$$

Again, denote the right-hand side of (42) by $y(t)$, verify that $y(t)$ is a solution of the inequality

$$\left\{ L_n y(t) + \frac{p(g^{-1}(t))}{g'\big(g^{-1}(t)\big)} f\big(y(t)\big) \right\} \operatorname{sgn} y(t) \leq 0\,.$$

Proceed as in Theorem 4, it can be shown that $\ell = k_i$ for some $i \in \{1, \ldots, m\}$ and $u(t)$ satisfies (P_{k_i}).

For $\ell = 0$ we can proceed exactly as in the proof of Theorem 4 and show that $u(t)$ satisfies (P_0).

Part (ii). Let $u(t)$ be a positive solution of (12) on $[t_0, \infty)$, which is a function of degree $\ell = n$ (in the case when $\ell < n$, we can proceed exactly as above). Let us assume that $\lim_{t\to\infty} \mathrm{D}_{n-1}(u;\, r_0, \ldots, r_{n-1})(t) = \text{const} > 0$. Then Theorem 3 implies that (21) holds for some $c \neq 0$. After the substitution $g(t) = s$ we have

$$\int^{\infty} \left| \frac{p(g^{-1}(s))}{g'\big(g^{-1}(s)\big)} f\big(cJ(s)\big) \right| \mathrm{d}s < \infty\,.$$

Theorem 3 shows that the last inequality is sufficient for the equation (40) to have a nonoscillatory solution $y(t)$ satisfying

$$\lim_{t\to\infty} |\,\mathrm{D}_{n-1}(y;\, r_0, \ldots, r_{n-1})(t)| = \text{const} > 0\,.$$

A contradiction. The proof is complete now.

The purpose of the following comparison theorem is to reduce the condition of smoothness and monotonicity on $g(t)$ imposed in Theorem 9.

Let function $Q(t)$ satisfies the conditions:

$$Q(t) \in C^1([t_0, \infty)), \quad Q'(t) > 0, \quad Q(t) \le g(t) \quad \text{and} \quad Q(t) \le t. \tag{43}$$

THEOREM 10. *Suppose that f is nondecreasing.*

(i) *Let $p(t) > 0$.*

Then the equation (12) has the property $A_{k_1,k_2,\dots,k_m}$ if equation

$$L_n u(t) + \frac{p(Q^{-1}(t))}{Q'(Q^{-1}(t))} f(u(t)) = 0, \tag{44}$$

has.

(ii) *Let $p(t) < 0$.*

Then the equation (12) has the property $B_{k_1,k_2,\dots,k_m}$ if the equation (44) has.

Proof. It follows immediately from Theorem 4 and Theorem 9.

COROLLARY 2. *The above comparison theorem remains valid also for the properties (A) and (B).*

Theorem 10 extends a result of W. Mahfoud (Theorem 3 in [4]) concerning equation $y^{(n)}(t) + p(t) f(y(g(t))) = 0$.

Applying Theorem 10 to the equation (2) we see that this equation can "inherit" the property $A_{k_1,k_2,\dots,k_m}$ ($B_{k_1,k_2,\dots,k_m}$) from equation

$$L_n u(t) + \frac{p(Q^{-1}(t))}{Q'(Q^{-1}(t))} u(t) = 0. \tag{45}$$

On the other hand, Theorems 5 – 8 provide sufficient conditions for the equation (45) to have the property $A_{k_1,k_2,\dots,k_m}$ ($B_{k_1,k_2,\dots,k_m}$). Consequently, the following theorem holds:

Let us denote

$$\beta_1 = (n-1) \liminf_{t \to \infty} R^{n-1}(Q(t)) \int_t^\infty p(s)\,\mathrm{d}s,$$

and

$$\gamma_1 = (n-1) \limsup_{t \to \infty} R^{n-1}(Q(t)) \int_t^\infty p(s)\,\mathrm{d}s.$$

THEOREM 11. *Let* (39) *and* (43) *hold.*

(i) *If* $\beta_1 > M_1$, *then equation* (2) *has the property* (A).

(ii) *If* $\gamma_1 < N_1$ *and* $n \geq 3$, *then equation* (2) *has the property* (B).

(iii) *If* $M_i > \beta_1 > M_{i+1}$, *then equation* (2) *has the property* $\mathrm{A}_{k_1,k_2,\dots,k_m}$.

(iv) *If* $N_i < \gamma_1 < N_{i+1}$, *then equation* (2) *has the property* $\mathrm{B}_{k_1,k_2,\dots,k_m}$,

where the numbers i *and* $k_1, k_2, \dots, k_m$ *are the same as in Theorems* 6–7.

Example 1. Let us consider the equation

$$y^{(4)}(t) + \frac{0.6}{t^2\sqrt{t}}\, y(\sqrt{t}) = 0\,, \qquad t \geq 1\,. \tag{46}$$

Note that M. Naito's result (see [5]) cannot be applied for (46). For this equation we can put $r(t) = 1$ and so $R(t) = t - 1$ and

$$\beta_1 = 3 \liminf_{t\to\infty} Q^3(t) \int_t^\infty p(s)\,\mathrm{d}s = 1.2\,.$$

As $M_1 = 1$ for the polynomial $P_4(k) = -k(k-1)(k-2)(k-3)$, Theorem 11 secures that the equation (46) has the property (A) (i.e. (46) is oscillatory).

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Received July 26, 1990

Revised September 23, 1991

Department of Mathematical Analysis
Šafárik University
Jesenná 5
041 54 Košice
Czecho-Slovakia