Martin Škoviera

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THE DECAY NUMBER AND THE MAXIMUM GENUS OF A GRAPH

MARTIN ŠKOVIERA

ABSTRACT. Let $\zeta(G)$ be the minimum number of components in a cotree of a connected graph G. In the paper this number is computed for cubic graphs, and for 2-connected graphs of diameter 2 a sharp upper bound is obtained. Using these results, a formula of Bouchet determining the maximum genus of truncated cubic graphs is reproved and the bounds $\lceil\beta(G)/2\rceil - 2 \le \gamma_M(G) \le \lfloor\beta(G)/2\rfloor$ for the maximum genus of 2-connected graphs of diameter 2 with possibly loops and multiple edges are established. This completes the results of [8], where the maximum genus of graphs of diameter 2 with connectivity 1 and the maximum genus of loopless graphs of diameter 2 is computed.

1. Introduction

The well-known theorem of Xuong [9] states that to compute the maximum genus of a graph G it suffices to determine the minimum number $\xi(G)$ of components of a cotree of G that have odd size (= number of edges). This is easy if G is known to have a connected cotree but otherwise the computation may be more problematic. Sometimes it is still possible to find an optimal cotree (one with $\xi(G)$ odd-size components) by applying successively appropriate transformations to some initial cotree of G (see, e.g., [7, 8]). In the opposite case, one can perhaps bound the minimum simply by the minimum number $\zeta(G)$ of components in a cotree of G. Although this number, called decay number here, may be of independent interest, in this paper we pursue mainly its relationship with the maximum genus.

We give two applications of this concept to obtain results on the maximum genus of a graph. We first reprove a theorem due to Bouchet [2] which determines the maximum genus of truncated cubic graphs, using a formula for the decay number of cubic graphs established in Section 3. In the next section we show that the decay number of a 2-connected graph of diameter 2 does not exceed 4. As a consequence we obtain that for a 2-connected graph G

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(homeomorphic to a graph) of diameter 2 with possibly multiple edges and loops added to some vertices one has

$$\lceil \beta(G)/2 \rceil - 2 \le \gamma_M(G) \le \lfloor \beta(G)/2 \rfloor .$$

This result does not extend to graphs with connectivity 1; however, in this case an exact formula for the maximum genus is available [8]. On the other hand, a graph of diameter 2 without loops is always upper embeddable, i.e., $\gamma_M(G) = \lfloor \beta/2 \rfloor$, [8].

The terminology adopted will be essentially that of [1]. In particular, we use *size* for the number of edges. If not explicitly stated otherwise, a graph $G = \big(V(G), E(G)\big)$ will be non-empty, finite and undirected, with possibly multiple edges and loops. Without loss of clarity, all paths considered here will be encoded by the corresponding vertex-sequence only.

The *maximum genus* $\gamma_M(G)$ of a connected graph G is the largest genus of an orientable surface on which G has a 2-cell embedding.

If $\beta(G) = |E(G)| - |V(G)| + 1$ is the *Betti number* (= *cycle rank*) of G, then the invariant $\xi(G) = \beta(G) - 2\gamma_M(G)$ is referred to as the *Betti deficiency* of G.

There are two results due to Xuong and Nebeský, respectively, giving together a complete combinatorial characterization of $\xi(G)$. For a spanning tree T of G, let $\xi(G,T)$ denote the number of odd-size components in the cotree $G - E(T)$ of T. Then $\xi(G) = \min\{\xi(G,T);\ T \text{ a spanning tree of } G\}$, [9]. To state the other result, for $A \subseteq E(G)$ denote by $p(G - A)$ ($i(G - A)$, respectively) the number of components of $G - A$ with even (odd) Betti number. Then $\xi(G) = \max\{p(G - A) + 2i(G) - |A| - 1;\ A \subseteq E(G)\}$, [6].

2. The decay number of a graph

Let G be a connected graph and T one of its spanning trees. Denote by $\zeta(G,T)$ the number of components of the corresponding cotree $G - E(T)$, and define

$$\zeta(G) = \min\{\zeta(G,T);\ T \text{ a spanning tree of } G\} .$$

We shall call this number the *decay number* of G. Note that by a result of Kundu [5], $\zeta(G) = 1$ whenever G is 4-edge-connected. Therefore the decay number is interesting only for graphs with small connectivity.

The value of $\zeta(G,T)$, and in particular $\zeta(G)$ itself, is closely related to the Betti number of $G - E(T)$. Since $\beta(H) = |E(H)| - |V(H)| + k$ if H is a graph with k components, for $C = G - E(T)$ we have:

$$\beta(C) = |E(C)| - |V(C)| + \zeta(G,T) = |E(G)| - 2|V(G)| + \zeta(G,T) + 1 .$$

Hence,

$$\zeta(G,T) = \beta(G - E(T)) + 2|V(G)| - |E(G)| - 1\,.$$

It follows that to minimize $\zeta(G,T)$ one has to minimize $\beta(G - E(T))$. Thus we have the following:

PROPOSITION 2.1. *Let G be a connected graph. Then*

$$\zeta(G) = \min(\beta(G - E(T)) + 2|V(G)| - |E(G)| - 1)\,,$$

where the minimum is taken over all spanning trees T of G. □

This equality can be especially useful when the graph in question is known to have an acyclic cotree.

COROLLARY 2.2. *If G is a graph with p vertices and q edges that has an acyclic cotree, then*

$$\zeta(G) = 2p - q - 1\,.$$

□

From X u o n g 's characterization theorem [9] mentioned above it follows that $\xi(G) \leq \zeta(G)$ for every connected graph G. Thus the decay number can serve to obtain lower bounds for the maximum genus as, for instance, in Section 4. Another, a less direct, connection between these two concepts will be described in the next section.

3. Cubic graphs

In this section we establish an exact formula for the decay number of a cubic graph and subsequently apply it to obtain a result concerning the maximum genus. It turns out, however, that the straightforward application of the result to bound the Betti deficiency is not very useful since better bounds are known ([3]). Instead, we show that it can be used to give a simple proof of a theorem of B o u c h e t [2] determining the maximum genus of truncated cubic graphs.

First of all observe that every cotree of a cubic graph is necessarily the union of paths and cycles. Moreover, if the graph in question has no loops and more than two vertices, then we can successively get rid of cycles, eventually obtaining an acyclic cotree. This interesting fact was first noted by K h o m e n k o and G l u k h o v [4] for simple graphs. We present a different proof which allows us to include multiple edges.

THEOREM 3.1. *Every connected loopless cubic graph G with $n \geq 4$ vertices admits a spanning tree with acyclic cotree. Consequently, its decay number is*

$$\zeta(G) = n/2 - 1\,.$$

Proof. Let T be a spanning tree of G. If $G - E(T)$ is acyclic, there is nothing to prove. If not, let $u_1u_2\ldots u_k = q$ ($k \geq 2$) be a cycle in $G - E(T)$. Since G is cubic, each vertex u_i is adjacent to some vertex v_i such that the edge u_iv_i is in T. (Note that v_i and v_j need not be distinct for $i \neq j$.) Clearly, no v_i lies on q for otherwise $T = u_iv_i$ and G would have only two vertices, contrary to assumption. Thus u_iv_i, $1 \leq i \leq k$, are mutually distinct edges. Now replace the edges u_iv_i of T, $2 \leq i \leq k$, by u_iu_{i+1}, $1 \leq i \leq k-1$, to obtain a new spanning tree S of G. By checking all the involved edges it is easily seen that the cycle q has been destroyed but no new cycle in the cotree has been formed. Thus the number of cycles of $G - E(S)$ is smaller than that of $G - E(T)$. The required spanning tree is now obtained by repeating this process as many times as necessary. The formula for $\zeta(G)$ follows immediately, using Corollary 2.2. □

The following more general result can easily be obtained from Theorem 3.1 by induction on the number of loops. Details are left to the reader.

COROLLARY 3.2. *Let G be a connected cubic graph of order n containing s loops. Then*

$$\zeta(G) = \begin{cases} 1, & \text{if } n = 2 \text{ and } s = 0, \\ n/2 + s - 1, & \text{otherwise.} \end{cases}$$

□

We now show how the above theorem can be used to determine the maximum genus of truncated cubic graphs. The result was previously proved by Bouchet employing rotation systems. Below we give a simple combinatorial proof revealing a relationship of this result with the decay number. Before stating the theorem, recall that the *truncation* of a cubic graph G is the cubic graph $T(G)$ obtained from G by replacing every vertex v of G with a triangle and joining the edges originally incident with v to distinct vertices of the triangle.

THEOREM 3.3. *Let G be a connected loopless cubic graph with $n \geq 4$ vertices. Then the Betti deficiency of $T(G)$ is*

$$\xi\big(T(G)\big) = n/2 - 1 = \zeta(G)\,,$$

and consequently

$$\gamma_M(T(G)) = n/2 + 1 .$$

P r o o f . Denote $T(G)$ by G'. For each vertex v of G let t_v be the triangle of G' obtained from v. The edges of G' not lying in these triangles correspond bijectively to the edges of G. Let e' be the edge of G' corresponding to $e \in E(G)$.

First we construct a spanning tree S' of G' with $\xi(G', S') \le n/2-1$. Clearly, for each spanning tree U of G one can extend the set $U' = \{e';\ e \in E(U)\}$ to a spanning tree of G' by adding to U' any two edges of each triangle t_v, $v \in V(G)$. By Theorem 3.1 we can choose in G a spanning tree W the cotree of which is a forest with $n/2 - 1$ components. Thus each non-trivial component P of $G - E(W)$ is a path. For e in $E(P)$ choose an orientation such that P will be oriented from one end-vertex into the other. Assign e' the orientation agreeing with that of e. In addition to $W' = \{e';\ e \in E(W)\}$, for each edge $e = uv$ of P include to S' the edge of t_u not incident with the initial vertex of e' and one of the two remaining edges of t_u (see Fig. 1). Thus e' is paired in $G' - E(S')$ with the third edge of t_u. For each component of $G - E(W)$ there is one triangle t_x not used so far, namely the one corresponding to its last (possibly isolated) vertex. For each of these triangles include to S' two edges arbitrarily. It is easily seen from our construction that odd-size components in $G' - E(S')$ can only arise in this way. Hence $\xi(G') \le \xi(G', S') \le n/2 - 1$. On the other hand, taking $E' = \{e';\ e \in E(G)\}$ we have $\xi(G') \ge p(G' - E') + 2i(G' - E') - |E'| - 1 = 0 + 2n - 3n/2 - 1 = n/2 - 1$. Thus $\xi(G') = n/2 - 1$, completing the proof. □

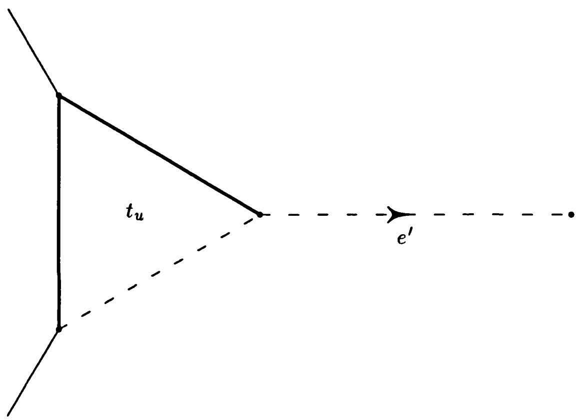


Figure 1. Dashed lines $= G' - E(S')$.

The above construction of S' can easily be modified to show (using elementary counting arguments) that G' admits a spanning tree whose cotree is the disjoint union of $n/2-1$ independent edges, $n/2+1$ paths of length 2 and $n/2-1$ isolated vertices.

Finally, the reader may observe that the proof of Theorem 3.3 uses only the easy part of the Nebesky theorem [6], namely the inequality $\xi(G) \geq p(G-A)+2i(G-A)-A-1$. This supports our claim that the proof is indeed simple.

4. Graphs of diameter two

The aim of this section is to establish

THEOREM 4.1. *Let G be a 2-connected graph of diameter 2 with possibly multiple edges and some loops added, or a graph homeomorphic to it. Then $\xi(G) \leq 4$ and consequently*

$$\lceil \beta(G)/2 \rceil - 2 \leq \gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor .$$

In view of the inequality $\xi(G) \leq \zeta(G)$ it suffices to show the following to be true:

THEOREM 4.2. *Every 2-connected graph G of diameter 2 has*

$$\zeta(G) \leq 4 .$$

The rest of this section is devoted to proving this theorem. In order to do so, we first develop the necessary technique.

Let P be a path in G with initial vertex u and terminal vertex v. To emphasize its end-vertices we shall often write $P = P(u,v)$. More generally, if x and y are two vertices on P, then $P(x,y)$ will denote the $x-y$ segment of P, and $P(y,x)$ the path obtained from $P(x,y)$ by reversing the order. The length of P will be denoted by $l(P)$ and the usual distance between vertices u and v in a subgraph $H \subseteq G$ by $d_H(u,v)$. We reserve the symbol $D(u,v)$ to denote a shortest path from u to v in a graph of diameter 2. Finally, if T is a tree, then $T(u,v)$ will denote the unique $u-v$ path in T.

Let T be a spanning tree of a connected graph K and let $C_1, C_2, \dots$ be the components of the corresponding cotree $K - E(T)$. Define a vertex labelling on K by putting $\psi_T(v) = i$ if and only if v is contained in C_i (the subscript T will be suppressed whenever T is known from context). If L is a subgraph of K, let $|\psi(L)|$ be the number of different labels used in L. Further, define an *m-fragment* of T to be a subtree $F \subseteq T$ of minimum order such that $|\psi(F)| \geq m$. For instance, a 2-fragment is an edge of T whose end-vertices belong to different components of $K - E(T)$. If F is an m-fragment for some m but m is irrelevant, then we simply refer to F as a *fragment* of T.

We summarize the basic properties of fragments in

LEMMA 4.3. *Let F be an m-fragment of a spanning tree T in a graph K. Then:*

(i) $|\psi(F)| = m$.
(ii) *Distinct end-vertices of F have different labels under ψ.*
(iii) *If u is an end-vertex and v an internal vertex of F, then $\psi(u) \neq \psi(v)$.*
(iv) *If $m \geq 3$, then F has at most $m-1$ end-vertices.*

Proof.

(i) – (iii): Assume the contrary. Then for some end-vertex w of F, $F-w$ too has $|\psi(F-w)| \geq m$, contradicting the minimality of F.

(iv): If $m \geq 3$, then some vertex of F is internal. Since at least one of m values of $\psi(F)$ is occupied by an internal vertex, from (iii) we deduce that at most $m-1$ values are assigned to end-vertices of F. Combining this with part (ii) we obtain the desired result. □

We shall say that a path of length 2 is *alternating* (with respect to a given spanning tree T of K) if one of its edges belongs to T but the other does not. As mentioned earlier, a shortest $u-v$ path in a graph of diameter 2 will always be denoted by $D(u,v)$.

LEMMA 4.4. *Let T be a spanning tree of a graph K, and let u and v be end-vertices of a subtree $S \subseteq T$ with $d_S(u,v) \geq 3$ and $\psi_T(u) \neq \psi_T(v)$. Then each $u-v$ path D of length ≤ 2 is alternating. Moreover, if S is a fragment of T, then the internal vertex of D does not belong to S.*

Proof. As $d_T(u,v) \geq 3$, D is not contained in T. On the other hand, D is not contained in $G - E(T)$ for $\psi(u) \neq \psi(v)$. Thus D is alternating. Finally, if S is a fragment, then the rest follows from Lemma 4.3 (iii). □

Part (iv) of Lemma 4.3 implies that, in particular, any 3-fragment is homeomorphic to K_2, any 4-fragment is homeomorphic to K_2 or $K_{1,3}$. For convenience we shall say that an m-fragment $F \subseteq T$ is of *type* M if F is homeomorphic to M. Further, for any vertex x of K let $F \circ x$ denote the subtree of T of minimum order containing both F and x.

Our last lemma, showing that certain positions of type $K_{1,3}$ fragments in graphs of diameter 2 are not possible, will prove important for establishing Theorem 4.2.

LEMMA 4.5. *Let G be a simple graph of diameter 2, let $T \subseteq G$ be a spanning tree with $\zeta(G,T) = \zeta(G)$ and let $F \subseteq T$ be a 4-fragment of type $K_{1,3}$ with* *end-vertices* y_0, y_1 *and* y_2. *Then at most one of the following two conditions can be satisfied (cf. Figs.* 2 *and* 3*):*

(i) G *contains alternating paths* $x_0x_1y_0$ *and* $x_1x_2y_1$, *where* x_0 *is not in* F *and both* x_1 *and* x_2 *are not in* $F \circ x_0$.

(ii) *For every vertex* f *of* F *with* $\psi(f) = \psi(x_0)$ *we have that*

$$d_{G-E(T)}(f, x_0) \leq 2.$$

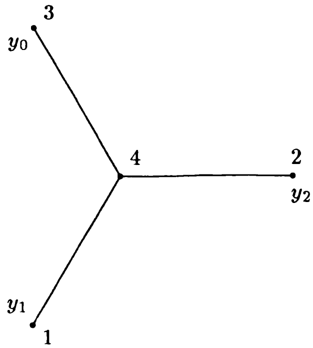


Figure 2.

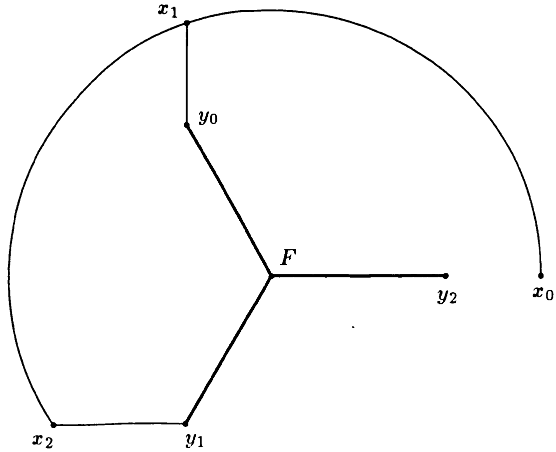


Figure 3.

P r o o f . Assume that F is a 4-fragment of type $K_{1,3}$ satisfying both (i) and (ii). In order to prove this lemma we shall inductively construct an infinite strictly increasing chain $G^0 \subset G^1 \subset \dots$ of subgraphs of G, contradicting the fact that G is a finite graph. Put $G^0 = F \circ x_0$. Further, for every $i \geq 0$, put $y_{i+3} = y_i$, where y_0, y_1 and y_2 are the end-vertices of F (see Fig. 2). We wish to define G^{n+1} by setting $G^{n+1} = G^n + x_nx_{n+1}y_n$, where the paths $x_0x_1y_0$ and $x_1x_2y_1$ are given (see Fig. 3) and for all $n \geq 2$

(a) $d_{G^n}(x_n, y_n) \geq 3$,

(b) the path $x_nx_{n+1}y_n$ is alternating, and

(c) the vertex x_{n+1} does not belong to G^n.

Assume inductively that the graphs $G^0, G^1, \dots, G^n$, $n \geq 2$, have already been constructed. We shall show that it is possible to construct G^{n+1} (cf. Fig. 4). Clearly, $V(G^n) - V(G^0) = \{x_1, x_2, \dots x_n\}$. Property (c) implies that $\deg_{G^n}(x_n) = 2$ and that x_i, $1 \leq i \leq n-1$, is adjacent in G^n just to x_{i-1}, x_{i+1} and y_{i-1}. Thus it is clear that $d_{G^n}(x_n, y_n) \geq 3$, confirming (a). It follows that no path of length ≤ 2 joining x_n to y_n is contained in G^n. We shall show that $x_nx_{n+1}y_n = D(x_n, y_n)$ is alternating and its internal vertex x_{n+1} does not belong to G^n, thereby verifying (b) and (c).

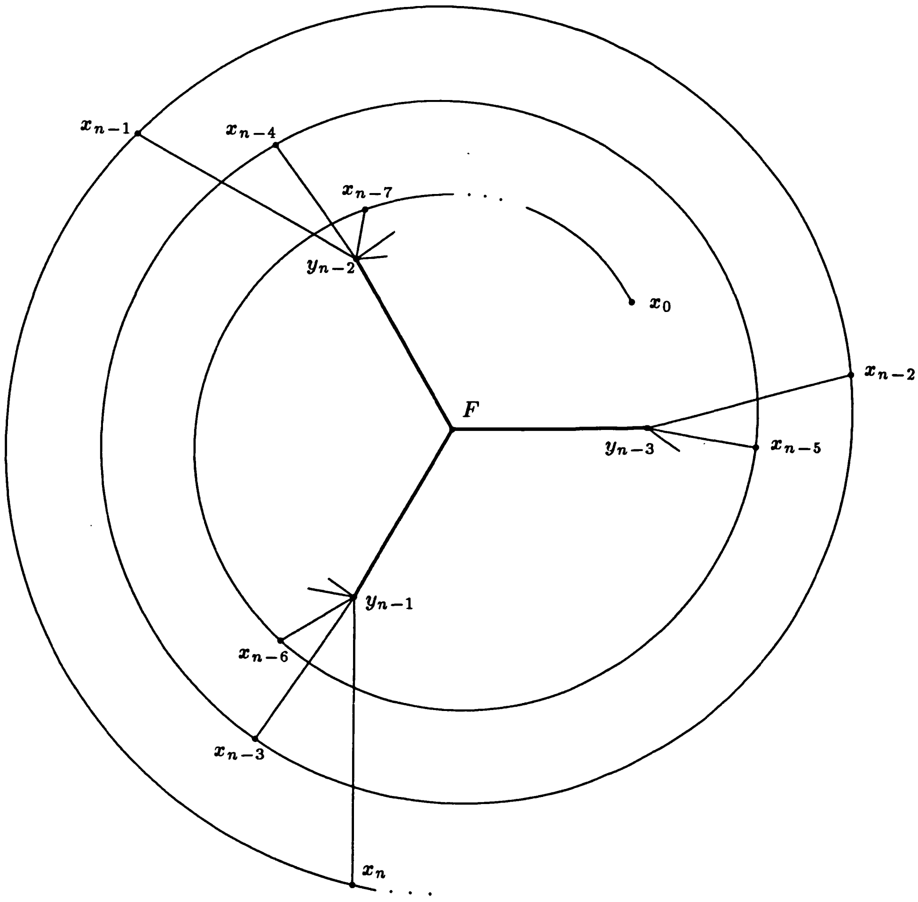


Figure 4. The construction of G_{n+1} .

Proof of (b): Considering the alternating path $x_{i-1}x_iy_{i-1}$, we see that x_i is joined in G^n to $\{x_{i-1}, y_{i-1}\}$ by an edge of T and by an edge of $C = G - E(T)$. Thus there exist paths $T(x_i)$ in T and $C(x_i)$ in C joining x_i to G_0, both being of the form $x_ix_{i-1}\dots x_{j+1}x_jz$, where $z = y_{j-1}$ or $z = x_0$. Consider the paths $T(x_n)$ and $C(x_n)$ and suppose that their terminal vertices are p and q, respectively. If $D(x_n, y_n)$ is contained in T, then $D(y_n, x_n)T(x_n)G^0(p, y_n)$ is a cycle in T, a contradiction. On the other hand, if $D(x_n, y_n)$ is contained in C, then we show that there is a cycle L in C containing x_nx_{n-1}. Consider the walk $D(y_n, x_n)C(x_n)$ in C. First assume that $C(x_n) = x_nx_{n-1}\dots x_jy_{j-1}$. In this case $\psi(y_n) = \psi(y_{j-1})$, but this is only possible when $y_n = y_{j-1}$ since they are both end-vertices of F. Thus we may choose L to be the cycle $D(y_n, x_n)C(x_n)$. If $q = x_0$, then $C(x_n) = x_n x_{n-1} \dots x_1 x_0$, whence $\psi(y_n) = \psi(x_0)$. By our assumption, $d_C(y_n, x_0) \leq 2$, so there is a path $D(x_0, y_n)$ in C. On the other hand, our construction implies that the path $D(y_n, x_n)C(x_n)$ has the length at least 3. There are three cases to consider according to the length of $D(x_0, y_n)$, which is 1 or 2, and to the position of the internal vertex h of $D(x_0, y_n)$ if the length is 2.

Case 1. Either the length of $D(x_0, y_n)$ is equal to 1 or it is equal to 2 and h does not lie in $C(x_n)$. Then put $L = D(y_n, x_n)C(x_n)D(x_0, y_n)$.

Case 2. $h = x_i$ for some i: $1 \leq i \leq n-1$. Then choose the segment $x_n x_{n-1} \dots x_{i+1} x_i$ of the path $C(x_n)$ and put

$$L = D(y_n, x_n) x_n x_{n-1} \dots x_{i+1} x_i y_n .$$

Case 3. $h = x_n$. Then take $L = C(x_n) x_0 x_n$.

Clearly h cannot belong to F, so the above cases cover all the possibilities. We have thus constructed a cycle L in C containing the edge $x_n x_{n-1}$. Since $\psi(y_{n-1}) \neq \psi(y_{n-2})$, the path $F(y_{n-1}, y_{n-2})$ contains a 2-fragment of T, which is an edge that we denote by e. Setting $S_1 = T + x_n x_{n-1} - e$ we obtain that $\zeta(G, S_1) < \zeta(G, T)$, a contradiction. As a result, the path $D(x_n, y_n)$ is contained neither in T nor in C, and so it is alternating.

Proof of (c): Suppose that the internal vertex of $D(x_n, y_n)$ (denoted by x_{n+1}) does belong to G^n. Since the path $x_n x_{n+1} y_n$ is alternating, one of its edges belongs to $T \cap G^n$. Several cases occur, depending on which of the two edges belongs to T, and on the position of x_{n+1} in G^n. It is not difficult to see that it is enough to consider the following situations:

1. x_{n+1} is identical with some x_j adjacent to y_n (thus $j \equiv n+1 \pmod 3$);
2. x_{n+1} is a vertex of F adjacent to y_n;
3. $x_{n+1} = y_{n-1}$;
4. $x_{n+1} = x_{n-1}$.

We shall consider only the first (and at the same time most difficult) case.

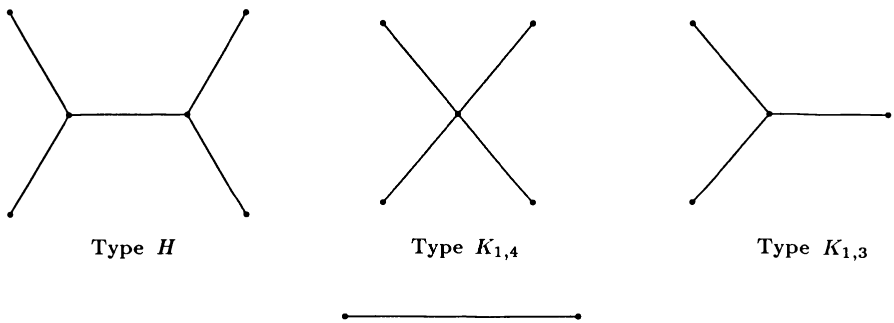
Assume that $x_{n+1} = x_j$, where $j \equiv n+1 \pmod 3$. It follows that the edge $x_j y_{j-1} = x_{n+1} y_n$ belongs to T and, consequently, $x_n x_j$ and $x_j x_{j-1}$ belong to C.

First suppose that each edge $x_i y_{i-1}$ with $1 \leq i \leq j-1$ is in T. Then the path $x_n x_j x_{j-1} \dots x_1 x_0$ is in C. Now consider the edges $x_i y_{i-1}$ with $j+1 \leq i \leq n$. If all of them belong to T, then $x_n x_{n-1} \dots x_{j+1} x_j x_n$ is a cycle in C containing the edge $x_n x_{n-1}$. As in the proof of (b), we can show that there exists a spanning tree S_2 of G with $\zeta(G, S_2) < \zeta(G, T) = \zeta(G)$, a contradiction. Thus at least one of the edges $x_i y_{i-1}$ with $j+1 \le i \le n$ belongs to C. Let $x_l x_{l-1}$ be the edge with the largest index. Then the path $y_{l-1} x_l x_{l-1} \dots x_1 x_0$ is in C. Hence $\psi(y_{l-1}) = \psi(x_0)$, and so $d_C(y_{l-1}, x_0) \le 2$. A similar method as in the proof of (b) can now be used to construct a spanning tree S_3 with $\zeta(G, S_3) < \zeta(G)$, again a contradiction.

Therefore suppose that some edge $x_i y_{i-1}$, $1 \le i \le j-1$, belongs to C. As above, take the one with the greatest index, say $x_l x_{l-1}$. Then the path $y_{l-1} x_l x_{l+1} \dots x_{j-1} x_j x_n$ is contained in C. Consider the edge $x_n y_{n-1}$. If it belongs to C, then $\psi(y_{l-1}) = \psi(y_{n-1})$, whence $y_{l-1} = y_{l-1}$. Thus we obtain a cycle in C that can be used to construct a spanning tree S_4 with $\zeta(G, S_4) < \zeta(G)$. Therefore $x_n y_{n-1}$ belongs to T. By analogous considerations, each edge $x_i y_{i-1}$ with $j+1 \le i \le n$ belongs to T. But then $x_n x_{n-1} \dots x_j x_n$ is a cycle in C containing the edge $x_n x_{n-1}$. Again, there is a spanning tree S_5 with $\zeta(G, S_5) < \zeta(G)$, a contradiction.

Thus we see that the subgraph $G^{n+1} = G^n + D(x_n, y_n)$ of G satisfies (a), (b), and (c). This concludes the induction step as well as the proof of the lemma. □

Proof of Theorem 4.2. It is obviously sufficient to prove the theorem for simple graphs. By way of contradiction, let G be a 2-connected simple graph of diameter 2 with $\zeta(G) \ge 5$, and let T be a spanning tree of G with $\zeta(G,T) = \zeta(G)$. Clearly, T contains a 5-fragment, say F. From Lemma 4.3 it follows that F has at most four end-vertices. Thus F is homeomorphic to one of the trees depicted in Fig. 5. We shall show that none of these possibilities occurs, deriving the required contradiction.



Type H Type $K_{1,4}$ Type $K_{1,3}$

Type K_2

Figure 5.

Claim 1. *T contains no 5-fragment of type H.*

P r o o f . Assume that T does contain a 5-fragment F of type H. Further assume F to be labelled consistently with Fig. 6. By Lemma 4.3, the labelling is correct and unique up to permutation. As $d_F(a,d) \geq 3$ and $d_F(b,c) \geq 3$, there exist paths $D(a,d) = axd$ and $D(b,c) = byc$ in G. Lemma 4.4 implies that both of them are alternating, $x \neq y$ and, since F is a fragment, neither x nor y belongs to F. In view of symmetry, two cases occur.

Case 1. ax and by belong to T. It follows that $\psi(x) = 4 = \psi(d)$ and $\psi(y) = 3 = \psi(c)$. Then $T(a,b) + ax + by$ is a 5-fragment with fewer vertices than F.

Case 2. ax and yc belong to T. Thus $\psi(x) = 4 = \psi(d)$ and $\psi(y) = 2 = \psi(a)$. Since $d_F(a,c) \geq 3$, Lemma 4.4 yields that there exists an alternating path $D(a,c) = azc$ with $z \notin F$; obviously, z is distinct from both x and y. We may clearly suppose that az is in T. Then $\psi(z) = 4$ and $T(a,b) + ax + az$ is a 5-fragment with fewer vertices than F.

In both cases we have a contradiction, proving the claim.

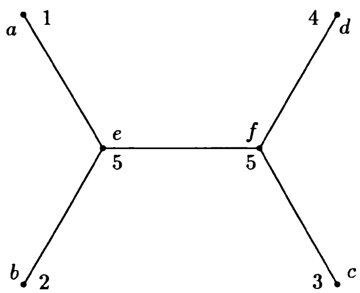


Figure 6.

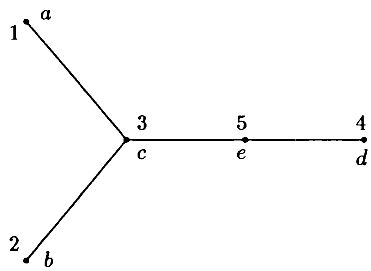


Figure 7.

CLAIM 2. *T contains no 5-fragment of type $K_{1,3}$.*

P r o o f . Suppose this is false and let F be a 5-fragment of type $K_{1,3}$ in T. Let a, b and d be the end-vertices of F, and let c be the internal vertex of degree 3 in F. Since F has at least five vertices, there exists an end-vertex of F, say d, such that the path $T(c,d)$ contains a vertex e whose label is different from $\psi(a)$, $\psi(b)$, $\psi(c)$ and $\psi(d)$. Thus we may assume that the labelling of F agrees with Fig. 7. Since $d_T(a,d) \geq 3$ and $\psi(a) \neq \psi(d)$, there exists an alternating path $D(a,d) = ax_1d$ with x_1 not in F. Clearly, $d_T(x_1,b) \geq 3$ and $\psi(x_1) \neq \psi(b)$, so the first part of Lemma 4.4 shows that there exists an alternating path $D(x_1,b) = x_1x_2b$. In addition, it is easy to see that x_2 does not belong to $F \cup x_1$. But now the 4-fragment $F(a,b) \circ e$ satisfies both (i) and (ii) of Lemma 4.5 with $x_0 = d$. By Lemma 4.5 this is impossible, a contradiction.

Using the previous claim and similar considerations as above one can easily prove:

CLAIM 3. *T has no 5-fragment of type K_2.*

Thus it remains to prove

CLAIM 4. *T has no 5-fragment of type $K_{1,4}$.*

Proof. Let F be a 5-fragment of type $K_{1,4}$ in T, labelled as in Fig. 8 (obviously, the labelling is correct). Since G is 2-connected, for any two distinct end-vertices x and y of F there exists an $x-y$ path avoiding c, the unique vertex of degree 4 in F. We shall call such a path *eccentric.* Without loss of generality suppose that among all 5-fragments of type $K_{1,4}$ in G, F is one admitting a shortest eccentric path, denoted by $Q = aq_1q_2 \dots q_{n-1}b$.

Clearly none of the vertices q_i $(1 \le i \le n-1)$ can be an end-vertex of F, for otherwise Q could be shortened. On the other hand, it is a straightforward matter to show (using Lemma 4.4) that if q_i is an internal vertex of F, then there exists a 5-fragment with fewer vertices. This implies that no internal vertex of Q belongs to F (and, at the same time, that $|V(F)| = 5$).

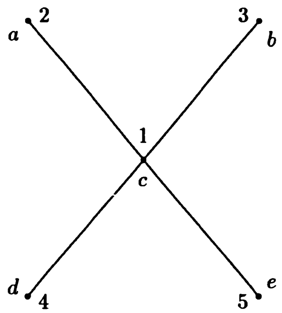


Figure 8.

Now we establish several intermediate results which, combined together, yield the required final contradiction.

(I). *If $l(Q) \ge 4$, then $\psi(q_i) = 1$ $(1 \le i \le n-1)$, aq_1, $q_{n-1}b \in E(T)$ and q_1c, $q_{n-1}c \in E(C)$. Moreover, if $l(Q) \ge 5$, then each q_i is adjacent to c.*

Proof of (I). By Claim 2, $\psi(q_1) = 1$ or 2. However, $\psi(q_1) = 2$ implies that $aq_1D(q_1, b)$ is a shorter eccentric path. Thus $\psi(q_1) = 1$, $D(q_1, b) = q_1cb$, $aq_1 \in E(T)$ and $q_1c \in E(C)$. Similar facts for q_{n-1} can be shown analogously. If $l(Q) = 4$, then obviously every q_i is labelled 1. If $l(Q) \ge 5$, then for every q_i there exists $u \in \{a, b\}$ with $d_Q(q_i, u) \ge 3$. By the minimality of Q, $D(q_i, u) = q_icu$ so that q_i is adjacent to c. If, however, $\psi(q_i) \ne 1$, then $F + q_ic$ contains a 5-fragment of type $K_{1,4}$ with a shorter eccentric path. Therefore $\psi(q_i) = 1$, completing the proof of (I).

(II). $3 \le l(Q) \le 4$.

Proof of (II). Since $\psi(a) \neq \psi(b)$, we have $l(Q) \ge 2$. If $l(Q) = 2$, then Q is an alternating path and $F + Q$ contains a 5-fragment of type $K_{1,3}$, contrary to Claim 2. Therefore $l(Q) \ge 3$.

To prove the second inequality, assume that $l(Q) \ge 5$. We show that Q contains a subpath $Q(x,y)$ such that $cxQ(x,y)yc$ is a cycle in C. Let s_1 (resp., t_1) be the first internal vertex on $Q(a,b)$ ($Q(b,a)$) incident with an edge of $C \cap Q$. If $Q(s_1,t_1)$ is contained in C, then it is itself the desired path. Otherwise let a_1 (resp., b_1) be the first internal vertex on $Q(s_1,t_1)$ ($Q(t_1,s_1)$) incident with an edge of $T \cap Q$. If a_1c or b_1c is in C, then $Q(s_1,a_1)$ or $Q(t_1,b_1)$ is the desired path. Therefore assume that both a_1c and b_1c are in T. Then we can repeat the above consideration about $Q(a,b)$ and $Q(a_1,b_1)$. In this way we obtain a sequence (s_1,t_1), (a_1,b_1), $(s_2,t_2), \ldots$ with $d_Q(s_i,t_i) > d_Q(s_{i+1},t_{i+1})$. This process must terminate with some (s_k,t_k) at $d_Q(s_k,t_k) > 0$. But then $Q(s_k,t_k)$ is contained in C and the desired path is $Q(s_k,t_k)$. Now form the spanning trees $T^k = T$ and $T^{i-1} = T^i + s_ic - a_{i-1}c$, where $1 \le i \le k$ and $a_0 = a$. Then $\psi_{T^0}(a) = \psi_{T^0}(c)$, whence $\zeta(G,T^0) < \zeta(G)$. Similar contradictions can be obtained using any other path $Q(x,y)$ constructed above. This proves (II).

(III). $l(Q) \neq 4$.

Proof of (III). Let $l(Q) = 4$. Then, using (I) and the minimality of T, it is easy to see that one of the edges q_1q_2 and q_2q_3 belongs to C but not both. We may assume that q_1q_2 is in T. Then $\psi(q_2) = 1 = \psi(q_3)$ and, again by the minimality of T, q_2 is not adjacent to c. Putting $x_0 = q_2$, $y_0 = d$, $y_1 = e$ and $y_2 = a$ it can be verified that $F - b$ is a 4-fragment satisfying both (i) and (ii) of Lemma 4.5. But according to Lemma 4.5 this is impossible, a contradiction. Consequently, $l(Q) \neq 4$.

(IV). $l(Q) \neq 3$.

Proof of (IV). Let $l(Q) = 3$. We consider two cases according to whether q_1q_2 belongs to C or not.

Case 1. $q_1q_2 \in E(C)$. Claim 2 then implies that both aq_1 and q_2b belong to T and $\psi(q_1) = 1 = \psi(q_2)$. By the minimality of T, either q_1 or q_2 is not adjacent to c. But now we can derive a contradiction using Lemma 4.5 as in the proof of (III).

Case 2. $q_1q_2 \in E(T)$. By Claim 2, $\psi(q_1) = 1$ or 2 and $\psi(q_2) = 1$ or 3. If $\psi(q_1) = 1$, then aq_1 belongs to T and $\psi(q_2) = 3$. Now, considering the paths $D(q_2,d)$ and $D(q_2,e)$ it is easy to find a 5-fragment of type $K_{1,3}$ in G, a contradiction. Thus $\psi(q_1) = 2$ and, analogously, $\psi(q_2) = 3$. Let us consider the paths $D(q_1, d)$, $D(q_1, e)$, $D(q_2, d)$ and $D(q_2, e)$. It is easily seen that they are all alternating, with the possible exception of, say, $D(q_1, d)$, which may be contained in T. Let $q_3, \ldots, q_6$ be their internal vertices, respectively (where q_3 possibly does not exist). Clearly, these vertices are mutually distinct. Moreover, Claim 2 yields that $\psi(q_3) = 4 = \psi(q_5)$ and $\psi(q_4) = 5 = \psi(q_6)$. Setting $x_0 = q_4$, $y_0 = q_5$, $y_1 = q_6$ and $y_2 = q_1$, we now obtain the 4-fragment $q_1 q_2 q_5 + q_2 q_6$ satisfying both (i) and (ii) of Lemma 4.5. Since, according to Lemma 4.5, this is impossible, we arrive at a contradiction proving (IV), Claim 4 and thereby Theorem 4.2. □

R e m a r k s . The bounds given in Theorem 4.1 are best possible in the sense that they are attained by infinitely many graphs. It suffices to exhibit 2-connected graphs of diameter 2 with $\zeta = 4$, each admitting an acyclic cotree, and to attach a loop to every vertex. It is easily seen that the resulting graphs have $\xi = 4$. Such examples are shown in Fig. 9, all having connectivity 2. The Petersen graph is a 3-connected graph of diameter 2 with $\zeta = 4$. Strangely enough, it is the only such example known to the author.

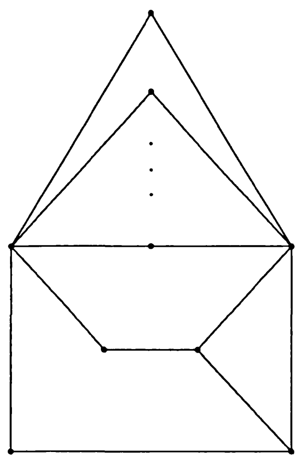


Figure 9.

Theorem 4.2 has no obvious analogue for graphs of diameter greater than 2. Indeed, consider the graph $R_{n,d}$ of diameter d consisting of n internally disjoint paths of length d joining two vertices of degree n. Since $R_{n,d}$ admits an acyclic cotree, Corollary 2.2 implies that $\zeta(R_{n,d}) = n(d-2) + 3$. Thus 2-connected graphs of diameter $d > 2$ may have an arbitrarily large decay number.

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Department of Computer Science
Faculty of Mathematics and Physics
Comenius University
842 15 Bratislava
Czecho-Slovakia