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MEASURES OF NONCOMPACTNESS IN BANACH SEQUENCE SPACES

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ABSTRACT. We construct a measure of noncompactness in the sequence space $l^p(E_i)$ which turns out to be regular but not equivalent to the Hausdorff measure of noncompactness. Apart from that a formula for the Hausdorff measure in the sequence space $c_0(E_i)$ is derived.

1. Introduction

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The theory connected with this notion was initiated by Kuratowski [12] and Darbo [7], but the main applications of measures of noncompactness were pointed out by Sadovskii [15], Ambrosetti [2], Nussbaum [14] and Daneš [6], among others. The current state of this theory and its applications are presented in the books [1, 3], for example.

In this paper we shall study measures of noncompactness in some Banach sequence spaces.

At the beginning we establish some notation.

Assume that E is a Banach space with the norm $\|\cdot\|$ and the zero element θ. By $K(x, r)$ we denote the closed ball centered at x and radius r. The unit ball $K(\theta, 1)$ will be denoted by B_E or shortly by B. Moreover, $\overline{X}$, $\operatorname{Conv} X$ denote the closure and the convex closure of a set X, respectively.

Finally, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

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DEFINITION 1. *A function $\mu\colon \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ will be called a measure of noncompactness in E if it satisfies the following conditions:*

1° $\mu(X) = 0 \iff X \in \mathfrak{N}_E$,
2° $X \subset Y \implies \mu(X) \le \mu(Y)$,
3° $\mu(\operatorname{Conv} X) = \mu(X)$,
4° $\mu(X \cup Y) = \max\{\mu(X),\, \mu(Y)\}$,
5° $\mu(X+Y) \le \mu(X) + \mu(Y)$,
6° $\mu(cX) = |c|\mu(X)$, $c \in \mathbb{R}$.

Notice that in book [3] measures of noncompactness defined above are called regular.

Remark 1. Let us mention that Definition 1 implies that the measure μ has also the following property:

7° If $X_n \supset X_{n+1}$ and $\overline{X}_n = X_n$ for $n = 1,2,\dots$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the set $\bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

This fact may be proved in the same fashion as Theorem 5 in [4].

Recall that the functions χ, $I\colon \mathfrak{M}_E \to \mathbb{R}_+$ defined by

$$\chi(X) = \inf\{r > 0\colon \text{ there exists a finite set } Y \subset E \text{ such that } X \subset Y + rB\},$$
$$I(X) = \sup_{(x_n)\subset X} \{\inf\{\|x_j - x_i\|_i\colon\ i \neq j,\ i,j = 1,2,\dots\}\}$$

are measures of noncompactness in the sense of Definition 1 possessing also some additional properties [1]. For example, $\chi(B) = 1$.

The function χ is said to be the *Hausdorff measure* and it seems to be the most convenient in applications (cf. [1, 3]). The function I is referred to as the *Istrătescu measure of noncompactness.* Recall that this function was introduced by Istrătescu in [9]. In paper [5] Daneš raised the question whether I is a measure of noncompactness in the above defined sense.

This question was answered in the affirmative by Erzakova [8] (cf. also [1]).

Let us observe that the measure χ and I are equivalent, i.e. the following inequality holds

$$\chi(X) \le I(X) \le 2\chi(X) \tag{1}$$

for any $X \in \mathfrak{M}_E$ (cf. [1]).

2. Auxiliary facts concerning Banach sequence spaces

Assume that $(E_i, \|\cdot\|_i)$ is a sequence of Banach spaces. Denote by $l^p(E_i)$, or briefly by l^p, $1 \le p \le \infty$, the space of all sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \dots$ such that $\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty$. Similarly, let $c_0 = c_0(E_i)$ denote the space of all sequences $x = (x_i)$, $x_i \in E_i$ with the property $\|x_i\|_i \to 0$ as $i \to \infty$. It is well known [11, 13] that both $l^p(E_i)$ and $c_0(E_i)$ form Banach spaces under the norms

$$\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p\right)^{1/p},$$

$$\|x\|_0 = \max\{\|x_i\|_i:\ i = 1, 2, \dots\},$$

respectively. Similarly we can define the space $l^\infty(E_i)$.

In the case when $E_i = E$ for all $i = 1, 2, \dots$ we shall write $l^p(E)$ and $c_0(E)$. Such a case was discussed in [13], for example.

For further purposes denote by p_n the projection operator $p_n\colon\ l^p(E_i) \to E_n$ (or $p_n\colon\ c_0(E_i) \to E_n$), $p_n(x) = p_n(x_1, x_2, \dots) = x_n$ $(n = 1, 2, \dots)$.

Moreover, let us recall the following theorem [13].

THEOREM 1. *A set $X \subset l^p(E_i)$ is relatively compact if and only if*

a) *X is bounded,*

b) *the set $p_n(X)$ is relatively compact in E_n for any $n = 1, 2, \dots$, and*

c) *for every $\varepsilon > 0$ there exists a positive integer n_0 such that*

$$\sum_{i=n}^{\infty} \|x_i\|_i^p < \varepsilon \quad \textit{for all } x = (x_i) \in X \quad \textit{whenever } n \ge n_0 .$$

3. Measures of noncompactness in $l^p(E_i)$ and $c_0(E_i)$

In this section we introduce rather convenient formulas for some measures of noncompactness in the Banach sequence spaces $l^p(E_i)$ and $c_0(E_i)$. To do this assume that χ_i is the Hausdorff measure of noncompactness in the space E_i, $i = 1, 2, \dots$ and let χ_p denote Hausdorff measure in the space $l^p(E_i)$. Further, for a set $X \in \mathfrak{M}_{l^p}$ let us denote

$$a(X) = \sup\{\chi_i(p_i(X)):\ i = 1, 2, \dots\},$$

$$b(X) = \lim_{n\to\infty}\left(\sup\left\{\left(\sum_{i=n}^{\infty} \|x_i\|_i^p\right)^{1/p}:\ x = (x_i) \in X\right\}\right),$$

$$\mu_p(X) = \max\{a(X),\ b(X)\}.$$

Then we have

THEOREM 2. *The function* μ_p *is a measure of a noncompactness in the space* $l^p(E_i)$ *such that*

$$\mu_p(X) \leq \chi_p(X)$$

for any $X \in \mathfrak{M}_{l^p}$.

P r o o f . The proof of the first part is very simple and is therefore omitted.

To prove the second part denote $\chi_p(X) = r$. Then for an arbitrary $\varepsilon > 0$ we can find a finite set $Y \subset l^p$ such that $X \subset Y + (r+\varepsilon)B_{l^p}$. Hence, using the equality $\mu_p(B_{l^p}) = 1$ we infer that $\mu_p(X) \leq r + \varepsilon$. The arbitrariness of ε completes the proof.

In what follows we show that there does not exist a constant $c > 0$ such that

$$c\chi_p(X) \leq \mu_p(X) \tag{2}$$

for any $X \in \mathfrak{M}_{l^p}$, provided the spaces E_i ($i = 1,2,\dots$) are assumed to be infinite dimensional.

Consider namely the sequence of subsets of $l^p(E_i)$ defined in the following way

$$X_n = \left\{x = (x_1, x_2, \dots, x_n, \theta, \theta, \dots) : \ x_k \in B_{E_k} \quad \text{for} \quad k = 1,2,\dots,n\right\}.$$

Obviously, we have

$$\mu_p(X_n) = 1 \tag{3}$$

for any $n = 1,2,\dots$.

On the other hand, by the Riesz lemma (see the improved version in [10]) for any i , $1 \leq i \leq n$, we can select a sequence $(x_k^i)_{k=1,2,\dots}$ of points from B_{E_i} such that $\|x_k^i - x_m^i\| > 1$ for $k \neq m$, $k,m = 1,2,\dots$.

Now, let us fix a natural number n and consider the sequence (x^k) of points from X_n of the form

$$x^k = (x_k^1, x_k^2, \dots, x_k^n, \theta, \theta, \dots),$$

where $k = 1,2,\dots$. For $k \neq m$ we have

$$\|x^k - x^m\|_p = \left(\sum_{i=1}^{n} \|x_k^i - x_m^i\|_i^p\right)^{1/p} > n^{1/p},$$

what implies that $I(x_n) \geq n^{1/p}$. Consequently, in virtue of (1) we get

$$\chi_p(X_n) \geq \frac{1}{2} n^{1/p}.$$

Combining this inequality and (3) we see that the inequality (2) does not hold for any $c > 0$.

Remark 2. In 1978 K. Goebel raised the question if each regular measure of noncompactness (i.e. a measure of noncompactness in the sense of Definition 1) has to be equivalent to the Hausdorff measure χ (cf. also [3]). The example of the measure μ_p described above answers this question in the negative.

In what follows we shall deal with a measure of noncompactness in the space $c_0(E_i)$. Let us denote by χ_0 the Hausdorff measure of noncompactness in c_0. Next, let $\mu_0 \colon \mathfrak{M}_{c_0} \to \mathbb{R}_+$ be the function defined by the formula

$$\mu_0(X) = \max\{a(X),\, c(X)\},$$

where $a(X)$ was defined previously and $c(X)$ is given by

$$c(X) = \lim_{n\to\infty}\Big(\sup_{x=(x_i)\in X}\Big(\max\{\|x_k\|_k \colon\ k \geq n\}\Big)\Big).$$

Then we have the following theorem.

THEOREM 3. *$\chi_0(X) = \mu_0(X)$ for any $X \in \mathfrak{M}_{c_0}$.*

Proof. First, let us observe that $\mu_0(B_{c_0}) = 1$. It is also easily seen that the function μ_0 satisfies the conditions $2^\circ - 6^\circ$ of the Definition 1. Moreover, from the definition of μ_0 it follows that $\mu_0(Y) = 0$ for any finite subset Y of $c_0(E_i)$. Thus, similarly as in the proof of Theorem 2 we infer that

$$\mu_0(X) \leq \chi_0(X). \tag{4}$$

In order to prove the converse inequality let us denote $\mu_0(X) = r$. Then, for an arbitrary $\varepsilon > 0$ we can find a positive integer n such that

$$\|x_k\|_k \leq r + \varepsilon \tag{5}$$

for any $k \geq n$ and for each $x = (x_i) \in X$.

On the other hand $\chi_i(p_i(X)) \leq r$ for $i = 1, 2, \ldots$. So, fixing k, $1 \leq k \leq n$, we can find a finite $(r+\varepsilon)$-net $\{y_1^k, y_2^k, \ldots, y_{q_k}^k\}$ of the set $p_k(X)$ in the space E_k.

Further, consider the set

$$Y = \Big\{y = (y_{i_1}^1, y_{i_2}^2, \ldots, y_{i_n}^n, \theta, \theta, \ldots) \colon\ 1 \leq i_1 \leq q_1,\ 1 \leq i_2 \leq q_2, \ldots, 1 \leq i_n \leq q_n\Big\}.$$

Obviously, Y is a finite set consisting of $q_1 q_2 \ldots q_n$ elements. Observe, that for an arbitrary $x = (x_i) \in X$ we can find $y = \left(y^1_{i_1}, y^2_{i_2}, \ldots, y^n_{i_n}, \theta, \theta, \ldots\right) \in Y$ with the property

$$\max\left\{\|x_k - y^k_{i_k}\| : 1 \le k \le n\right\} \le r + \varepsilon .$$

Hence, keeping in mind (5) we deduce that for any $x \in X$ there exists $y \in Y$ such that

$$\|x - y\|_0 \le r + \varepsilon ,$$

which means that Y is a finite $(r+\varepsilon)$-net of the set X in the space $c_0(E_i)$. Thus

$$\chi_0(X) \le \mu_0(X) .$$

This inequality in conjunction with (4) completes the proof.

Remark 3. Let us notice that in the classical cases of $l^p(\mathbb{R})$ and $c_0(\mathbb{R})$ both μ_p and μ_0 are equal to the Hausdorff measure of noncompacteness (cf. [3]).

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