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3RD ORDER DIFFERENTIAL INVARIANTS OF COFRAMES

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ABSTRACT. The aim of this paper is to characterize all 3th order differential invariants of linear coframes on smooth manifold. These differential invariants are described in terms of bases of invariants.

1. Introduction

In this paper, we mean by a *left G-manifold* a smooth manifold endowed with a left action of a Lie group G. A mapping between two left G-manifolds transforming G-orbits into G-orbits is said to be *G-equivariant.* As usual, we denote by $\mathbb{R}$ the field of real numbers. The *rth differential group* L^r_n of $\mathbb{R}^n$ is the Lie group of invertible r-jets with source and target at the origin $0 \in \mathbb{R}^n$; the group multiplication in L^r_n is defined by the composition of jets. Note that $L^1_n = GL_n(\mathbb{R})$. For generalities on spaces of jets and their mappings, differential groups, their actions, etc., we refer to Nijenhuis [13], Krupka and Janyška [9], and Kolář, Michor and Slovák [5].

Let P and Q be two left L^r_n-manifolds. Recall that a smooth L^r_n-equivariant mapping $F\colon U \to Q$, where U is an open, L^r_n-invariant set in P, is called a *differential invariant*.

Let X be an n-dimensional manifold. By an *r-frame* at a point $x \in X$ we mean an invertible r-jet with source $0 \in \mathbb{R}^n$ and target at x. The set of r-frames together with its natural structure of a principal L^r_n-bundle with base X is denoted by F^rX, and is called the *bundle of r-frames* over X. If $r = 1$, we speak of the *bundle of linear frames*, and write $F^1X = FX$. If Q is a

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left L^r_n-manifold, then the fiber bundle with fiber Q, associated with F^rX is denoted by F^r_QX.

It is well known that differential invariants can equivalently be described as natural transformations of the lifting functors F^r_P and F^r_Q ([7]). A principal meaning of differential invariants for differential geometry consists in their independence of local coordinates on a manifold over which they are considered.

Most of differential invariants appearing in differential geometry correspond with the case when Q is an L^1_n-manifold. These differential invariants can be described as follows. Let $K^{r,s}_n$ be the *kernel* of the canonical group morphism $\pi^{r,s}\colon L^r_n \to L^s_n$, where $r \geq s$. If L^r_n acts on Q via its subgroup L^1_n, each *continuous*, L^r_n-equivariant mapping $F\colon U \to Q$ has the form $F = f \circ \pi$, where $P/K^{r,1}_n$ is the space of $K^{r,1}_n$-orbits, $\pi\colon P \to P/K^{r,1}_n$ is the quotient projection and $f\colon P/K^{r,1}_n \to Q$ is a continuous, L^1_n-equivariant mapping. Indeed, in this scheme $P/K^{r,1}_n$ is considered with the quotient topology, but is not necessarily a smooth manifold. The quotient projection π is continuous but not necessarily smooth. If $P/K^{r,1}_n$ has a smooth structure such that π is a submersion, we call π the *basis* of differential invariants on P (for more general concepts of a basis, see [11]).

In [8], a method based on this observation, was applied to the problem of finding invariants of a linear connection. The initial problem was reduced to a more simple problem of the classical invariant theory (see e.g. [14], [15]) to describe all L^1_n-equivariant mappings from $P/K^{r,1}_n$ to L^1_n-manifolds. Our aim in this paper is to study invariants of linear coframes by the same method.

In this paper, we consider by the same method the problem of characterizing all 3rd order differential invariants of coframes with values in L^3_n-, L^2_n- and L^1_n-manifolds. According to the prolongation theory of manifolds endowed with a Lie group action [4], [6] (see also [5], [9]), we first introduce the domain of these differential invariants, i.e., the L^4_n-manifold of $P = T^3_nL^1_n$ of 3-jets with source $0 \in \mathbb{R}^n$ and target in L^1_n, and describe the frame and coframe actions of L^4_n on $T^3_nL^1_n$. Then we construct the corresponding orbit spaces of the normal subgroups $K^{4,3}_n$, $K^{4,2}_n$, $K^{4,1}_n \subset L^4_n$. We show that these orbit spaces can be identified with Cartesian products of $T^2_nL^1_n$, $T^1_nL^1_n$, and L^1_n, respectively, with some tensor spaces over $\mathbb{R}^n$; in this way the corresponding differential invariants are described in terms of their bases. These results extend the results recently obtained by the first author, who described all 2nd order invariants of coframes (see [1]).

It should be pointed out, however, that the factorization method which is used to compute all 3rd order invariants of coframes, leads to difficult calculations which cannot be effectively extended to higher-order cases. Thus, the problem of characterizing all 4th- and higher order differential invariants remains open.

The geometric interpretation of the new invariants is also an open question (see, however, the remark Added in proofs in this paper).

Note that there is a correspondence between the frame and coframe actions of L^1_n on itself, which is also discussed below. This correspondence allows us to compute differential invariants of frames as functions of differential invariants of coframes, and vice versa. M. Krupka [12] considered 1st order invariants of velocities, the objects which are more general than frames. Garcia and Munoz [3] described higher order $\mathbb{R}$-valued differential invariants of frames in terms of integrals of a canonical differential system.

2. Equivariant mappings with respect to the quotient group

In this section, we recall some general concepts on equivariant mappings, related with a normal subgroup. These remarks will be applied later to the differential groups. We follow, with only minor modifications, the paper [8].

Let G be a Lie group, K a normal subgroup $\tau\colon G \to G/K$, the quotient projection, and let Q be a G-manifold. Denote by $[q]_K$ the K-orbit in Q passing through a point $q \in Q$. Let Q/K be the set of K-orbits, and $\rho\colon Q \to Q/K$ the quotient projection. We define for each $h \in G/K$

$$h \cdot [q]_K = [g \cdot q]_K \,, \tag{1}$$

where $g \in G$ is any element such that $\tau(g) = h$. (1) defines a left action of G/K on Q/K, which is said to be *induced* by the action of G on Q.

The action (1) is defined correctly. Indeed, if $\tau(g') = h = \tau(g)$, then there exists an element $k \in K$ such that $g' = k \cdot g$. If $[q']_K = [q]_K$, then there exists an $k' \in K$ such that $q' = k' \cdot q$. Thus, $[g' \cdot q']_K = [k \cdot g \cdot k' \cdot q]_K = [k \cdot g \cdot k' \cdot g^{-1} \cdot g \cdot q]_K = [g \cdot q]_K$, since $k \cdot g \cdot k' \cdot g^{-1} \in K$ because K is a normal subgroup.

LEMMA 1. *Assume that the group G/K acts on a set M on the left. Let $F\colon Q \to M$ be a mapping such that for each $g \in G$ and $q \in Q$,*

$$F(g \cdot q) = \tau(g) \cdot F(q) \,. \tag{2}$$

Then F is of the form

$$F = f \circ \rho \,, \tag{3}$$

where $f\colon Q/K \to M$ is a uniquely determined G/K-equivariant mapping.

Proof. Let $p \in Q/K$. Choose $q \in Q$ such that $[q] = p$. Setting $f(p) = F(q)$ we obtain, by (2), a mapping $f\colon Q/K \to M$. If $h \in G/K$, then $f(h \cdot p) = f(h \cdot [q]_K) = f([g \cdot q]_K)$ where $\tau(g) = h$. Thus, $f(h \cdot p) = F(g \cdot q) = \tau(g) \cdot F(q) = h \cdot F(q)$. □

LEMMA 2. *Let G be a Lie group, K a normal Lie subgroup, and let Q be a G-manifold. Assume that the equivalence "there exists $k \in K$ such that $q_1 = k \cdot q_2$" is a closed submanifold of $Q \times Q$. Then the quotient Q/K has an orbit manifold structure, the induced left action of the quotient group G/K on Q/K is smooth, and*

$$\rho(g \cdot q) = \tau(g) \cdot \rho(q) . \tag{4}$$

If in addition K acts freely on Q, then Q is a left principal K-bundle.

P r o o f. Since our assumption guarantees existence of the orbit manifold structure on the set Q/K, and of the left principal K-bundle structure on Q ([2]), and (4) holds by (1), it remains to show that the group action $G/K \times Q/K \ni \left(h, [q]_K\right) \to h \cdot [q]_K \in Q/K$ is smooth. This is, however, immediately seen by using local sections of the submersions ρ and τ. □

COROLLARY. *Let M be a G/K-manifold. Under the hypothesis of Lemma 2, each smooth mapping $F \colon Q \to M$ satisfying (2) is of the form (3), where $f \colon Q/K \to M$ is a uniquely determined smooth, G/K-equivariant mapping.*

3. Jet prolongations of L^1_n-manifolds

In this section, the general prolongation theory of left G-manifolds is applied to the case of the Lie group $G = L^1_n = GL_n(\mathbb{R})$. We use the prolongation formula derived in [6], and the terminology and notation of the book [9].

Recall that the *rth differential group* L^r_n of $\mathbb{R}^n$ is the group of invertible r-jets with source and target at the origin $0 \in \mathbb{R}^n$. The group multiplication in L^r_n is defined by the composition of jets. $K^{r,s}_n$ denotes the *kernel* of the canonical group morphism $\pi^{r,s} \colon L^r_n \to L^s_n$, where $r \geq s \geq 1$. The *first canonical coordinates* $a^i_{j_1}, a^i_{j_1 j_2}, \dots, a^i_{j_1 j_2 \cdots j_r}$ where $1 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n$, on L^r_n are defined as follows. Let $J^r_0 \alpha \in L^r_n$, where α is a diffeomorphism of a neighborhood U of the origin $0 \in \mathbb{R}^n$ into $\mathbb{R}^n$ such that $\alpha(0) = 0$; in components, $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$. We define

$$a^i_{j_1 j_2 \cdots j_k}(J^r_0 \alpha) = D_{j_1} D_{j_2} \cdots D_{j_k} \alpha^i(0) , \qquad 1 \leq k \leq r . \tag{5}$$

The *second canonical coordinates* $b^i_{j_1}, b^i_{j_1 j_2}, \dots, b^i_{j_1 j_2 \cdots j_r}$ on L^r_n, where $1 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n$, are then defined by

$$b^i_{j_1 j_2 \cdots j_k}(J^r_0 \alpha) = a^i_{j_1 j_2 \cdots j_k}(J^r_0 \alpha^{-1}) , \qquad 1 \leq k \leq r . \tag{6}$$

Indeed, $a^i_j b^j_k = \delta^i_k$ (the Kronecker symbol).

Let us consider a left L^1_n-manifold P, and denote by $T^r_n P$ the manifold of r-jets with source $0 \in \mathbb{R}^n$ and target in P. According to the general theory of prolongations of left G-manifolds, $T_n^r P$ has a (canonical) structure of a left L_n^{r+1}-manifold. To define this structure, denote by t_x the translation of $\mathbb{R}^n$ defined by $t_x(y) = y - x$. Consider an element $J_0^{r+1}\alpha \in L_n^{r+1}$, and denote $\alpha_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$, $\bar{\alpha}(x) = J_0^1 \alpha_x$, $A = J_0^r \alpha$, and $S = J_0^r \bar{\alpha}$. The action of L_n^{r+1} on $T_n^r P$ is then defined by

$$J_0^{r+1}\alpha \cdot q = S \cdot (q \circ A^{-1}) = J_0^r\big(\bar{\alpha} \cdot (\gamma \circ \alpha^{-1})\big), \tag{7}$$

where $q = J_0^r \gamma \in T_n^r P$, and the dot in the parentheses on the right denotes the group multiplication in L_n^1.

The left L_n^{r+1}-manifold $T_n^r P$ is called the *r-jet prolongation* of the left L_n^1-manifold P.

4. Frames and coframes

Let X be an n-dimensional manifold. Recall that an *r-frame* at a point $x \in X$ is an invertible r-jet with source $0 \in \mathbb{R}^n$ and target at x. The set of r-frames, denoted by $F^r X$, will be considered with its natural structure of a principal L_n^r-bundle over X. We write $FX = F^1 X$; FX is the bundle of *linear* frames.

Thus, the structure group of FX is the group $L_n^1 = GL_n(\mathbb{R})$. FX can also be regarded as a fiber bundle with fiber L_n^1, *associated* with FX, if we let act the group L_n^1 on itself by left translations. Namely this structure of FX appears in the theory of differential invariants. The left translation defined by the group multiplication $L_n^1 \ni (J_0^1\alpha, J_0^1\mu) \to J_0^1(\alpha \circ \mu) = J_0^1\alpha \circ J_0^1\mu \in L_n^1$ is given in the canonical coordinates by $p_j^i\big(J_0^1(\alpha \circ \mu)\big) = a_k^i(J_0^1\alpha) p_j^k(J_0^1\mu)$. We write these equations simply by

$$\bar{p}_j^i = a_k^i p_j^k, \tag{8}$$

where p_j^i (resp. a_j^i) stand for the first canonical coordinates on the fiber L_n^1 of FX, (resp. on the structure group L_n^1 of FX). (8) is called the *frame action* of L_n^1 on itself.

$J^r FX$ denotes the r-jet prolongation of FX. It follows from the general theory of jet prolongations of fiber bundles that $J^r FX$ can be considered as a fiber bundle over X with fiber $T_n^r L_n^1$, associated with $F^{r+1}X$. Equations of the group action of L_n^{r+1} on $T_n^r L_n^1$ can be obtained from (7) and (8).

An *r-coframe* at $x \in X$ is an invertible r-jet with source $x \in X$ and target $0 \in \mathbb{R}^n$. If $r = 1$, we speak of *linear* coframes. The set of linear coframes, denoted by $F^* X$, has a natural structure of a fiber bundle with structure group L_n^1, associated with the bundle of frames FX. This structure of $F^* X$ is defined by the left action of L_n^1 on itself given by $p_j^i\big(J_0^1(\mu \circ \alpha^{-1})\big) = p_k^i(J_0^1\mu) a_j^k(J_0^1\alpha^{-1}) =$

$p^i_k(J^1_0\mu)b^k_j(J^1_0\alpha)$. As before, it is convenient to express this action by the equations

$$\bar{p}^i_j = p^i_k b^k_j \,. \tag{9}$$

Here p^i_j are the first canonical coordinates on the fiber L^1_n of F^*X, and b^i_j are the second canonical coordinates on the structure group L^1_n. (9) is called the *coframe action* of L^1_n on itself.

One can easily compare the actions (8) and (9) of L^1_n on L^1_n. Let (5) (resp. (6)) be expressed by $\Phi(g,h) = g \cdot h$ (resp. $\Psi(g,h) = h \cdot g^{-1}$). Then $\Phi(g,h^{-1}) = \Psi(g,h)^{-1}$. If $\vartheta\colon L^1_n \to L^1_n$ is the mapping $g \to g^{-1}$ and L_g (resp. R_g) is the left (resp. right) translation on L^1_n by g, then $L_g \circ \vartheta = \vartheta \circ R_{\vartheta(g)}$, i.e., $R_{\vartheta(g)} = \vartheta \circ L_g \circ \vartheta$.

The r-jet prolongation J^rF^*X of F^*X can be considered as a fiber bundle over X with fiber $T^r_nL^1_n$, associated with $F^{r+1}X$. Equations of the group action of L^{r+1}_n on $T^r_nL^1_n$ can be obtained from (7) and (9).

5. The 3rd jet prolongation of the coframe action

Now we investigate the action (7) of the group L^4_n on $T^3_nL^1_n$, associated with (9). We prove three lemmas which are fundamental for the discussion of the corresponding orbit spaces.

Let U be a neighborhood of the origin $0 \in \mathbb{R}^n$. Let α be a diffeomorphism of U onto $\alpha(U) \subset \mathbb{R}^n$ such that $\alpha(0) = 0$. Then $\bar{\alpha}(x) = J^1_0\alpha_x$, where $\alpha_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$. Let $\gamma\colon U \to L^1_n$ be a mapping. We denote $\psi(x) = \bar{\alpha}(x)\cdot\gamma(\alpha^{-1}(x))$, where $x \in \alpha(U)$, and the dot on the right hand side means the multiplication in the group L^1_n. In coordinates,

$$p^i_j(\psi(x)) = p^i_j(\bar{\alpha}(x)\cdot\gamma(\alpha^{-1}(x))) = p^i_k(\gamma(\alpha^{-1}(x)))b^k_j(\bar{\alpha}(x)) \,. \tag{10}$$

Note that in this formula,

$$\begin{aligned} b^k_j(\bar{\alpha}(x)) &= b^k_j(J^1_0\alpha_x) = a^k_j(J^1_0\alpha_x^{-1}) \\ &= D_j(\alpha_x^{-1})^k(0) = D_j\big(t_{\alpha_x^{-1}(x)} \circ \alpha^{-1} \circ t_{-x}\big)^k(0) \\ &= D_p t^k_{\alpha_x^{-1}(x)}\big((\alpha^{-1}\circ t_{-x})(0)\big) D_q(\alpha^{-1})^p(t_{-x}(0)) D_j t^q_{-x}(0) \\ &= \delta^k_p D_q(\alpha^{-1})^p(x)\delta^q_j = D_j(\alpha^{-1})^k(x) \,. \end{aligned} \tag{11}$$

Now the chart expression of the coframe action is obtained by expressing the r-jet $J^r_0\psi = J^{r+1}_0\alpha \cdot J^r_0\gamma$ (7) in coordinates.

Consider the case $r = 3$. Our aim is to compute the 3-jet $J^4_0\psi = J^4_0\alpha \cdot J^3_0\gamma$ in the associated coordinates on $T^3_nL^1_n$ and L^4_n.

LEMMA 1. *The group action of L^4_n on $T^3_nL^1_n$ induced by the coframe action of L^1_n on L^1_n is defined by the equations*

$$\begin{aligned}
\bar{p}^i_j &= p^i_s b^s_j\,,\\
\bar{p}^i_{j,k} &= p^i_{s,t} b^t_k b^s_j + p^i_s b^s_{jk}\,,\\
\bar{p}^i_{j,kl} &= p^i_{s,tu} b^u_l b^t_k b^s_j + p^i_{s,t}(b^t_{kl} b^s_j + b^t_k b^s_{jl} + b^t_l b^s_{jk}) + p^i_s b^s_{jkl}\,,\\
\bar{p}^i_{j,klm} &= p^i_{s,tuv} b^v_m b^u_l b^t_k b^s_j\\
&\quad + p^i_{s,tu}\big((b^u_{lm} b^t_k b^s_j + b^u_l b^t_{km} b^s_j + b^u_l b^t_k b^s_{jm}) + b^u_m (b^t_{kl} b^s_j + b^t_k b^s_{jl} + b^t_l b^s_{jk})\big)\\
&\quad + p^i_{s,t}(b^t_{klm} b^s_j + b^t_{km} b^s_{jl} + b^t_{kl} b^s_{jm} + b^t_k b^s_{jlm} + b^t_{lm} b^s_{jk} + b^t_l b^s_{jkm} + b^t_m b^s_{jkl})\\
&\quad + p^i_s b^s_{jklm}\,.
\end{aligned} \tag{12}$$

P r o o f. Since the proof is routine and long, we shall only verify the first two equations. The first equation (12) is immediately obtained by taking $x = 0$ in (10). To get the second equation, we use the definition of the canonical coordinates on $T^3_nL^1_n$ and on L^4_n, and apply (11). We obtain

$$\begin{aligned}
p^i_{j,l}(J^r_0\psi) &= D_l(p^i_j \circ \psi)(0)\\
&= D_l(p^i_k \circ \gamma \circ \alpha^{-1})(0) b^k_j(J^1_0\alpha) + p^i_k\big(\gamma(0)\big) D_l(b^k_j \circ \bar{\alpha})(0)\\
&= D_s(p^i_k \circ \gamma)(0) D_l(\alpha^{-1})^s(0) b^k_j(J^1_0\alpha) + p^i_k\big(\gamma(0)\big) D_l D_j(\alpha^{-1})^k(0)\,.
\end{aligned}$$

Substituting $x = 0$ yields the second equation (12).

To get the remaining equations, we differentiate (10) two resp. three times, and then substitute $x = 0$. □

Now we restrict the action (12) to the subgroups $K^{4,1}_n$, $K^{4,2}_n$, and $K^{4,3}_n$ of L^4_n. The following result is fundamental for the discussion of the corresponding orbit spaces.

LEMMA 2.

(a) *The group action of $K^{4,1}_n$ on $T^3_nL^1_n$ induced by the coframe action of L^1_n on L^1_n is defined by the equations*

$$\begin{aligned}
\bar{p}^i_j &= p^i_j\,,\\
\bar{p}^i_{j,k} &= p^i_{j,k} + p^i_s b^s_{jk}\,,\\
\bar{p}^i_{j,kl} &= p^i_{j,kl} + p^i_{j,t} b^t_{kl} + p^i_{s,k} b^s_{jl} + p^i_{s,l} b^s_{jk} + p^i_s b^s_{jkl}\,,\\
\bar{p}^i_{j,klm} &= p^i_{j,klm}\\
&\quad + p^i_{j,ks} b^s_{lm} + p^i_{j,sl} b^s_{km} + p^i_{s,kl} b^s_{jm} + p^i_{j,sm} b^s_{kl} + p^i_{s,km} b^s_{jl}\\
&\quad + p^i_{s,lm} b^s_{jk} + p^i_{j,t} b^t_{klm} + p^i_{s,t} b^t_{km} b^s_{jl} + p^i_{s,t} b^t_{kl} b^s_{jm} + p^i_{s,k} b^s_{jlm}\\
&\quad + p^i_{s,t} b^t_{lm} b^s_{jk} + p^i_{s,l} b^s_{jkm} + p^i_{s,m} b^s_{jkl} + p^i_s b^s_{jklm}\,.
\end{aligned} \tag{13}$$

(b) *The group action of* $K_n^{4,2}$ *on* $T_n^3L_n^1$ *induced by the coframe action of* L_n^4 *on* L_n^4 *is defined by the equations*

$$\begin{aligned} \bar{p}^i_j &= p^i_j \,, \\ \bar{p}^i_{j,k} &= p^i_{j,k} \,, \\ \bar{p}^i_{j,kl} &= p^i_{j,kl} + p^i_s b^s_{jkl} \,, \\ \bar{p}^i_{j,klm} &= p^i_{j,klm} + p^i_{j,t} b^t_{klm} + p^i_{s,k} b^s_{jlm} + p^i_{s,l} b^s_{jkm} + p^i_{s,m} b^s_{jkl} + p^i_s b^s_{jklm} \,. \end{aligned} \tag{14}$$

(c) *The group action of* $K_n^{4,3}$ *on* $T_n^3L_n^1$ *induced by the coframe action of* L_n^4 *on* L_n^4 *is defined by the equations*

$$\begin{aligned} \bar{p}^i_j &= p^i_j \,, \\ \bar{p}^i_{j,k} &= p^i_{j,k} \,, \\ \bar{p}^i_{j,kl} &= p^i_{j,kl} \,, \\ \bar{p}^i_{j,klm} &= p^i_{j,klm} + p^i_s b^s_{jklm} \,. \end{aligned} \tag{15}$$

Proof.

(a) We take $b^i_j = \delta^i_j$ in (12).

(b) We take $b^i_{jk} = 0$ in (13).

(c) We take $b^i_{jkl} = 0$ in (14). □

COROLLARY. *Each of the actions* (13), (14), *and* (15) *is free.*

Proof. Taking $\bar{p}^i_j = p^i_j$, $\bar{p}^i_{j,k} = p^i_{j,k}$, $\bar{p}^i_{j,kl} = p^i_{j,kl}$, $\bar{p}^i_{j,klm} = p^i_{j,klm}$ in either of these actions yields the identity of the corresponding group. □

Now we describe *orbits* of the group actions (13), (14), and (15). Let us introduce some notation. Using the second canonical coordinates on $T_n^3L_n^1$, we denote by q^i_k the inverse matrix of the matrix p^i_k; thus, $q^i_k \colon T_n^3L_n^1 \to \mathbb{R}$ are functions such that $q^i_s p^s_j = \delta^i_j$.

S_n^0 denotes the vector subspace of the tensor product $\bigotimes^2 \mathbb{R}^{n*} = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$, defined in the canonical coordinates on $\mathbb{R}^n$ by the equations

$$x_{jk} + x_{kj} = 0 \,. \tag{16}$$

S_n^1 denotes the vector subspace of the tensor product $\bigotimes^3 \mathbb{R}^{n*}$, defined by the equations

$$x_{ijk} + x_{ikj} = 0 \,, \qquad x_{ijk} + x_{kij} + x_{jki} = 0 \,. \tag{17}$$

Similarly, S_n^2 is the vector subspace of the tensor product $\bigotimes^4 \mathbb{R}^{n*}$, defined by the equations

$$x_{jklm} + x_{jlkm} = 0 \,, \qquad x_{jklm} + x_{ljkm} + x_{kljm} = 0 \,, \qquad x_{jklm} + x_{jmkl} + x_{jlmk} = 0 \,. \tag{18}$$

Special notation for *symmetrization* and *antisymmetrization* of indexed families of functions through selected indices is needed. Symmetrization (resp. antisymmetrization) in some indices $i, j, k, \dots$ is denoted by writing a bar (resp. a tilde) over these indices, i.e., by writing $\bar{i}, \bar{j}, \bar{k}, \dots$ (resp. $\tilde{i}, \tilde{j}, \tilde{k}, \dots$).

Finally, we introduce the following functions on $T^3_n L^1_n$:

$$
\begin{aligned}
I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= p^i_{\tilde{j},\tilde{k}}\,, \\
I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= p^i_{j,kl} - q^s_t(p^i_{\bar{l},\tilde{k}} p^t_{\tilde{j},\bar{s}} - p^i_{\bar{s},\tilde{j}} p^t_{\tilde{k},\bar{l}})\,, \\
I^i_{jklm}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= p^i_{j,klm} - q^s_n(p^i_{\tilde{j},\bar{s}} p^n_{\tilde{k},\bar{l}\bar{m}} - p^i_{\tilde{k},\bar{s}} p^n_{\tilde{j},\bar{l}\bar{m}} + p^n_{\bar{l},\bar{m}} p^i_{\tilde{j},\tilde{k}s} \\
&\quad + p^n_{\tilde{k},\bar{m}} p^i_{\tilde{j},\bar{s}l} + p^n_{\tilde{k},\bar{l}} p^i_{\tilde{j},\bar{s}m} + p^n_{\tilde{j},\bar{m}} p^i_{\bar{s},\tilde{k}l} + p^n_{\tilde{j},\bar{l}} p^i_{\bar{s},\tilde{k}m}) \\
&\quad + \tfrac{1}{3} q^s_n q^t_u \big(p^i_{\tilde{j},\bar{s}}(p^n_{\bar{t},\tilde{k}} p^u_{\bar{l},\bar{m}} + p^n_{\bar{t},\bar{l}} p^u_{\tilde{k},\bar{m}} + p^u_{\bar{t},\bar{m}} p^u_{\tilde{k},\bar{l}}) \\
&\quad + p^i_{s,k}(p^n_{\tilde{j},\bar{t}} p^u_{\bar{l},\bar{m}} + p^n_{\bar{t},\bar{l}} p^u_{\tilde{j},\bar{m}} + p^n_{\bar{t},\bar{m}} p^u_{\tilde{j},\bar{l}})\big) \\
&\quad + q^s_n q^t_u p^i_{\bar{s},\bar{t}}(p^n_{\tilde{j},\bar{l}} p^u_{\tilde{k},\bar{m}} + p^n_{\tilde{j},\bar{m}} p^u_{\tilde{k},\bar{l}}) \\
&\quad + q^s_n q^t_u \big(p^i_{\tilde{j},\bar{s}}(p^n_{\bar{t},\tilde{k}} p^u_{\bar{l},\bar{m}} + p^n_{\bar{t},\bar{l}} p^u_{\tilde{k},\bar{m}} + p^n_{\bar{t},\bar{m}} p^u_{\tilde{k},\bar{l}}) \\
&\quad + p^i_{\bar{s},\tilde{k}}(p^n_{\bar{t},\tilde{j}} p^u_{\bar{l},\bar{m}} + p^n_{\bar{t},\bar{l}} p^u_{\tilde{j},\bar{m}} + p^n_{\bar{t},\bar{m}} p^u_{\tilde{j},\bar{l}})\big)\,.
\end{aligned}
$$

It is easily seen by verifying equations (16) (resp. (17), resp. (18)) that in canonical coordinates, the functions I^i_{jk} (resp. I^i_{jkl}, resp. I^i_{jklm}) define a mapping of $T^3_n L^1_n$ into $\mathbb{R}^n \otimes S^0_n$ (resp. $\mathbb{R}^n \otimes S^1_n$, resp. $\mathbb{R}^n \otimes S^2_n$).

LEMMA 3.

(a) *$K^{4,1}_n$-orbits in $T^3_n L^1_n$ induced by the coframe action of L^1_n on L^1_n are defined by the equations*

$$
\begin{aligned}
p^i_j &= c^i_j\,, \\
I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jk}\,, \\
I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jkl}\,, \\
I^i_{jklm}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jklm}\,,
\end{aligned}
\tag{19}
$$

where $c^i_j, c^i_{jk}, c^i_{jkl}, c^i_{jklm} \in \mathbb{R}$ are arbitrary constants such that $\det c^i_j \neq 0$.

(b) *$K^{4,2}_n$-orbits in $T^3_n L^1_n$ induced by the coframe action of L^1_n on L^1_n are defined by the equations*

$$
\begin{aligned}
p^i_j &= c^i_j\,, \\
p^i_{jk} &= c^i_{jk}\,, \\
I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jkl}\,, \\
I^i_{jklm}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jklm}\,.
\end{aligned}
$$

(c) *$K_n^{4,3}$-orbits in $T_n^3 L_n^1$ induced by the coframe action of L_n^1 on L_n^1 are defined by the equations*

$$\begin{aligned} p^i_j &= c^i_j\,, \\ p^i_{jk} &= c^i_{jk}\,, \\ p^i_{jkl} &= c^i_{jkl}\,, \\ I^i_{jklm}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde}) &= c^i_{jklm}\,. \end{aligned}$$

P r o o f.

(a) Consider the action (13). Writing this action in the form

$$\begin{aligned} \bar{p}^i_j &= p^i_j\,, \\ \bar{p}^i_{j,k} &= p^i_{j,k} + p^i_s b^s_{jk}\,, \\ \bar{p}^i_{j,kl} &= p^i_{j,kl} + \chi^i_{j,kl} + p^i_s b^s_{jkl}\,, \\ \bar{p}^i_{j,klm} &= p^i_{j,klm} + \chi^i_{j,klm} + p^i_s b^s_{jklm}\,, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \chi^i_{j,kl} &= p^i_{j,t} b^t_{kl} + p^i_{s,k} b^s_{jl} + p^i_{s,l} b^s_{jk}\,, \\ \chi^i_{j,klm} &= p^i_{j,ks} b^s_{lm} + p^i_{j,sl} b^s_{km} + p^i_{s,kl} b^s_{jm} + p^i_{j,sm} b^s_{kl} + p^i_{s,km} b^s_{jl} \\ &\quad + p^i_{s,lm} b^s_{jk} + p^i_{j,t} b^t_{klm} + p^i_{s,t} b^t_{km} b^s_{jl} + p^i_{s,t} b^t_{kl} b^s_{jm} + p^i_{s,k} b^s_{jlm} \\ &\quad + p^i_{s,t} b^t_{lm} b^s_{jk} + p^i_{s,l} b^s_{jkm} + p^i_{s,m} b^s_{jkl}\,, \end{aligned} \tag{21}$$

we get from the first equation

$$\bar{q}^i_j = q^i_j\,, \tag{22}$$

and from the remaining ones

$$\begin{aligned} b^s_{jk} &= q^s_i (\bar{p}^i_{\bar{j},\bar{k}} - p^i_{\bar{j},\bar{k}})\,, \\ b^s_{jkl} &= q^s_i (\bar{p}^i_{\bar{j},\bar{k}\bar{l}} - p^i_{\bar{j},\bar{k}\bar{l}} - \chi^i_{\bar{j},\bar{k}\bar{l}})\,, \\ b^s_{jklm} &= q^s_i (\bar{p}^i_{\bar{j},\bar{k}\bar{l}\bar{m}} - p^i_{\bar{j},\bar{k}\bar{l}\bar{m}} - \chi^i_{\bar{j},\bar{k}\bar{l}\bar{m}})\,. \end{aligned} \tag{23}$$

Substituting for b^s_{jk}, b^s_{jkl}, b^s_{jklm} (23) in (20) and using (22) we obtain after a long and tedious calculation

$$\begin{aligned} I^i_{jk}(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}, \bar{p}^a_{b,cde}) &= I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde})\,, \\ I^i_{jkl}(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}, \bar{p}^a_{b,cde}) &= I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde})\,, \\ I^i_{jklm}(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}, \bar{p}^a_{b,cde}) &= I^i_{jklm}(p^a_b, p^a_{b,c}, p^a_{b,cd}, p^a_{b,cde})\,. \end{aligned}$$

With an obvious convention, these equations are written in the form

$$\bar{p}^i_j = p^i_j\,, \qquad \bar{I}^i_{jk} = I^i_{jk}\,, \qquad \bar{I}^i_{jkl} = I^i_{jkl}\,, \qquad \bar{I}^i_{jklm} = I^i_{jklm}\,. \tag{24}$$

Indeed, these equations mean that the functions p^i_j, $I^i_{j,k}$, $I^i_{j,kl}$, $I^i_{j,klm}$ (19) have the same values at the transformed points $(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}, \bar{p}^a_{b,cde}) \in T^3_n L^1_n$ as at the initial points $(p^i_j, p^i_{j,k}, p^i_{j,kl}, p^i_{j,klm}) \in T^3_n L^1_n$ or, which is the same, that they are constant along each $K^{4,1}_n$-orbit in $T^3_n L^1_n$.

Since the system (23), (24) is equivalent with (13), assertion (a) is proved.

(b), (c) We proceed in the same way. □

6. 3rd order invariants of coframes

Consider a point in $T^3_n L^1_n$ whose initial coordinates satisfy $p^{\bar{i}}_{\bar{j},\bar{k}} = 0$, $p^{\bar{i}}_{\bar{j},\bar{k}\bar{l}} = 0$ and $p^{\bar{i}}_{\bar{j},\bar{k}\bar{l}\bar{m}} = 0$. If b^s_{jk}, b^s_{jkl}, b^s_{jklm} are coordinates of an element of the group $K^{4,1}_n$, then by (23), the transformed point whose coordinates are denoted by p^i_j, $p^i_{j,k}$, $p^i_{j,kl}$, $p^i_{j,klm}$ satisfies

$$b^s_{jk} = q^s_i p^{\bar{i}}_{\bar{j},\bar{k}}\,, \qquad b^s_{jkl} = q^s_i (p^{\bar{i}}_{\bar{j},\bar{k}\bar{l}} - \chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}})\,, \qquad b^s_{jklm} = q^s_i (p^{\bar{i}}_{\bar{j},\bar{k}\bar{l}\bar{m}} - \chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}\bar{m}})\,, \tag{25}$$

where by (21),

$$\begin{aligned}
\chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}} &= p^{\bar{i}}_{\bar{j},s} b^s_{\bar{k}\bar{l}} + p^{\bar{i}}_{s,\bar{k}} b^s_{\bar{j}\bar{l}} + p^{\bar{i}}_{s,\bar{l}} b^s_{\bar{j}\bar{k}}\,, \\
\chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}\bar{m}} &= p^{\bar{i}}_{\bar{j},\bar{k}\bar{s}} b^s_{\bar{l}\bar{m}} + p^{\bar{i}}_{\bar{j},s\bar{l}} b^s_{\bar{k}\bar{m}} + p^{\bar{i}}_{s,\bar{k}\bar{l}} b^s_{\bar{j}\bar{m}} + p^{\bar{i}}_{\bar{j},s\bar{m}} b^s_{\bar{k}\bar{l}} + p^{\bar{i}}_{s,\bar{k}\bar{m}} b^s_{\bar{j}\bar{l}} \\
&\quad + p^{\bar{i}}_{s,\bar{l}\bar{m}} b^s_{\bar{j}\bar{k}} + p^{\bar{i}}_{\bar{j},t} b^t_{\bar{k}\bar{l}\bar{m}} + p^{\bar{i}}_{s,\bar{k}} b^s_{\bar{j}\bar{l}\bar{m}} + p^{\bar{i}}_{s,\bar{l}} b^s_{\bar{j}\bar{k}\bar{m}} + p^{\bar{i}}_{s,\bar{m}} b^s_{\bar{j}\bar{k}\bar{l}}\,.
\end{aligned}$$

Using the first two equations (25), we get after some calculation,

$$\begin{aligned}
\chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}} &= \tfrac{1}{3} q^t_s (p^{\bar{i}}_{\bar{t},\bar{j}} p^{\bar{s}}_{\bar{k},\bar{l}} + p^{\bar{i}}_{\bar{t},\bar{k}} p^{\bar{s}}_{\bar{j},\bar{l}} + p^{\bar{i}}_{\bar{t},\bar{l}} p^{\bar{s}}_{\bar{k},\bar{j}})\,, \\
\chi^{\bar{i}}_{\bar{j},\bar{k}\bar{l}\bar{m}} &= \tfrac{1}{2} q^t_s (p^{\bar{i}}_{\bar{t},\bar{j}} \chi^s_{k,lm} + p^{\bar{i}}_{\bar{t},\bar{k}} \chi^s_{j,lm} + p^{\bar{i}}_{\bar{t},\bar{l}} \chi^s_{j,km} + p^{\bar{i}}_{\bar{t},\bar{m}} \chi^s_{j,kl}) - 2 q^t_s p^{\bar{s}}_{\bar{l},\bar{m}} I^{\bar{i}}_{\bar{j}t\bar{k}}\,.
\end{aligned}$$

Now consider the product $L^1_n \times (\mathbb{R}^n \otimes S^0_n) \times (\mathbb{R}^n \otimes S^1_n) \times (\mathbb{R}^n \otimes S^2_n) \times K^{4,1}_n$ as a trivial principal $K^{4,1}_n$-bundle with base $L^1_n \times (\mathbb{R}^n \otimes S^0_n) \times (\mathbb{R}^n \otimes S^1_n) \times (\mathbb{R}^n \otimes S^2_n)$. We can now summarize the discussion of Section 5 in the following three theorems, describing differential invariants on $T^3_n L^1_n$ with values in L^1_n-, L^2_n-, and L^3_n-manifolds.

Theorem 1.

(a) *The coframe action defines on $T^3_n L^1_n$ the structure of a left principal $K^{4,1}_n$-bundle.*

(b) *The mapping $\psi\colon T^3_n L^1_n \to L^1_n \times (\mathbb{R}^n \otimes S^0_n) \times (\mathbb{R}^n \otimes S^1_n) \times (\mathbb{R}^n \otimes S^2_n) \times K^{4,1}_n$, defined in components by*

$$\psi = \left(p^i_j, q^i_s I^s_{jk}, q^i_s I^s_{jkl}, q^i_s I^s_{jklm}, (\delta^i_j, b^i_{jk}, b^i_{jkl}, b^i_{jklm})\right), \tag{26}$$

is an isomorphism of left principal $K^{4,1}_n$-bundles.

P r o o f .

(a) Since we have already proved that the action (13) of $K_n^{4,1}$ on $T_n^3 L_n^1$ is free, in order to show that $T_n^3 L_n^1$ is a principal $K_n^{4,1}$-bundle it remains to show that the equivalence "there exists an element $J_0^4 \alpha \in L_n^r$ such that $J_0^3 \bar{\gamma} = J_0^4 \alpha \cdot J_0^3 \gamma$" is a closed submanifold in $T_n^3 L_n^1 \times T_n^3 L_n^1$. However, using (13) in the form of (23), (24), we see at once that this submanifold is defined by the equations

$$p_j^i(J_0^3\bar{\gamma}) - p_j^i(J_0^3\gamma) = 0\,, \qquad I_{jk}^i(J_0^3\bar{\gamma}) - I_{jk}^i(J_0^3\gamma) = 0\,,$$
$$I_{jkl}^i(J_0^3\bar{\gamma}) - I_{jkl}^i(J_0^3\gamma) = 0\,, \qquad I_{jklm}^i(J_0^3\bar{\gamma}) - I_{jklm}^i(J_0^3\gamma) = 0\,,$$

and is therefore closed.

(b) This assertion can be proved by a direct computation. □

Consider the product $T_n^1 L_n^1 \times (\mathbb{R}^n \otimes S_n^1) \times (\mathbb{R}^n \otimes S_n^2) \times K_n^{4,2}$ as a trivial left principal $K_n^{4,2}$-bundle with base $T_n^1 L_n^1 \times (\mathbb{R}^n \otimes S_n^1) \times (\mathbb{R}^n \otimes S_n^2)$.

THEOREM 2.

(a) *The coframe action defines on $T_n^3 L_n^1$ the structure of a left principal $K_n^{4,2}$-bundle.*

(b) *The mapping $\psi\colon T_n^3 L_n^1 \to T_n^1 L_n^1 \times (\mathbb{R}^n \otimes S_n^1) \times (\mathbb{R}^n \otimes S_n^2) \times K_n^{4,2}$, defined in components by*

$$\psi = (p_j^i, p_{j,k}^i, q_s^i I_{jkl}^s, q_s^i I_{jklm}^s, \delta_j^i, 0, b_{jkl}^i, b_{jklm}^i)\,, \tag{27}$$

is an isomorphism of left principal $K_n^{4,2}$-bundles.

P r o o f . We proceed as in the proof of Theorem 1. □

Finally, consider the product $T_n^2 L_n^1 \times (\mathbb{R}^n \otimes S_n^2) \times K_n^{4,3}$ as a trivial left principal $K_n^{4,2}$-bundle with base $T_n^2 L_n^1 \times (\mathbb{R}^n \otimes S_n^2)$.

THEOREM 3.

(a) *The coframe action defines on $T_n^3 L_n^1$ the structure of a left principal $K_n^{4,3}$-bundle.*

(b) *The mapping $\psi\colon T_n^3 L_n^1 \to T_n^2 L_n^1 \times (\mathbb{R}^n \otimes S_n^2) \times K_n^{4,3}$, defined in components by*

$$\psi = (p_j^i, p_{j,k}^i, p_{j,kl}^i, q_s^i I_{jklm}^s, \delta_j^i, 0, 0, b_{jklm}^i)\,, \tag{28}$$

is an isomorphism of left principal $K_n^{4,3}$-bundles.

P r o o f . We proceed as in the proof of Theorem 1. □

Theorems 1, 2, and 3 say that every 3rd order differential invariant of coframes factors through the corresponding bundle projection (see (26), (27), (28)). Consider the components of the bundle projections defined by $p_j^i\colon T_n^3 L_n^1 \to L_n^1$, $I_{jk}^s\colon T_n^3 L_n^1 \to \mathbb{R}^n \otimes S_n^0$, $I_{jkl}^s\colon T_n^3 L_n^1 \to \mathbb{R}^n \otimes S_n^1$ and $I_{jklm}^s\colon T_n^3 L_n^1 \to \mathbb{R}^n \otimes S_n^2$. We have the following results.

COROLLARY 1. *The mappings* p^i_j, I^s_{jk}, I^s_{jkl}, I^s_{jklm} *represent a basis of* 3*rd order invariants of coframes with values in left* L^1_n*-manifolds.*

COROLLARY 2. *The mappings* p^i_j, $p^i_{j,k}$, $p^i_{j,kl}$, I^s_{jklm} *represent a basis of* 3*rd order invariants of coframes with values in left* L^2_n*-manifolds.*

COROLLARY 3. *The mappings* p^i_j, $p^i_{j,k}$, I^s_{jkl}, I^s_{jklm} *represent a basis of* 3*rd order invariants of coframes with values in left* L^3_n*-manifolds.*

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Added in proofs:

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the reader can find a discussion on the geometric meaning of invariants of frames and coframes.

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