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Krzysztof Banaszewski

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ALGEBRAIC PROPERTIES OF FUNCTIONS WITH THE CANTOR INTERMEDIATE VALUE PROPERTY

KRZYSZTOF BANASZEWSKI

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ABSTRACT. We prove that:

1. Every function can be expressed as a sum or product of two functions with Cantor intermediate value property ($CIVP$) and the pointwise or transfinite limit of functions with week Cantor intermediate value property ($CIVP$).

2. The maximal additive and multiplicative classes for the family $CIVP$ are equal to the family of all constant functions.

3. The uniform closure of the class $CIVP$ is equal to $\mathcal{U} \cap WCIVP$ (where $\mathcal{U}$ denotes the uniform closure of Darboux functions [Bruckner, A. M.—Ceder, J. G. —Weiss, M.: *Uniform limits of Darboux function*, Colloq. Math. **15** (1966), 65–77]).

1. Introduction

We shall consider only real functions of a real variable. We will use the following notations:

$Const$ — the class of constant functions,

$\mathcal{C}$ — the class of all continuous functions,

$\mathcal{D}$ — the class of Darboux functions,

$\mathcal{B}_1$ — the family of all functions of the first Baire class,

$\mathcal{PR}$ — the class of all functions having a bilateral perfect road at each point of the domain [6] (cf. [2]),

$CIVP$ — the class of functions f having the *Cantor intermediate value property*, i.e., functions for which the following condition is satisfied: for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$, there is a Cantor set C between x and y such that $f(C) \subset K$ ([4]),

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$WCIVP$ — the class of functions f having the *weak Cantor intermediate value property*, i.e., functions for which the following condition is satisfied: for every $x, y \in \mathbb{R}$ such that $f(x) < f(y)$, there is a Cantor set C between x and y such that $f(C) \subset \big(f(x), f(y)\big)$ ([5]),

$\mathcal{U}_0$ — the set of all functions f such that $f(J)$ is dense in the interval $\left[\inf_J f, \sup_J f\right]$ for each interval $J \subset \mathbb{R}$ ([3]),

$\mathcal{U}$ — the class of all functions f such that for every interval J and every set A of cardinality less than c (c means the cardinality of the reals), the set $f(J \setminus A)$ is dense in the interval $\left[\inf_J f, \sup_J f\right]$ ([3]).

R. G. Gibson and F. Roush prove that $CIVP \subset \mathcal{PR}$ ([4]), and therefore $CIVP \cap \mathcal{B}_1 = \mathcal{DB}_1$. It is clear that $CIVP$ is a proper subset of $WCIVP$. R. G. Gibson [5] showed that $\mathcal{D} \setminus WCIVP \neq \emptyset$. Let $\mathcal{X}$ be a class of real functions. The family of functions $\mathcal{M}_a(\mathcal{X}) = \{f \in \mathcal{X};\ \forall g \in \mathcal{X} \quad f + g \in \mathcal{X}\}$ is called the *maximal additive family* for $\mathcal{X}$. Similarly, we define $\mathcal{M}_m(\mathcal{X})$, the *maximal multiplicative family* of $\mathcal{X}$.

We will use the fact that we can represent every Cantor set C as the uncountable union of pairwise disjoint Cantor sets $\bigcup_{\alpha<c} C_\alpha$. Because there exists a homeomorphism $\phi\colon C \to C \times C$, we can find such a family $\{C_\alpha\}_{\alpha<c}$.

Towards the end of this paper, we state that the symbols $K^-(f,x)$, $K^+(f,x)$ denote the cluster sets from the left-hand side and from the right-hand side of the function f at a point x, respectively, and $K(f,x) = K^-(f,x) \cap K^+(f,x)$. Denote by $\mathcal{C}(f)$ the set of all points of continuity of f, and $\mathcal{D}(f) = \mathbb{R} \setminus \mathcal{C}(f)$.

Let x be a real and $A \subset \mathbb{R}$. Mark by $x + A = \{x + a;\ a \in A\}$ and $xA = \{xa;\ a \in A\}$. A symbol such as $[a,b]$ will always denote the interval with endpoints a and b whether or not $a < b$.

2. Algebraic properties

We shall say that a function f is nowhere constant on a set J if $f|_{(I \cap J)}$ is constant for no interval I such that $I \cap J \neq \emptyset$. Denote by A_f the set of all points $x \in J$ for which there is an open interval I such that $x \in I$, $I \cap J \neq \emptyset$, and $f|_{(I \cap J)}$ is constant. Then the set A_f is open in J. Let $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of open components of A_f. Denote by A^f the set $\bigcup_{n=1}^{\infty} \overline{J_n}$. Notice that $\mathbb{R} \setminus A^f$ is a G_δ set which is c-dense in itself, and for each open interval I such that $I \cap J \neq \emptyset$, $I \setminus A^f$ is a Baire space.

REMARK 2.1. *$\mathcal{D} \setminus CIVP \neq \emptyset$ and $CIVP \setminus \mathcal{D} \neq \emptyset$.*

Proof. Let $\{x_\alpha\}_{\alpha<c}$ be a transfinite sequence of all reals different from zero, and $\{I_n\}_{n=1}^{\infty}$ be a sequence of all open intervals having rational endpoints. We can find a family of pairwise disjoint Cantor sets $\{C_n\}_{n=1}^{\infty}$ such that $C_n \subset I_n$ for $n \in \mathbb{N}$. Represent each Cantor set C_n as a union of pairwise disjoint Cantor sets $\bigcup_{\alpha<c} C_{n,\alpha}$ for $n \in \mathbb{N}$. Let B be a Bernstein set in $\bigcup_{n=1}^{\infty} C_n$. Then for each $n \in \mathbb{N}$ the set $B \cap C_n$ is uncountable. Denote by ϕ_n a bijection between $B \cap C_n$ and $\mathbb{R}$. Put

$$f(x) = \begin{cases} x_\alpha & \text{if } x \in C_{n,\alpha}\,, \ n \in \mathbb{N}, \ \alpha < c\,, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} \phi_n(x) & \text{if } x \in B \cap C_n\,, \ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in CIVP \setminus \mathcal{D}$ and $g \in \mathcal{D} \setminus CIVP$. □

LEMMA 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous non constant function, and I be an open set such that $I \setminus A^f \neq \emptyset$. Then for each Cantor set $K \subset \mathbb{R} \setminus f(A^f)$, for any set P which is of the first category in $I \setminus A^f$, and for each real number y there is a Cantor set $C \subset I \setminus [A^f \cup P]$ which is of the first category in $I \setminus [A^f \cup P]$ and such that $(y + f(C)) \cap K = \emptyset$. If $0 \notin K$, there exists a Cantor set $C \subset I \setminus [A^f \cup P]$ of first category in $I \setminus [A^f \cup P]$ for which $(yf(C)) \cap K = \emptyset$.*

Proof. Let $g = f|_{(I \setminus A^f)}$. Then g is a continuous nowhere constant function, $(y+g)^{-1}(K)$ is a closed and nowhere dense set in $I \setminus A^f$. $L = (I \setminus A^f) \setminus [P \cup (y+g)^{-1}(K)]$ is a residual set in $I \setminus A^f$. Since $I \setminus A^f$ is a Baire space, L is a non-empty Borel set in $\mathbb{R}$. So we can find a Cantor set $C \subset L$ of the first category in $I \setminus [A^f \cup P]$. Therefore $C \subset I \setminus [A^f \cup P]$ and $C \cap (y+g)^{-1}(K) = \emptyset$. Since $y + f(C) = y + g(C)$, $[y + f(C)] \cap K = \emptyset$.

If $0 \notin K$, then in the same way, we can prove that there is a Cantor set C for which $(yf(C)) \cap K = \emptyset$. □

LEMMA 2.2. *Assume CH. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous non constant function, $K \subset \mathbb{R} \setminus f(A^f)$ be a Cantor set such that $0 \notin K$, and $\{y_\alpha\}_{\alpha<\omega_1}$ be a net of real numbers. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of all intervals having rational endpoints such that $I_n \setminus A^f \neq \emptyset$. We can choose an uncountable family of pairwise disjoint Cantor sets $\{C_{\alpha,n}\}_{\alpha<\omega_1,\, n\in\mathbb{N}}$ such that $C_{\alpha,n} \subset I_n \setminus A^f$ is of the first category in $I_n \setminus A^f$ for $\alpha < \omega_1$, $n \in \mathbb{N}$ and $(y_\alpha + f)(x) \notin K$ for $x \in C_{\alpha,n}$ $((y_\alpha f)(x) \notin K$ for $x \in C_{\alpha,n})$.*

Proof. Assume that we can choose all Cantor sets $C_{\beta,n}$ for $\beta < \alpha$ and $n \in \mathbb{N}$. By CH and by Lemma 2.1, $P = \bigcup_{\beta<\alpha} \bigcup_{n\in\mathbb{N}} C_{\beta,n}$ is of the first category in $I_n \setminus A^f$ for all $n \in \mathbb{N}$. Moreover, by that lemma, there exist Cantor sets $C_{\alpha,n} \subset I_n \setminus \left(P \cup A^f \cup \bigcup_{m<n} C_{\alpha,n}\right)$ such that $(y_\alpha + f)(C_{\alpha,n}) \cap K = \emptyset$ for $n \in \mathbb{N}$.

The proof that there exists a family of Cantor sets $\{C_{\alpha,n}\,;\ \alpha < \omega_1\,,\ n \in \mathbb{N}\}$ for which $(y_\alpha f)(x) \notin K$ for $x \in C_\alpha$ is analogous. □

THEOREM 2.1. *Assume CH. For each non constant function $f \in CIVP$ there exists a function $g \in CIVP$ such that $f+g \notin CIVP$ and $f+g \notin \mathcal{D}$. If $f \in \mathcal{D}$, then $g \in \mathcal{D} \cap CIVP$.*

Proof.

(1) Let $f \in CIVP \setminus \mathcal{C}$, and x_0 be a point at which f is discontinuous from the right (if f is a discontinuous function from the left, the proof is similar). By [1; Lemma 3.2], there exists a finite number $c \in K^+(f, x_0) \setminus \{f(x_0)\}$. Put

$$g(x) = \begin{cases} -c & \text{if } x \le x_0\,, \\ -f(x) & \text{if } x > x_0\,. \end{cases}$$

We can easily see that $g \in CIVP$, $(f+g)(x_0) = f(x_0) - c \neq 0$ and $(f+g)(x) = 0$ for $x > x_0$. This implies that $f+g \notin CIVP$ and $f+g \notin \mathcal{D}$. Moreover, if $f \in \mathcal{D}$, then $g \in \mathcal{D}$.

(2) Assume that $f \in \mathcal{C} \setminus \mathcal{C}onst$. Denote by $K \subset \left(\mathbb{R} \setminus f(A_f)\right) \cap (0,1)$ a Cantor set, and by $\{I_n\}_{n\in\mathbb{N}}$ the family of all intervals having rational endpoints for which $I_n \setminus A^f \neq \emptyset$. Let $\{y_\alpha\}_{\alpha<\omega_1}$ be an uncountable sequence of all real numbers. By Lemma 2.2, we can find a family $\{C_{\alpha,n}\}_{n\in\mathbb{N},\,\alpha<\omega_1}$ of Cantor sets such that $C_{\alpha,n} \subset I_n$ and $(y_\alpha + f)(x) \notin K$ for $x \in C_{\alpha,n}$. Let $x_0 \in \mathbb{R} \setminus A^f$. Then x_0 is a point of bilateral accumulation of $\mathbb{R} \setminus A^f$. Let $c = f(x_0) - 1$, and put

$$g(x) = \begin{cases} y_\alpha & \text{if } x \in C_{\alpha,n}\,, \\ -c & \text{if } x = x_0\,, \\ -f(x) & \text{otherwise}. \end{cases}$$

We shall prove that $g \in CIVP$.

Let $g(x) \neq g(y)$, and let $C \subset \left(g(x), g(y)\right)$ be an arbitrary Cantor set. Denote by z some point from C. Notice that x and y do not belong to the same component of A^f. Thus there exists an interval $I_n \subset (x,y)$ such that $I_n \setminus A^f \neq \emptyset$, and a Cantor set $C_{\alpha,n} \subset I_n$ for which $g(C_{\alpha,n}) = \{z\} \subset C$.

In the same way, we can prove that $g \in \mathcal{D}$.

Now we prove that $(f+g) \notin CIVP$. Let $x_1 \in \mathbb{R} \setminus \bigcup_{\alpha<\omega_1} \bigcup_{n\in\mathbb{N}} C_{\alpha,n}$ and $x_0 < x_1$. Then $(f+g)(x_0) = 1$, $(f+g)(x_1) = 0$, and $K \subset \left((f+g)(x_1), (f+g)(x_0)\right)$. Since $(f+g)(x) \notin K$ for any $x \in \mathbb{R}$, $f(C) \not\subset K$ for each Cantor set $C \subset (x_0, x_1)$. It is clear that $f+g \notin \mathcal{D}$. □

THEOREM 2.2. *Assume CH. For each non constant function $f \in CIVP$ there exists a function $g \in CIVP$ such that $fg \notin CIVP$ and $fg \notin \mathcal{D}$. Moreover, if $f \in \mathcal{D}$, then $g \in \mathcal{D} \cap CIVP$.*

Proof.

(1) Assume that f is continuous and non constant.

Denote by $K \subset (\mathbb{R} \setminus f(A_f)) \cap (0,1)$ some Cantor set, and by $\{I_n\}_{n\in\mathbb{N}}$ the family of all intervals having rational endpoints for which $I_n \setminus A^f \neq \emptyset$. Let $\{y_\alpha\}_{\alpha<\omega_1}$ be a sequence of all real numbers. By Lemma 2.2, we can find a family $\{C_{\alpha,n}\}_{n\in\mathbb{N},\,\alpha<\omega_1}$ of Cantor sets such that $C_{\alpha,n} \subset I_n$ and $(y_\alpha f)(x) \notin K$ for $x \in C_{\alpha,n}$. Let $x_0 \in \mathbb{R} \setminus A^f$. Put

$$g(x) = \begin{cases} y_\alpha & \text{if } x \in C_{\alpha,n}\,, \\ 1/f(x) & \text{if } f(x) \neq 0 \text{ and } x \notin \bigcup\limits_{\alpha<\omega_1} \bigcup\limits_{n=1}^{\infty} C_{\alpha,n}\,, \\ 0 & \text{if } \Big(f(x) = 0 \text{ and } x \notin \bigcup\limits_{\alpha<\omega_1} \bigcup\limits_{n=1}^{\infty} C_{\alpha,n}\Big) \text{ or } x = x_0\,. \end{cases}$$

We shall prove that $g \in CIVP \cap \mathcal{D}$.

Let $g(x) \neq g(y)$, and let $C \subset (g(x), g(y))$ be an arbitrary Cantor set. Denote by z some point from C. Notice that x and y do not belong to the same component of A^f. Thus there is an interval $I_n \subset (x,y)$ such that $I_n \setminus A^f \neq \emptyset$, and we can find a Cantor set $C_{\alpha,n} \subset I_n$ for which $g(C_{\alpha,n}) = \{z\} \subset C$. Now we prove that $(fg) \notin CIVP$. Let

$$x_1 \in \Big(\mathbb{R} \setminus \bigcup_{\alpha<\omega_1} \bigcup_{n\in\mathbb{N}} C_{\alpha,n}\Big) \cap (x_0, \infty)\,.$$

Then $(fg)(x_0) = 0$, $(fg)(x_1) = 1$, and $K \subset \big((fg)(x_0), (fg)(x_1)\big)$. Since $(fg)(x) \notin K$ for any $x \in \mathbb{R}$, $f(C) \not\subset K$ for each Cantor set $C \subset (x_0, x_1)$. In the same way, we can prove that $fg \notin \mathcal{D}$.

(2) Suppose that $f \in CIVP \setminus \mathcal{C}$. By (1), we can assume that if I is a closed interval and $f|_I$ is continuous, then $f|_I$ is constant.

We shall prove that there is a point $x_0 \in \mathcal{D}(f)$ for which $f(x_0) \neq 0$.

Assume that $f(\mathcal{D}(f)) = \{0\}$. Then there exists a point z_0 of continuity of f at which $f(z_0) \neq 0$, and an interval J such that $z_0 \in J$ and $f|_J$ is continuous. Therefore we have that $f|_J$ is constant. Let $a = \inf\Big\{x\,;\ f|_{[x,\,z_0]} \text{ is continuous}\Big\}$ and $b = \sup\Big\{x\,;\ f|_{[z_0,\,x]} \text{ is continuous}\Big\}$. Because $f \notin \mathcal{C}$, $(a,b) \neq \mathbb{R}$. Assume that $a \neq -\infty$. So f is discontinuous at a and $f(a) \neq 0$. This is impossible because we assume that $f(\mathcal{D}(f)) = \{0\}$. Therefore we can assume that f is discontinuous at x_0 from the right, and $a = f(x_0) \neq 0$ (if f is discontinuous at x_0 from the left, the proof is similar).

We shall consider two cases.

(a) There exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \searrow x_0$ and $f(x_n) = 0$.

Define $g \colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2a & \text{for } x \le x_0 , \\ 2a - f(x) & \text{for } x > x_0 . \end{cases}$$

According to [1; Lemma 3.3 and Theorem 3.5], $g \in CIVP$, and if $f \in \mathcal{D}$, then $g \in \mathcal{D}$. On the other hand, $fg \notin CIVP$ and $fg \notin \mathcal{D}$ since $fg(x_0) = 2a^2$, and for $x > x_0$

$$fg(x) = \bigl(2a - f(x)\bigr)f(x) \le a^2 < 2a^2 .$$

(b) There exists $d > 0$ for which $f(x) \neq 0$ for $x \in (x_0, x_0 + d]$. Let $c \notin \{\pm f(x_0), 0\}$ be a point from $K^+(f, x_0)$. By [1; Lemma 3.2], such a point exists. Then we define

$$g(x) = \begin{cases} 1/|c| & \text{for } x \le x_0 , \\ 1/|f(x)| & \text{for } x \in (x_0, x_0 + d] , \\ 1/|f(x_0 + d)| & \text{for } x > x_0 + d . \end{cases}$$

By [1; Lemma 3.3 and Theorems 3.4, 3.5], $g \in CIVP$. Notice that $fg(x_0) = f(x_0)/|c| \neq \pm 1$, and $fg(x) = \pm 1$ for $x \in (x_0, x_0 + d]$. Thus $fg \notin CIVP$ and $fg \notin \mathcal{D}$. □

COROLLARY 2.1. $\mathcal{M}_a(CIVP) = \mathcal{M}_m(CIVP) = \mathcal{M}_a(\mathcal{D} \cap CIVP) = \mathcal{M}_m(\mathcal{D} \cap CIVP) = \mathcal{C}onst$.

THEOREM 2.3. *Assume CH. Let $\{f_\alpha\}_{\alpha < \omega_1}$ be a class of real functions. Then there is a function $f \in CIVP \cap \mathcal{D}$ such that $f + f_\alpha \in CIVP \cap \mathcal{D}$ for all $\alpha < c$.*

Proof. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of all intervals having rational endpoints. Let $\{C_{n,i,\alpha} : n \in \mathbb{N},\ i = 1, 2,\ \alpha < c\}$ be a class of Cantor sets which fulfils the conditions:

$$C_{n,i,\alpha} \cap C_{m,j,\beta} \neq \emptyset \qquad \text{for} \quad (n, i, \alpha) \neq (m, j, \beta)$$

and

$$C_{n,i,\alpha} \subset I_n , \qquad \text{where} \quad n \in \mathbb{N},\ i = 1, 2,\ \alpha < c .$$

Let $\{x_\alpha\}_{\alpha < c}$ be a sequence of all real numbers. Put

$$f(x) = \begin{cases} x_\alpha - f_\alpha(x) & \text{for } x \in C_{n,1,\alpha} , \\ x_\alpha & \text{for } x \in C_{n,2,\alpha} ,\ n \in \mathbb{N},\ \alpha < c , \\ 0 & \text{otherwise} . \end{cases}$$

Then $f, f + f_\alpha \in CIVP \cap \mathcal{D}$. □

COROLLARY 2.2. *Every $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed as the union of two CIVP functions.*

THEOREM 2.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$. Then there exist CIVP functions $g, h\colon \mathbb{R} \to \mathbb{R}$ such that $f = gh$.*

Proof. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of all open intervals whose endpoints are rationals. Then we can find families of pairwise disjoint Cantor sets $\{C_{k,1}\}_{k=1}^{\infty}$ and $\{C_{k,2}\}_{k=1}^{\infty}$ such that $C_{k,n} \subset I_k$ for each $k \in \mathbb{N}$, $n = 1, 2$. We can represent each set $C_{k,n}$, $k \in \mathbb{N}$, $n = 1, 2$, as the union pairwise disjoint Cantor sets $\bigcup_{\alpha<c} C_{k,n,\alpha}$. Let $\{r_\alpha\}_{\alpha<c}$ be a net of all real numbers different from zero. Put

$$g(x) = \begin{cases} r_\alpha & \text{if } x \in C_{k,1,\alpha}, \quad k \in \mathbb{N}, \quad \alpha < c, \\ f(x)/r_\alpha & \text{if } x \in C_{k,2,\alpha}, \quad k \in \mathbb{N}, \quad \alpha < c, \\ f(x) & \text{otherwise}, \end{cases}$$

and

$$h(x) = \begin{cases} f(x)/r_\alpha & \text{if } x \in C_{k,1,\alpha}, \quad k \in \mathbb{N}, \quad \alpha < c, \\ r_\alpha & \text{if } x \in C_{k,2,\alpha}, \quad k \in \mathbb{N}, \quad \alpha < c, \\ 1 & \text{otherwise}. \end{cases}$$

We can easily see that $gh = f$ and $f, g \in CIVP$. □

If $\{f_t\}_{t\in\mathcal{T}}$ is an arbitrary class of real functions, then $f = 0 = 0f_t \in CIVP$ for all $t \in \mathcal{T}$. Let us ask whether Theorem 2.3 is true if a sum becomes a composition and assume that $f \neq 0$. The answer is negative, even if the family f_α contains only one function.

REMARK 2.2. *There exists a function f such that $fg \notin CIVP$ for each function $g \in CIVP \setminus \{0\}$.*

Proof. Denote by A a Bernstein set. Put

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in \mathbb{R} \setminus A. \end{cases}$$

Assume that there exists a function $g \in CIVP \setminus \{0\}$ such that $fg \in CIVP$. Then we can find a point $y \in \mathbb{R} \setminus A$ such that $g(y) \neq 0$. Choose an $x \in A \cap (-\infty, y)$. Then does not exist a Cantor set $C \subset (x, y)$ such that $fg(C) \subset (0, fg(y))$. □

EXAMPLE 2.1. *There exists a family $\mathcal{F} \subset \mathcal{B}_1$ with $\operatorname{card}(\mathcal{F}) = c$ such that for each function $g\colon \mathbb{R} \to \mathbb{R}$ if $fg \in CIVP$ for each $f \in \mathcal{F}$, then $g = 0$.*

Proof. Let $\{x_\alpha\}_{\alpha<c}$ be a net of all reals. Put

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x = x_\alpha, \\ 0 & \text{if } x \neq x_\alpha, \end{cases}$$

If $g \neq 0$, then there exists a point a such that $g(a) \neq 0$. We can find an $\alpha < c$ such that $a = x_\alpha$, and we have $gf_\alpha = g(a)f_\alpha \notin CIVP$. □

THEOREM 2.5. *Assume CH. Let $\{f_k\}_{k=1}^{\infty}$ be a countable family of Lebesgue measurable functions. There exists a Lebesgue measurable function $f \in CIVP \cap \mathcal{D}$ such that $f \neq 0$ and $ff_k \in CIVP \cap \mathcal{D}$ for all $k \in \mathbb{N}$. Moreover, if for each $k \in \mathbb{N}$, f_k is with the Baire property, then f can be taken with the Baire property, too.*

Proof. Let $[f_k = 0] = \{x \in \mathbb{R};\ f_k(x) = 0\}$ and $[f_k \neq 0] = \{x \in \mathbb{R};\ f_k(x) \neq 0\}$ for $k \in \mathbb{N}$. Denote for each $k \in \mathbb{N}$ by S_k (P_k) the set of all points $x \in [f_k \neq 0]$ ($x \in [f_k = 0]$) for which there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap [f_k \neq 0]$ ($(x-\varepsilon, x+\varepsilon) \cap [f_k = 0]$) has measure zero, and let $S = \bigcup_{k=1}^{\infty} S_k$, $P = \bigcup_{k=1}^{\infty} P_k$. Fix a $k \in \mathbb{N}$. If $[f_k = 0]$ has positive measure, then let $A_k \subset [f_k = 0] \setminus P$ be a Borel set c-dense in itself such that the measure of $[f_k = 0] \setminus A_k$ is zero. Otherwise, let $A_k = \emptyset$. If $[f_k \neq 0]$ has positive measure, then let $B_k \subset [f_k \neq 0] \setminus S$ be a Borel set c-dense in itself such that the measure of $[f_k \neq 0] \setminus B_k$ is zero. Otherwise, let $B_k = \emptyset$. Let $\{I_n\}_{n=1}^{\infty}$ be a family of all open intervals having rational endpoints. By [7; Lemma 2], there exists a family of pairwise disjoint sets $\{C_{k,n}, K_{k,n}\}_{k,n \in \mathbb{N}}$ such that $C_{k,n} \subset I_n \cap A_k$, $K_{k,n} \subset I_n \cap B_k$,

$$C_{k,n} = \begin{cases} \text{is a Cantor set} & \text{if } I_n \cap A_k \neq \emptyset, \\ \emptyset & \text{if } I_n \cap A_k = \emptyset, \end{cases}$$

and

$$K_{k,n} = \begin{cases} \text{is a Cantor set} & \text{if } I_n \cap B_k \neq \emptyset, \\ \emptyset & \text{if } I_n \cap B_k = \emptyset. \end{cases}$$

We can represent each Cantor set $K_{k,n}$ as the union $\bigcup_{i=1}^{2} \bigcup_{\alpha<c} K_{k,n,\alpha}^i$ of pairwise disjoint Cantor sets for $k, n \in \mathbb{N}$. Similarly, let $C_{k,n} = \bigcup_{\alpha<c} C_{k,n,\alpha}$, where $\{C_{k,n,\alpha}\}_{\alpha<c}$ is a net of pairwise disjoint Cantor sets. Let $\{x_\alpha\}_{\alpha<c}$ be a transfinite sequence of all reals. Put

$$f(x) = \begin{cases} x_\alpha & \text{if } x \in C_{k,n,\alpha} \cup K_{k,n,\alpha}^1, \\ x_\alpha / f_k(x) & \text{if } x \in K_{k,n,\alpha}^2,\ k, n \in \mathbb{N},\ \alpha < c, \\ 0 & \text{otherwise}. \end{cases}$$

Then $f \in CIVP \cap \mathcal{D}$ is a Lebesgue measurable function, and

$$ff_s(x) = \begin{cases} x_\alpha f_s(x) & \text{if } x \in K_{k,n,\alpha}^1, \\ x_\alpha & \text{if } x \in K_{s,n,\alpha}^2, \\ x_\alpha f_s(x)/f_k(x) & \text{if } x \in K_{k,n,\alpha}^2,\ k \in \mathbb{N} \setminus \{s\}, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in \mathbb{N}$. Let $x < y$, and assume that $ff_s(x) < ff_s(y)$. Choose a Cantor set $K \subset \big(ff_s(x), ff_s(y)\big)$, and denote by z some point belonging to K. Then there is an interval $I_n \subset (x,y)$ such that $B_s \cap I_n \neq \emptyset$, $z = x_\alpha$ for some $\alpha < c$, and $ff_s(K^2_{s,n,\alpha}) = \{z\} \subset K$. Similarly, we can prove that $ff_s \in \mathcal{D}$.

If the functions f_k have the Baire property, then we can easy see that f has the Baire property. □

3. Uniform limits of $CIVP$ functions

THEOREM 3.1. *$CIVP \subset \mathcal{U}$.*

P r o o f . Fix an open set $U \subset \Big[\inf_J f, \sup_J f\Big] = [m, M]$, $f \in CIVP$, and $J = [\alpha, \beta] \subset \mathbb{R}$. Let A be a set of cardinality less then c, and (a,b) be an interval whose closure is contained in U. Then there are numbers $\alpha_1, \beta_1 \in J$ such that

$$m \le f(\alpha_1) \le a < b \le f(\beta_1) \le M .$$

Let $K \subset (a,b)$ be a Cantor set. Then there exists a Cantor set $C \subset (\alpha_1, \beta_1)$ such that $f(C) \subset K \subset (a,b)$. Because $C \setminus A \neq \emptyset$, $\emptyset \neq f(C \setminus A) \subset (a,b) \subset U$. □

THEOREM 3.2. *In the class of all Borel measurable functions, $\mathcal{U}$ is a proper subset $WCIVP$.*

P r o o f . Fix a Borel measurable function $f \in \mathcal{U}$ and points $a, b \in \mathbb{R}$ such that $f(a) < f(b)$. Assume that $a < b$. Then $B = f^{-1}\big((f(a), f(b)\big) \cap (a,b)$ is a Borel set whose cardinality is equal to the continuum, and we can find a Cantor set $C \subset B$. It is easy to observe that $f(C) \subset \big(f(a), f(b)\big)$.

To prove that $\mathcal{U} \neq WCIVP$, we consider a function

$$f(x) = \begin{cases} x & \text{if } x > 0\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

□

REMARK 3.1. *In the class of all Borel measurable functions, $\mathcal{U} \setminus CIVP \neq \emptyset$.*

P r o o f . Let $\{I_n\}_{n=1}^\infty$ be a family of all open intervals having rational endpoints, and let $\{C_{n,m}\}_{n,m=1}^\infty$ be a sequence of pairwise disjoint Cantor sets such that $C_{n,m} \subset I_n$. Denote by $\{q_m\}_{m=1}^\infty$ an enumeration of all rationals. Define

$$f(x) = \begin{cases} q_m & \text{if } x \in \bigcup\limits_{n=1}^\infty C_{n,m}\,, \quad m \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\overline{f(J \setminus A)} = \mathbb{R}$ for each interval J, for each a set A of cardinality less then c, so $f \in \mathcal{U}$. Let $K \subset \mathbb{R} \setminus Q$ be a Cantor set. Then $f(C) \cap K = \emptyset$ for each a Cantor set C. So we have $f \notin CIVP$. □

THEOREM 3.3. $\mathcal{U} \cap WCIVP = \mathcal{U}_0 \cap WCIVP$.

P r o o f . For the proof, we must show that if $f \in \mathcal{U}_0 \cap WCIVP$, then $f \in \mathcal{U}$. Assume that $a < b$, $f(a) < f(b)$. Put by $J = [a,b]$. Denote by $U \subset (f(a), f(b))$ some open interval and by A arbitrary set for which $\operatorname{card} A < c$. Because $f(J)$ is dense in U, then there exist points $x_1, x_2 \in J$ and $y_1, y_2 \in U$ such that $y_1 \neq y_2$, $f(x_1) = y_1$ and $f(x_2) = y_2$. We can find a Cantor set $C \subset (x_1, x_2)$ with $f(C) \subset (y_1, y_2)$. Since $C \setminus A$ is non-empty, then there exists $x \in C \setminus A$ and $f(x) \in U$. □

REMARK 3.2. *$\mathcal{U}_0 \cap CIVP$ is a proper subset of $\mathcal{U}_0 \cap WCIVP$.*

P r o o f . We need only prove that the inclusion is proper. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of all open intervals having rational endpoints. Choose a net of pairwise disjoint Cantor sets $\{C_{n,\alpha}\}_{n\in\mathbb{N},\, \alpha<c}$ with $C_{n,\alpha} \subset I_n$. Let K be a Cantor set, and $\{r_\alpha\}_{\alpha<c}$ be a net of all points of $\mathbb{R} \setminus K$. Define

$$f(x) = \begin{cases} r_\alpha & \text{if } x \in \bigcup\limits_{n=1}^{\infty} C_{n,\alpha}\,, \quad \alpha < c\,, \\ 0 & \text{otherwise}. \end{cases}$$

Then $f \in (\mathcal{U}_0 \cap WCIVP) \setminus (\mathcal{U}_0 \cap CIVP)$. □

REMARK 3.3. *$WCIVP$ is not uniformly closed.*

P r o o f . Put

$$f_n(x) = \begin{cases} -1 & \text{if } x \le 0\,, \\ x/n & \text{if } x \in (0,1)\,, \\ 1/n & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. Then $f_n \in WCIVP$, f_n is uniformly convergent to

$$f(x) = \begin{cases} -1 & \text{if } x \le 0\,, \\ 0 & \text{if } x > 0\,, \end{cases}$$

and $f \notin WCIVP$. □

THEOREM 3.4. *If f is the uniform limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of $CIVP$ functions, then $f \in \mathcal{U}_0 \cap WCIVP$.*

P r o o f . Let $J = [a,b]$. Without loss of generality, we can assume $f(a) < f(b)$. Let U be an open interval whose closure is contained in $(f(a), f(b))$. Express U as $(y-\varepsilon, y+\varepsilon)$. Then there exists an $n \in \mathbb{N}$ such that $|f_n(x) - f(x)|$

$< \varepsilon/4$ for $x \in J$, and $f_n(a) < y - \varepsilon$ and $f_n(b) > y + \varepsilon$. Since $f_n \in CIVP$, there exists a Cantor set $C \subset (a,b)$ such that $f_n(C) \subset (y - \varepsilon/4, y + \varepsilon/4)$. Then $f(C) \subset (y - \varepsilon/2, y + \varepsilon/2) \subset U$, and this implies that $f \in WCIVP$. Because $CIVP \subset \mathcal{U}$ and $\mathcal{U}$ is closed under the operation of uniform limit ([3]), $f \in \mathcal{U} \subset \mathcal{U}_0$. □

LEMMA 3.1. *Let $J = (a,b)$, $f \in \mathcal{U}_0 \cap WCIVP$, $A = f^{-1}(J)$, and denote by $\{I_m\}_{m=1}^{\infty}$ the set of all intervals having rational endpoints for which $I_m \cap A \neq \emptyset$. If $A \neq \emptyset$, then there exists a sequence of pairwise disjoint Cantor sets $\{C_m\}_{m=1}^{\infty}$ such that $C_m \subset A \cap I_m$ for $m \in \mathbb{N}$.*

Proof. Denote by $\{I_m\}_{m=1}^{\infty}$ the set of all open intervals having rational endpoints for which $I_m \cap A \neq \emptyset$. First we shall prove that if $f|_{(I_m \cap A)}$ is constant, then $I_m \subset A$.

Assume that $f|_{(I_m \cap A)}$ is constant, $x \in I_m \setminus A$ and $f(I_m \cap A) = \{z\}$. Then $f(x) \notin (a,b)$. Suppose that $f(x) \geq b$. Let $U = (z,b)$. Because $f(I_m) \cap U = \emptyset$, then $f \notin \mathcal{U}$.

Choose an $m \in \mathbb{N}$. If $f|_{(I_m \cap A)}$ is constant, then we can find a Cantor set $K_m \subset I_m \cap A$. Otherwise, there exist points $x_m, y_m \in I_m \cap A$ such that $f(x_m) < f(y_m)$. Then there is a Cantor set $K_m \subset (x_m, y_m)$, and $f(K_m) \subset \big(f(x_m), f(y_m)\big)$. Thus $K_m \subset f^{-1}\big(f(K_m)\big) \subset f^{-1}(J) = A$. By [7; Lemma 2], we can find a sequence of pairwise disjoint Cantor sets $\{C_m\}_{m=1}^{\infty}$ such that $C_m \subset K_m \subset I_m \cap A$ for each $m \in \mathbb{N}$. □

THEOREM 3.5. *If $f \in \mathcal{U}_0 \cap WCIVP$, then f is the uniform limit of some sequence of $CIVP \cap \mathcal{D}$ functions.*

Proof. Choose an $\varepsilon > 0$. It is enough to prove that there exists a function $g \in CIVP \cap \mathcal{D}$ with $\|f - g\| < \varepsilon$. If f is constant, then we can put $g = f$. If g is not constant, we can assume without loss of generality that the closure of range of f is $\mathbb{R}$. Now decompose $\mathbb{R}$ into disjoint half open intervals $\{J_n\}_{n=1}^{\infty}$ each of length $\varepsilon/2$. Put $A_n = f^{-1}\big(\mathrm{int}(J_n)\big)$. Choose an $n \in \mathbb{N}$. Denote by $\{I_{n,m}\}_{m=1}^{\infty}$ the family of all open intervals having rational endpoints for which $I_{n,m} \cap A_n \neq \emptyset$. By Lemma 3.1, there exists a sequence $\{C_{n,m}\}_{m=1}^{\infty}$ of pairwise disjoint Cantor sets such that $C_{n,m} \subset I_{n,m} \cap A_n$. We may decompress each $C_{n,m}$ into pairwise disjoint Cantor sets $\{C_{n,m,\alpha}\}_{\alpha<c}$. Denote by $\{r_{n,\alpha}\}_{\alpha<c}$ the net of all points of $\overline{J_n}$. Now define the function

$$g(x) = \begin{cases} r_{n,\alpha} & \text{for } x \in \bigcup\limits_{m=1}^{\infty} C_{n,m,\alpha}, \quad n \in \mathbb{N}, \quad \alpha < c, \\ f(x) & \text{otherwise.} \end{cases}$$

It is obvious that $\|f - g\| < \varepsilon$. To show that $g \in CIVP$, we suppose that $a < b$ and $g(a) < g(b)$. Choose a Cantor set $K \subset \big(g(a), g(b)\big)$. Then there exists a point $r_{n,\alpha} \in K \cap J_n$ for some $n \in \mathbb{N}$ and $\alpha < c$, and a Cantor set $C_{n,m,\alpha} \subset A_n \cap (a,b)$ for which $g(C_{n,m,\alpha}) = \{r_{n,\alpha}\} \subset K$. Similarly, we can prove that $g \in \mathcal{D}$. □

According to Theorems 3.4 and 3.5, we can prove the following Theorem.

THEOREM 3.6. *The uniform closure of CIVP is $\mathcal{U}_0 \cap WCIVP$.*[1)]

4. Pointwise and transfinite limits

THEOREM 4.1. *Every real function of real variable f is a pointwise limit of $CIVP \cap \mathcal{D}$ functions f_n. If f is measurable or has the Baire property, then f_n can be measurable or have the Baire property, too.*

Proof. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of all open intervals having rational endpoints. We can find a family $\{C_{k,n}\}_{k,n=1}^{\infty}$ of pairwise disjoint Cantor sets such that $C_{k,n} \subset I_k$ for $k, n \in \mathbb{N}$. Represent each $C_{k,n}$ as the union $\bigcup_{\alpha<c} C_{k,n,\alpha}$ of pairwise disjoint perfect sets. Let $(x_\alpha)_{\alpha<c}$ be a transfinite sequence of all reals. Put

$$D_{n,\alpha} = \bigcup_{k=1}^{\infty} C_{k,n,\alpha}$$

and

$$f_n(x) = \begin{cases} x_\alpha & \text{if } x \in D_{n,\alpha}, \ \alpha < c, \\ f(x) & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. Then $f_n \in CIVP \cap \mathcal{D}$. We shall show that

$$f(x) = \lim_{n\to\infty} f_n(x). \tag{1}$$

Choose an $x \in \mathbb{R}$. If $x \notin \bigcup_{n=1}^{\infty} \bigcup_{\alpha<c} D_{n,\alpha}$, then $f_n(x) = f(x)$ for each $n \in \mathbb{N}$, and (1) holds. Otherwise, $x \in D_{n_0,\alpha}$ for some $n_0 \in \mathbb{N}$, $\alpha < c$, and since $x \notin D_{n,\alpha}$ for $n > n_0$, $\alpha < c$, so $f_n(x) = f(x)$ for $n > n_0$, which completes the proof. If f is measurable or has the Baire property, then f_n are measurable or have the Baire property. □

Recall that a function f is the limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of functions if and only if for each positive $\varepsilon > 0$ and $x \in \mathbb{R}$ there exists an $\alpha < \omega_1$ such that $|f(x) - f_\beta(x)| < \varepsilon$ for all $\beta > \alpha$.

[1)] Note that $\mathcal{U}_0 \cap WCIVP = \mathcal{U} \cap WCIVP = \mathcal{U} \cap \mathcal{PR}$.

THEOREM 4.2. *Every function $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of CIVP functions with the Darboux property. Moreover, if f is measurable or has the Baire property, then each f_α can be measurable or have the Baire property.*

Proof. Let $(I_k)_{k=1}^{\infty}$ be a sequence of all open intervals with rational endpoints. We shall use the fact that in each interval I_k, we can choose a Cantor set C_k such that $C_k \cap C_n = \emptyset$ for $n < k$. Represent each C_k as a union $\bigcup\limits_{\alpha<\omega_1} \bigcup\limits_{\beta<c} C_{k,\alpha,\beta}$ of pairwise disjoint closed sets. Let $(x_\beta)_{\beta<c}$ be a net of all reals. Put

$$D_{\alpha,\beta} = \bigcup_{k=1}^{\infty} C_{k,\alpha,\beta}$$

and

$$f_\alpha(x)\begin{cases} x_\beta & \text{if } x \in D_{\alpha,\beta}\,,\ \beta < c\,, \\ f(x) & \text{otherwise} \end{cases}$$

for $\alpha < \omega_1$. Then each function $f_\alpha \in CIVP \cap \mathcal{D}$. We shall show that

$$f(x) = \lim_{\alpha\to\omega_1} f_\alpha(x)\,. \tag{2}$$

Choose an $x \in \mathbb{R}$. Then either $x \notin \bigcup\limits_{\alpha<\omega_1} \bigcup\limits_{\beta<c} D_{\alpha,\beta}$, so $f_\alpha(x) = f(x)$ for each $\alpha < \omega_1$. If $x \in D_{\alpha,\beta}$ for some $\beta < c$, then $x \notin D_{\alpha,\gamma}$ for $\gamma > \beta$, so $f_\alpha(x) = f(x)$ for $\gamma > \beta$. □

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Instytut Matematyki
Wyzsza Szkola Pedagogiczna
Chodkiewicza 30
PL-85-064 Bydgoszcz
POLAND