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ACYCLIC CHROMATIC INDEX AND LINEAR ARBORICITY OF GRAPHS

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ABSTRACT. In the paper it is shown the near relation between the linear arboricity and the acyclic chromatic index of graphs. Some new lower and upper bounds for the acyclic chromatic index of graphs are proved and an open combinatorial problem is formulated.

The aim of this paper is to show an interesting close connection between two graph invariants which were introduced by different authors in different years and to improve lower and upper bounds of the acyclic chromatic index for some graphs.

The first of the two mentioned invariants — the linear arboricity — was introduced by Harary [12] in 1970. The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests whose union is G, where a linear forest is a graph in which each component is a path. Determining the linear arboricity of an arbitrary graph seems to be a difficult problem and until now, the value of linear arboricity has been determined only for a few special classes of graphs, e.g. for trees, complete graphs and complete bipartite graphs (see [1], [2]). The fundamental question in the study of linear arboricity is a conjecture by Akiyama, Harary and Exoo [2].

Conjecture 1. *The linear arboricity of an r-regular graph is* $\lceil (r+1)/2 \rceil$.

At this time the conjecture has been proved only for the cases of $r = 2, 3, 4, 5, 6, 8$ and 10 by various authors in [2]—[4], 9, [13].

The second area of our interest in this paper are acyclic regular colourings of graphs. Acyclic colouring of vertices was studied at first by Grünbaum [8] in 1973. The investigation of acyclic regular colourings of edges was begun by Fiamčík in 1978 and he defined [5] the acyclic chromatic index $a(G)$ of a graph G as the least number of colours of an edge colouring of G in which any adjacent edges have different colours and no cycle is 2-coloured. Determining of $a(G)$ for general graphs is a difficult problem, too, and until now, only few partial results have been obtained. For a graph G with maximum degree $\Delta(G) = 1$ it is trivial that $a(G) = 1$, for $\Delta(G) = 2$ we have $a(G) = 2$ if G is a

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linear forest and $a(G) = 3$ if G contains a cycle. For the graphs with maximum degree $\Delta(G) = 3$ Fiamčík [6], [7] proved that $a(K_4) = a(K_{3,3}) = 5$ and if $\Delta(G) = 3$, G is connected and $G \neq K_4$, $G \neq K_{3,3}$ then $a(G) \leq 4$. He then further investigated the classes of graphs with $\Delta(G) = 3$ for which $a(G) = 3$ or $a(G) = 4$. For the other general graphs with $\Delta(G) \geq 4$ only some bounds are known. For the lower bound of $a(G)$

$$\Delta(G) \leq \chi'(G) \leq a(G)$$

trivially holds, where $\chi'(G)$ is the edge chromatic number of a graph G satisfying Vizing's inequality [14]:

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

For the upper bound of $a(G)$ Fiamčík [5] proved:

Theorem 1. *Let G be a graph with maximum degree $\Delta(G) = \Delta$. Then*
(i) *if $\Delta = 4$ then $a(G) \leq 9$*
(ii) *if $\Delta \geq 5$ then $a(G) \leq \Delta(\Delta - 1) + 1$.*

In the same paper Fiamčík introduced an interesting conjecture.

Conjecture 2. *Let G be a graph with maximum degree $\Delta(G)$. Then $a(G) \leq \Delta(G) + 2$.*

This conjecture has been verified up to now only for $\Delta(G) = 1$, 2 and 3.

Our first contribution to this problem area is an improvement of lower bounds of $a(G)$ for some type of graphs.

Theorem 2. *Let k be a positive integer and let $G = (V(G), E(G))$ be a graph with maximum degree $\Delta(G) = \Delta$, $|V(G)| = 2k$ and $|E(G)| > (\Delta + 1)(k - 1) + 1$. Then $a(G) \geq \Delta + 2$.*

Proof. By contradiction. Let us have an acyclic regular edge colouring of G by $(\Delta + 1)$ colours. Let $L_1, L_2, \ldots, L_{\Delta+1}$ be the monochromatic sets of edges of this colouring. For arbitrary $i \neq j$ it must hold that $|L_i \cup L_j| < 2k$ otherwise $L_i \cup L_j$ would contain a cycle because every acyclic grph must have fewer edges than vertices. From this it follows that for the maximum number of coloured edges we have

$$\sum_{i=1}^{\Delta+1} |L_i| \leq k + (k - 1)\Delta = (\Delta + 1)(k - 1) + 1$$

which is a contradiction to the assumption $|E(G)| > (\Delta + 1) \cdot (k - 1] + 1$ and so $(\Delta + 1)$ colours cannot suffice. So we have $a(G) \leqq \Delta + 2$.

From Theorem 2 it follows that there exists a large class of graphs with $a(G) \geq \Delta(G) + 2$. This disproves one other conjecture introduced by Fiamčík at the Czechoslovak Conference Graph Theory '84 at Kočovce where he conjectured that if a connected graph G has the maximum degree $\Delta(G) = \Delta$ and if $G \notin \{K_{\Delta+1}, K_{\Delta,\Delta}, (K_{\Delta+2} - F_1)\}$ then $a(G) \leq \Delta + 1$.

In the same way as in the proof of Theorem 2, i.e. by counting the maximum possible number of edges in an acyclic colouring we can prove two more bounds.

Theorem 3. *Let k be a positive integer. Let $G = (V(G), E(G))$ be a graph with maximum degree $\Delta(G) = \Delta, |V(G)| = 2k + 1$ and $|E(G)| > \Delta k$. Then $a(G) \geq \Delta + 1$.*

Theorem 4. *Let k be a positive integer. Let $G = (V(G), E(G))$ be a graph with maximum degree $\Delta(G) = \Delta$, $|V(G)| = 2k$ and $|E(G)| > (k-1)\Delta + 1$. Then $a(G) \geq \Delta + 1$.*

From Theorem 3 and Theorem 4 then it is easy to deduce the following corollary.

Corollary 1. *Let G be a Δ-regular graph with $\Delta > 1$. Then $a(G) \geq \Delta + 1$.*

The two graph invariants under consideration —acyclic chromatic index and linear arboricity are closely connected. We can see it in the next theorem.

Theorem 5. *Let G be a graph the edges of which can be regularly and acyclically coloured by $2x$ colours. Then G can be decomposed into x linear forests.*

Proof. We devide $2x$ colours arbitrarily into x pairs of colours and from the edges of every pair of colours we form one subgraph. Every such created subgraph is a forest because the original colouring is acyclic and every such forest is linear because the original colouring is regular. So we obtained x linear forests which contain all edges of G.

On the basis of Theorem 5 we can easily deduce next general inequality.

Corollary 2. *Let G be a graph. Then $la(G) \leq \left\lceil \frac{a(G)}{2} \right\rceil$.*

From this inequality then it follows that for all graphs G with $\Delta(G) = \Delta$ even the following interesting implication holds:

if $a(G) \leq \Delta + 2$ then $la(G) \leq \left\lceil \frac{a(G)}{2} \right\rceil = \left\lceil \frac{\Delta + 2}{2} \right\rceil = \left\lceil \frac{\Delta + 1}{2} \right\rceil$ i.e. for these graphs Conjecture 2 implies Conjecture 1.

By our attempts to get some better general upper bound for $a(G)$ than the inequality in Theorem 1 we obtained an interesting open general combinatorial problem.

Problem 1. *Let k, m be positive integers, $m > 1$. Determine the smallest integer $n = f_a(m, k)$ such that for arbitrary choose of m-tuple $(S_1, S_2, \dots, S_m)$ of subsets of the set $N = \{i\}_{i=1}^{n}$, with $|S_i| = k$ for all $i = 1, \dots, m$ there exists an m-tuple $(x_1, x_2, \dots, x_m)$ such that:*

(1) $x_i \in N$ *for all* $i = 1, \dots, m$

(2) $x_i \neq x_j$ *for* $i \neq j$

(3) $x_i \notin S_i$ *for all* $i = 1, \dots, m$

(4) *for all couples* (i, j), $1 \leq i, j \leq m$ *at least one of these two conditions holds*

(i) $x_i \notin S_j$

(ii) $x_j \notin S_i$.

The relation of this general combinatorial problem to the problem of determining of the acyclic chromatic index of a graph is given by the next theorem.

Theorem 6. *Let* G *be a graph with maximum degree* $\Delta(G) = \Delta > 1$. *Then* $a(G) \leq f_a(\Delta, \Delta - 1)$.

Proof. Let be $n = f_a(\Delta, \Delta - 1)$. Let be $N = \{i\}_{i=1}^{n}$. Now we prove that the edges of any graph with maximum degree not greater than Δ can be coloured regularly and acyclically by the set N of n colours. We will perform the proof by induction on number of vertices.

Step 1. For all $\Delta > 1$ we have $f_a(\Delta, \Delta - 1) \geq 3$. So the graph G with $|V(G)| = 2$ trivially satisfies the assertion for arbitrary $\Delta > 1$.

Step 2. Now let us assume that the assertion is true for all graphs with $|V(G)| \leq k$ and we will prove it for $k + 1$. Let us have a graph G with $\Delta(G) \leq \Delta$ and $|V(G)| = k + 1$. We choose arbitrary vertex $v \in V(G)$ and create a graph $G_1 = G - v$. Then $|V(G_1)| = k$ and by inductional assumption there exists a colouring α_1 of edges of G_1 which is regular and acyclic and all colours of α_1 are from N. Let $v_1, v_2, \ldots, v_m$ be vertices adjacent to v in G, so we have $m \leq \Delta$. Let for all $i = 1, \ldots, m$ S_i be a set of colours of the edges adjacent to v_i in G_1, so we have $S_i \subset N$, $|S_i| \leq \leq \Delta - 1$ for all i. As $n = f_a(\Delta, \Delta - 1)$ there exists an m-tuple of colours $(x_1, \ldots, x_m)$ satisfying the conditions (1) (4) of Problem 1. Now we create a colouring α of $E(G)$ in such a way that $\alpha = \alpha_1$ for all edges of $E(G_1)$ and $\alpha(v, v_i) = x_i$ for all $i = 1, \ldots, m$. Then the condition (1) implies that we used at most n colours. The conditions (2) and (3) imply that the new colouring α remains regular and the condition (4) implies that no new 2-coloured cycle could arise and so the colouring α remains acyclic, too. So we obtained in this way an acyclic regular colouring α of $E(G)$ by n colours.

Now we prove an useful bound for the function f_a.

Theorem 7. *Let* k *be a positive integer,* $k > 2$. *Then* $f_a(k, k - 1) \leq \left\lceil \frac{k^2}{2} \right\rceil$.

Proof. Let be $N = \{i\}_{i=1}^{\lceil k^2/2 \rceil}$. Let us be given arbitrary subsets $S_1, \ldots, S_k$ of the set N with the cardinality $|S_i| = k - 1$ for all $i = 1, \ldots, k$. In the proof we have to determine a k-tuple $(x_1, \ldots, x_k)$ satisfying the conditions (1) (4) of Problem 1. We will consider two cases in this proof.

Case A. Suppose that there exists at least one element $u \in N$ such that $u \notin \bigcup_{i=1}^{k} S_i$. As $\sum_{i=1}^{k} |S_i| = k(k-1)$ there must exist at least $\left\lceil \frac{k}{2} \right\rceil$ elements of N that are occurring at most in one of the sets $S_1, \ldots, S_k$. This implies that there exists a set $M \subset N$ such that $u \in M$, $|M| = \left\lceil \frac{k}{2} \right\rceil$ and every element from M occurs at most in one of the sets $S_1, \ldots, S_k$. As the order of the subsets S_i is not important in our problem we may assume without loss of generality that the subsets S_i are ordered in such a way that:

1. $S_i \cap M = \emptyset$ implies $S_{i-1} \cap M = \emptyset$ for all $1 < i \leq k$.

2. $\left| \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} S_i \right| < \frac{k(k-1)}{2}$

The second condition for an odd k is trivially satisfied, for an even k it is sufficient to have $S_1 \cap S_2 \neq \emptyset$. From the second condition then it follows that there exists an element $y \in N - \left(M \cup \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} S_i \right)$. We determine $x_{\lfloor \frac{k}{2} \rfloor} = y$. Then we sequentially choose the elements $x_{\lfloor \frac{k}{2} \rfloor - i} \in N$ for $i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1$ that $x_{\lfloor \frac{k}{2} \rfloor - i} \notin M \cup \bigcup_{j=1}^{\lfloor \frac{k}{2} \rfloor - i} S_j \cup \left\{ x_{\lfloor \frac{k}{2} \rfloor - j} \right\}_{j=0}^{i-1} = Z_i$. This choose is always possible because $|Z_i| \leq \left\lceil \frac{k}{2} \right\rceil + \left(\left\lfloor \frac{k}{2} \right\rfloor (k-1) \quad - i(k-1) \right) + i \leq \left\lceil \frac{k}{2} \right\rceil +$ $+ \frac{k(k-1)}{2} - i(k-2) < |N|$. Now we denote the elements of M by $m_1, m_2, \ldots$ $\ldots, m_{\lceil \frac{k}{2} \rceil}$ in such a way that $m_{\lceil \frac{k}{2} \rceil} = u$ and it will be true that:

1. if $1 \leq i < j \leq k$ and $m_i \in S_a$, $m_j \in S_b$ then $a \leq b$

2. if $1 \leq i < j \leq k$ and $m_i \notin \bigcup_{c=1}^{k} S_c$ then $m_j \notin \bigcup_{c=1}^{k} S_c$.

Now we determine $x_{\lfloor \frac{k}{2} \rfloor + i} = m_i$ for $i = 1, \ldots, \left\lceil \frac{k}{2} \right\rceil$. So we have obtained a k-tuple $(x_1, \ldots, x_k)$ which trivially satisfies the conditions (1) and (2) of Problem 1. As for all $i = 1, \ldots, k$ $x_i \notin \bigcup_{j=1}^{i} S_j$ holds, the conditions (3) and (4) are satisfied, too.

Case B. Assume now that there exists no $u \notin \bigcup_{i=1}^{k} S_i$, i.e. $\left|\bigcup_{i=1}^{k} S_i\right| = |N| = \left\lceil \frac{k^2}{2} \right\rceil$. Let $M_1 \subset N$ be a set of elements which occur exactly in one of the sets $S_1, \ldots, S_k$. From the assumption of Case B it implies that $|M_1| \geq k$, because $|M_1| \geq \left\lceil \frac{k^2}{2} \right\rceil - \left(k(k-1) - \left\lceil \frac{k^2}{2} \right\rceil\right) \geq k$. We choose now the set $M \subset M_1$ so that $|M| = k - 1$ and not all the elements of M are in the same set S_i for some i. Without loss of generality we may assume that the sets S_i are so ordered that $S_i \cap M = \emptyset$ implies $S_{i-1} \cap M = \emptyset$ for $1 < i \leq k$ and that $|S_k \cap M| = s \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. Now we denote the elements of M by $m_1, \ldots, m_{k-1}$ so that for all i, j $1 \leq i < j \leq k - 1$ and $m_i \in S_a$, $m_j \in S_b$ holds $a \leq b$. Now we determine $x_i = m_i$ for $i = 1, \ldots, k - 1$ and choose arbitrary vertex $x_k \in N - \left(M \cup \bigcup_{i=k-s}^{k} S_i\right)$. Such a choice is always possible because $\left|M \cup \bigcup_{i=k-s}^{k} S_i\right| = |M| + \left|\bigcup_{i=k-s}^{k} S_i\right| - \left|M \cap \bigcup_{i=k-s}^{k} S_i\right| \leq (s+1)(k-1) + (k-1) - 2s < \left\lceil \frac{k^2}{2} \right\rceil = |N|$, because we have $s \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. It is easy to verity that the k-tuple $(x_1, \ldots, x_k)$ satisfies the conditions (1)—(4) because $x_i \notin \bigcup_{j=1}^{i} S_i$ holds for all $i = 1, \ldots, k - 1$ and more $x_i \notin S_k$ for $1 \leq i \leq k - s - 1$ and $x_k \notin S_i$ for $i = k - s, \ldots, k$.

The bound in Theorem 7 is exact for $k = 4$, i.e. $f_a(4,3) = 8$, because it is easy to see that 7 colours does not suffice in this case. The counterexample is $S_1 = S_2 = \{1, 2, 3\}$, $S_3 = S_4 = \{4, 5, 6\}$. So this bound for $f_a(\Delta, \Delta - 1)$ is quite good and it enables us to improve the upper bounds of the acyclic chromatic index of graphs given in Theorem 1.

Corollary 3. *Let G be a graph with maximum degree $\Delta(G) = \Delta$. Then*
(i) *if $\Delta = 4$ then $a(G) \leq 8$*
(ii) *if $\Delta \geq 5$ then $a(G) \leq \left\lceil \frac{\Delta^2}{2} \right\rceil$*

Although this general bound is twice as good as the bound given in Theorem 1, it is probably still rather far from the true values.

In the conclusion we introduce one another formulation of Problem 1 which is more understandible in some sense and which represents one variant of the extremal problem of determining the set of distinct representatives of some subsets.

Problem 1'. *Let k, m be positive integers, $m > 1$. Determine the smallest integer $n = f_a(m, k)$ such that for arbitrary choose of m-tuple $(\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_m)$ of subsets of the set $N = \{i\}_{i=1}^{n}$, with $|\bar{S}_i| = n - k$ for all $i = 1, \ldots, m$ there exists an m-tuple $(x_1, x_2, \ldots, x_m)$ such that:*

(1) $x_i \in N$ *for all* $i = 1, \ldots, m$

(2) $x_i \neq x_j$ *for* $i \neq j$

(3) $x_i \in \bar{S}_i$ *for all* $i = 1, \ldots, m$

(4) *for all couples* (i, j), $1 \leq i, j \leq m$ *at least one of these two conditions holds*

(i) $x_i \in \bar{S}_j$

(ii) $x_j \in \bar{S}_i$.

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