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ON ISOMETRIES OF NON-ABELIAN LATTICE ORDERED GROUPS

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K. L. Swamy [5] defined an isometry of an abelian lattice ordered group G to be a one-to-one mapping f of G onto G such that the relation

(1) $|x-y|=|f(x)-f(y)|$ for each $x, y \in G$ is valid. Cf. also Swamy [6].

In [3] the isometry of a lattice ordered group G (that need not be abelian) has been defined as a one-to-one mapping of G onto G fulfilling the relation (1) and the relation

(2) $f([x \wedge y, x \vee y])=[f(x) \wedge f(y), f(x) \vee f(y)]$ for each $x, y \in G$.

It has been shown in [3] that if G is abelian and if f is a one-to-one mapping of G onto G, then (1) implies (2). Thus in the case of abelian lattice ordered groups the above definitions of isometry are equivalent.

An isometry f is said to be an 0-isometry if $f(0)=0$. Each isometry can be represented as a composition of an 0-isometry and a translation. In [3] it has been shown that there exist a one-to-one correspondence between 0-isometries of G and direct factors of G.

In this note it will be shown that for each lattice ordered group G and each one-to-one mapping f of G onto G the implication (1)$\Rightarrow$(2) holds. Hence the condition (2) can be cancelled in the definition of isometry of a (non-abelian) lattice ordered group.

For the terminology and denotations, cf. Conrad [1] and Fuchs [2]. Let G be a lattice ordered group. Let f be a one-to-one mapping of G onto G fulfilling (1).

1. Lemma. *Let $a, b \in G$, $a \leqq b$, $x \in G$. Then the following conditions are equivalent:*

(i) $x \in [a, b]$;

(ii) $|a-b|=|b-x|+|x-a|$.

Proof. The implication (i)$\Rightarrow$(ii) is obvious. Suppose that (ii) is valid. Denote $a_1=a \wedge x$, $b_1=b \wedge x$, $a_2=a \vee x$, $b_2=b \vee x$, $r=a \vee b_1=b \wedge a_2$. Then

$$|a-b|=|b-a|=b-a=(b-r)+(r-a)=$$
$$=(b_2-a_2)+(b_1-a_1),$$

$$|b-x|=b_2-b_1=(b_2-a_2)+(a_2-b_1),$$
$$|x-a|=a_2-a_1=(a_2-b_1)+(b_1-a_1).$$

From this and from (ii) we obtain $a_2-b_1=0$, whence $x=r$, and thus $x\in[a, b]$.

2. Lemma. *Let $a, b\in G$, $a\leqq b$, $u=f^{-1}(f(a)\wedge f(b))$, $v=f^{-1}(f(a)\vee f(b))$. Then $u\wedge v=a$, $u\vee v=b$.*

Proof. We have $|f(a)-f(b)|=f(b)-f(u)+f(a)-f(u)$, hence

$$|f(a)-f(b)|=|f(b)-f(u)|+|f(u)-f(a)|$$

and thus $|a-b|=|b-u|+|u-a|$. In view of Lemma 1 we get $u\in[a, b]$. Similarly we obtain $v\in[a, b]$. Then $b-a = |b-a| = |f(b)-f(a)| = |f(v)-f(u)| = |v-u| = v\vee u-v\wedge u$. On the other hand, from $a\leqq u\wedge v\leqq u\vee v\leqq b$ we obtain

$$b-a=(b-u\vee v)+(u\vee v-u\wedge v)+(u\wedge v-a),$$

thus $a=u\wedge v$, $b=u\vee v$.

The relation (1) implies $|x-y|=|f^{-1}(x)-f^{-1}(y)|$ (i.e., the mapping f^{-1} fulfils (1) as well). From this and from Lemma 1 we obtain immediately:

3. Lemma. *Let $a, b\in G$. Suppose that $a\leqq b$ and $f(a)\leqq f(b)$. Then $f([a, b]) = [f(a), f(b)]$.*

4. Lemma. *Let $a, b\in G$. Suppose that $a\leqq b$ and $f(a)\geqq f(b)$. Then $f([a, b]) = [f(b), f(a)]$.*

Proof. Since f^{-1} fulfils (1) it suffices to verify that $f([a, b]) \subseteq [f(b), f(a)]$ is valid. Let $x\in[a, b]$. According to Lemma 2 there are $u, v\in G$ such that

$$u\wedge v=a,\quad u\vee v=x,$$
$$f(a)\wedge f(x)=f(u),\quad f(a)\vee f(x)=f(v).$$

Since $v\in[a, b]$, we have $|b-a|=|b-v|+|v-a|$, hence

$$|f(b)-f(a)|=|f(b)-f(v)|+|f(v)-f(a)|$$

and thus

$$f(a)-f(b)=f(v)-f(b)+f(v)-f(a),$$
$$f(a)-f(b)=(f(v)-f(a))+(f(a)-f(b))+(f(v)-f(a)).$$

If $f(v)\neq f(a)$, then $f(v)-f(a)>0$ and hence

$$f(a)-f(b)<(f(v)-f(a))+(f(a)-f(b))+(f(v)-f(a)),$$

which is a contradiction. Therefore $f(v)=f(a)$. This implies $f(x)=f(u)$ and thus $f(x)\leqq f(a)$.

The proof of the relation $f(b)\leqq f(x)$ is analogous.

By summarizing, we obtain:

5. Lemma. *Let $a, b \in G$, $a \leqq b$. Suppose that $f(a)$ and $f(b)$ are comparable. Then $f([a, b]) = [f(a) \wedge f(b), f(a) \vee f(b)]$.*

Let M_1 be the set of all intervals $[p, q] \subseteq G$ with $f(p) \leqq f(q)$. Further let M_2 be the set of all intervals $[p_1, q_1]$ of G with $f(q_1) \leqq f(p_1)$. From Lemma 2 we obtain:

6. Corollary. *Let $a, b \in G$, $a \leqq b$. There are elements $u, v \in [a, b]$ such that $[a, v], [u, b] \in M_1$, $[a, u], [v, b] \in M_2$.*

7. Lemma. *Let $a, b, x \in G$, $a \leqq b$, $x \in [a, b]$. Let u, v be as in Lemma 2. Denote $x \wedge v = a_1$, $x \vee v = b_1$, $x \wedge u = a_2$, $x \vee u = b_2$. Then $[a_2, b_1] \in M_1$ and $[a_1, b_2] \in M_2$.*

Proof. According to Corollary 6 there exists $y \in [a_2, b_1]$ such that $[a_2, y] \in M_2$ and $[y, b_1] \in M_1$. Then for $y_1 = v \wedge y$ we have $a \leqq y_1 \leqq v$, hence according to Lemma 5, $[a, y_1] \in M_1$. On the other hand, $y_1 \in [a, y] \in M_2$, thus (again by Lemma 5) we obtain $[a, y_1] \in M_2$. Therefore $a = y_1$ and hence $y = a_2$, implying $[a_2, b_1] \in M_1$. Similarly, according to Corollary 6 there is $z \in [a_1, b_2]$ with $[a_1, z] \in M_1$, $[z, b_2] \in M_2$. Put $z \wedge u = z_1$. Then $[a, z] \in M_1$, $[a, z_1] \subseteq [a, z]$, whence $[a, z_1] \in M_1$. At the same time, $[a, z_1] \subseteq [a, u] \in M_2$, thus $[a, z_1] \in M_2$. Hence $z_1 = a$ and so $z = a_1$. Therefore $[a_1, b_2] \in M_2$.

8. Lemma. *Let $a, b \in G$, $a \leqq b$, $x \in [a, b]$. Let u, v be as in Lemma 2. Then $f(x) \in [f(u), f(v)]$.*

Proof. Let a_1, a_2 be as in Lemma 7. According to Lemma 7 and Lemma 5 we have

$$f(u) \leqq f(a_2) \leqq f(x) \leqq f(a_1) \leqq f(v).$$

9. Lemma. *Let $a_1, b_1 \in G$, $a_1 \wedge b_1 = a$, $a_1 \vee b_1 = b$. Let u, v be as in Lemma 2. Then $f(a_1) \wedge f(b_1) = f(u)$, $f(a_1) \vee f(b_1) = f(v)$.*

Proof. Put $f(a_1) \wedge f(b_1) = u_1$, $f(a_1) \vee f(b_1) = v_1$. According to Lemma 8 we have $u_1, v_1 \in [f(u), f(v)]$. Assume that either $f(u) < u_1$ or $v_1 < f(v)$. Then

$$\begin{aligned} |f(b_1) - f(a_1)| &= |b_1 - a_1| = |b - a| = |f(b) - f(a)| = f(v) - f(u) = \\ &= (f(v) - v_1) + (v_1 - u_1) + (u_1 - f(u)) > v_1 - u_1 = |f(b_1) - f(a_1)|, \end{aligned}$$

which is a contradiction. Hence $f(a_1) \wedge f(b_1) = f(u)$ and $f(a_1) \vee f(b_1) = f(v)$.

10. Lemma. *Let $a_1, b_1 \in G$. Then $f([a_1 \wedge b_1, a_1 \vee b_1]) \subseteq [f(a_1) \wedge f(b_1), f(a_1) \vee f(b_1)]$.*

This is an immediate consequence of Lemma 8 and Lemma 9.

Since f^{-1} fulfils (1), by using Lemma 10 for the mapping f^{-1} we obtain:

11. Corollary. *Let $a_1, b_1 \in G$. Then $f^{-1}([f(a_1) \wedge f(b_1), f(a_1) \vee f(b_1)]) \subseteq [a_1 \wedge b_1, a_1 \vee b_1]$.*

By summarizing, Lemma 10 and Corollary 11 yield:

12. Proposition. *Let G be a lattice ordered group. Let f be a one-to-one mapping of G onto G fulfilling the condition* (1). *Then G fulfils the condition* (2) *as well.*

For which types of lattice ordered groups are the conditions (1) and (2) equivalent? A partial answer to this question is given by the following proposition:

13. Proposition. *Let $G \neq \{0\}$ be a complete and completely distributive lattice ordered group. Then the following conditions are equivalent:*

(i) *If f is a one-to-one mapping of G onto G, then* $(1) \Leftrightarrow (2)$.

(ii) *G is isomorphic with the additive group of all integers (with the natural linear order).*

Proof. In view of Proposition 12, the relation $(1) \Leftrightarrow (2)$ in (i) can be replaced by $(2) \Rightarrow (1)$. The proof of the implication (ii)$\Rightarrow$(i) is easy. Assume that (i) is valid. Since G is complete, it can be expressed as $G = A \times B$, where A is a singular lattice ordered group and B is a vector lattice (cf. e.g., Conrad [1]). For $g \in G$ we denote by $g(A)$ and $g(B)$ the component of g in A or B, respectively. For each $g \in G$ we put

$$f(g) = g(A) + \tfrac{1}{2} g(B).$$

Then f is a one-to-one mapping of G onto G fulfilling the condition (2). If $B \neq \{0\}$, then f would fail to fulfil the condition (1); thus $B = \{0\}$ and so $G = A$.

Let $0 < s$ be a singular element of G. Then $[0, s]$ is a Boolean algebra. Since G is complete and completely distributive, the Boolean algebra $[0, s]$ is complete and completely distributive. Hence $[0, s]$ is atomic. Therefore for each $0 < g \in G$ there exists $a_i \in G$ such that $a_i \leqq g$ and a_i covers 0 in G. Let $\{a_i\}_{i \in I}$ be the set of all elements of G covering 0. For each $i \in I$ let $A_i = \{a_i\}^{\delta\delta}$ be the polar of G generated by a_i (cf. Šik [7]). Then A_i is linearly ordered; since it is complete and contains an element covering 0, A_i is isomorphic with the additive group A_0 of all integers with the natural linear order. Since G is archimedean, A_i fails to be bounded and hence (cf. [4]) A_i is a direct factor of G.

Assume that card $I > 1$. Choose $i, j \in I$, $i \neq j$. Then G can be written as $G = A_i \times A_j \times C$. For $g \in G$ let $g(A_k)$ $(k \in \{i, j\})$ and $g(C)$ be the corresponding components of g. Let φ_k $(k \in \{i, j\})$ be the isomorphism of A_k onto A_0. For each $g \in G$ we set

$$f_1(g) = \varphi_i^{-1}(\varphi_j(g(A_j)) + \varphi_j^{-1}(\varphi_i(g(A_i)) + g(C).$$

Then f_1 is a one-to-one mapping of G onto G fulfilling (2) that fails to fulfil the condition (1), which is a contradiction. Thus I is a one-element set, say $I = \{i\}$. From this it follows that $G = A_i$, completing the proof.

REFERENCES

[1] CONRAD, P.: Lattice ordered groups, Tulane University 1970.

[2] ФУКС, Л.: Частично упорядоченные алгебраические системы, Москва 1965.

[3] JAKUBÍK, J.: Isometries of lattice ordered groups. Czech. Math. J. 30, 1980, 142—152.

[4] JAKUBÍK, J.: Konvexe Ketten in l-Gruppen. Časopis pěst. mat. 84, 1959, 53—63.
[5] SWAMY, K. L.: Isometries in autometrized lattice ordered groups, Alg. univ. 8, 1977, 58—64.
[6] SWAMY, K. L.: Isometries in autometrized lattice ordered groups, II, Math. Seminar Notes Kobe Univ. 5, 1977, 211—214.
[7] ШИК, Ф.: К теории структурно упорядоченных групп. Czech. Math. J. 6, 1965, 1—25.

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ОБ ИЗОМЕТРИЯХ НЕАБЕЛЕВЫХ РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУПП

Я. Якубик

Резюме

Пусть G-решеточно упорядоченная группа. Предположим, что f будет одно-однозначное отображение множества G на G такое, что $|f(x) - f(y)| = |x - y|$ для всех x, $y \in G$. (Отображения f с этим свойством исследовал К.Л. Свами для случая абелевых решеточно упорядоченных групп.) В этой заметке доказано, что имеет место $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$ для всех x, $y \in G$.