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GRAPH ISOMORPHISM OF ORDERED SETS

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ABSTRACT. Two discrete (semi)lattices having isomorphic graphs, are compatible (semi)lattice orders of each other if and only if all their sub(semi)lattices of certain types are preserved or reversed. In the paper, we show that all connected compatible orderings of a lattice have graphs isomorphic to the graph of the lattice; and then we characterize all compatible orderings of a lattice in term of subgraphs of the lattice. It turns out that the consideration of certain types of sublattices of a lattice $\mathbf{L}$ leads to necessary and sufficient conditions for all ordered sets whose graphs are isomorphic to $\mathbf{L}$ to be compatible orderings of $\mathbf{L}$. The results cover all the cases of compatible lattice orderings.

An ordered set is called *discrete* if all its bounded chains are finite. All ordered sets which are dealt with in this paper are assumed to be discrete.

Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set. For $a, b \in P$ with $a \leq b$, the *interval* $[a, b]$ is the set $\{x \in P : a \leq x \leq b\}$; for the case when $[a, b] = \{a, b\}$ and $a \neq b$ we will write $a \prec b$ or $b \succ a$ and we say that a is *covered* by b or b *covers* a, respectively.

A subset X of an ordered set $\mathbf{P} = \langle P; \leq \rangle$ is called a *c-subset* if, whenever $a, b \in X$ and $a \prec b$ in $(X; \leq)$, then $a \prec b$ in $\mathbf{P}$. The definition of *c-sublattice* is analogous.

Let u, v, $x_1, \dots, x_m$, $y_1, \dots, y_n$ be distinct elements in P such that

(i) $u \prec x_1 \prec \dots \prec x_m \prec v$, $u \prec y_1 \prec \dots \prec y_n \prec v$,

(ii) either v is the least upper bound of x_1 and y_1 (denoted by $v = x_1 \vee y_1$) or u is the greatest lower bound of x_m and y_n (denoted by $u = x_m \wedge y_n$).

Then the set $C = \{u, v, x_1, \dots, x_m, y_1, \dots, y_n\}$ is said to be a *cell* of $\mathbf{P}$. If $x_1 \vee y_1 = v$, we call C a *cell of type* $\vee(m, n)$. Dually, if $x_m \wedge y_n = u$, we call C a *cell of type* $\wedge(m, n)$. If $x_1 \vee y_1 = v$ and $x_m \wedge y_n = u$, we call C a *cell of type* $\Diamond(m, n)$. A cell C is called *proper* if $m > 1$ or $n > 1$.

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By the *graph* $G(\mathbf{P})$, we mean the (undirected) graph whose vertex set is P and whose edges are those pairs $\{a, b\}$ which satisfy either $a \prec b$ or $b \prec a$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be ordered sets. It is said that $G(\mathbf{P})$ is *isomorphic* to $G(\mathbf{Q})$ if there is a bijection $\psi\colon P \to Q$ such that for all $a, b \in P$, $\{a, b\}$ is an edge of $G(\mathbf{P})$ if and only if $\{\psi(a), \psi(b)\}$ is an edge of $G(\mathbf{Q})$. Without loss of generality, throughout this paper we may assume that $P = Q$ and that ψ is the identity map if $G(\mathbf{P})$ is isomorphic to $G(\mathbf{Q})$, whence $G(\mathbf{P}) = G(\mathbf{Q})$; in this case, ψ is called a *graph isomorphism of* $\mathbf{P}$ *onto* $\mathbf{Q}$.

Let ψ be a graph isomorphism of $\mathbf{P}$ onto $\mathbf{Q}$ and let $X \subseteq P$. We say that X is *preserved* (*reversed*) under ψ if, whenever $x, y \in X$ and $x \prec y$, then $\psi(x) \prec \psi(y)$ (or $\psi(x) \succ \psi(y)$, respectively).

J. Jakubík proved in [3] that if $\mathbf{L}$ and $\mathbf{M}$ are discrete modular lattices, then $G(\mathbf{L}) = G(\mathbf{M})$ if and only if the following Condition (a) holds.

(a) There are lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ and a direct product representation via which $\mathbf{L}$ is isomorphic to $\mathbf{L}_1 \times \mathbf{L}_2$ and $\mathbf{M}$ is isomorphic to $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$.

Note that this yields a solution to Birkhoff's problem ([1; Problem 8]) within the class of discrete modular lattices since a modular lattice $\mathbf{L}$ will be uniquely determined by its graph if and only if every direct factor of $\mathbf{L}$ is self dual.

J. Jakubík also proved in [4] that for discrete lattices (with no assumption of modularity), Condition (a) is equivalent to Condition (b).

(b) $\mathbf{L}$ and $\mathbf{M}$ have isomorphic graphs and all proper cells of $\mathbf{L}$ and all proper cells of $\mathbf{M}$ are either preserved or reversed.

In [5] and [7], the preservation of certain types of sublattices of the lattices $\mathbf{L}$ and $\mathbf{M}$ was given for the case when $\mathbf{L}$ and $\mathbf{M}$ are semimodular.

In [6], Kolibiar proved that for discrete semimodular semilattices $\mathbf{S}$ and $\mathbf{S}_1$ on the same underlying set S, the graphs $G(\mathbf{S})$ and $G(\mathbf{S}_1)$ are isomorphic if and only if the following Conditions (c) holds.

(c) There exist a lattice $\mathbf{A} = (A; +, \cdot)$, a semilattice $\mathbf{B} = (B; \vee)$ and a map $\psi\colon S \to A \times B$ via which ψ is a subdirect embedding of $\mathbf{S}$ into $\mathbf{A} \times \mathbf{B}$ and $\mathbf{S}_1$ into $\mathbf{A}^{\partial} \times \mathbf{B}$.

In [8], we gave a new characterization of Condition (c) by proving that Condition (c) holds if and only if $G(\mathbf{S}) = G(\mathbf{S}_1)$ and the graph isomorphism preserves the order on some special types of cells and proper cells.

An order $\leq$ is said to be a *compatible ordering* of a (semi)lattice $\mathbf{L}$ if $\leq$ is a sub(semi)lattice of $\mathbf{L}^2$. If a compatible ordering $\leq$ of a (semi)lattice $\mathbf{L}$ is also a (semi)lattice order, we call $\leq$ a *compatible* (*semi*)*lattice order* of $\mathbf{L}$.

In [9], we characterized all compatible orderings of a lattice. In this paper, we will show that all connected compatible orderings of a lattice $\mathbf{L}$ have graph isomorphic to $G(\mathbf{L})$, and then we characterize all compatible orderings of a lattice in terms of subgraphs of the lattice. It turns out that consideration of the types of sublattices of a lattice which are mentioned in [4] and [5] leads to necessary and sufficient conditions for all ordered sets whose graphs are isomorphic to $G(\mathbf{L})$ to be compatible orderings of $\mathbf{L}$. The results shown in [4] and [5] become a special case when those orders are compatible lattice orders.

A 4-element subset $\{a, b, c, d\}$ of an ordered set $\mathbf{P}$ is said to be a *quadrilateral* if $a \prec b \prec d$ and $a \prec c \prec d$; and it is called a *crisscross* if $a, b \prec c, d$. We will denote these by $\langle a, b, c, d\rangle$ and $\langle ab; cd\rangle$ respectively. If $G(\mathbf{P}) = G(\mathbf{Q})$, then a quadrilateral of $\mathbf{P}$ can either be preserved, be reversed, be rotated through $90°$, or be bent into a crisscross in $\mathbf{Q}$. We have the following lemma.

Lemma 1. *Let $\mathbf{P}$ and $\mathbf{Q}$ be ordered sets with $G(\mathbf{P}) = G(\mathbf{Q})$ and let $\langle a, b, c, d\rangle$ be a quadrilateral of $\mathbf{P}$. If $\mathbf{Q}$ contains no crisscross, then the set $\{a, b\}$ is preserved (reversed) if and only if the set $\{c, d\}$ is preserved (reversed).*

Corollary 1. ([3], [4], [5]) *Let $\mathbf{P}$ and $\mathbf{Q}$ be lattices with $G(\mathbf{P}) = G(\mathbf{Q})$. If $\langle a, b, c, d\rangle$ is a quadrilateral in $\mathbf{P}$, then the set $\{a, b\}$ is preserved (reversed) if and only if the set $\{c, d\}$ is preserved (reversed).*

Corollary 2. *Let $\mathbf{P}$ and $\mathbf{Q}$ be ordered sets with $G(\mathbf{P}) = G(\mathbf{Q})$. If $\mathbf{Q}$ contains no crisscross, then every c-subset of $\mathbf{P}$ which is isomorphic to $\mathbf{M}_n$ (the ordered set shown in Figure 1) is preserved or reversed in $\mathbf{Q}$.*

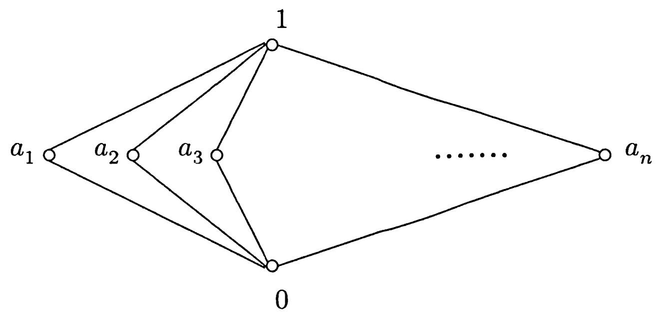

Figure 1.

P r o o f. It is enough to prove that the subset $\{0, 1, a_1, a_2, a_3\}$ of Figure 1 is preserved or reversed. We may assume that $\{0, a_1\}$ is preserved. It follows from Lemma 1 that $\{a_2, 1\}$ and $\{0, a_3\}$ are preserved since $\langle 0, a_1, a_2, 1\rangle$ and $\langle 0, a_2, a_3, 1\rangle$ are quadrilaterals. Now consider the quadrilateral $\langle 0, a_1, a_3, 1\rangle$. The preservation of $\{0, a_1\}$ and $\{0, a_3\}$ implies the preservation of $\{a_1, 1\}$ and $\{a_3, 1\}$. Hence, in the quadrilateral $\langle 0, a_1, a_2, 1\rangle$, the preservation of $\{a_1, 1\}$ implies the preservation of $\{0, a_2\}$. So $\{0, 1, a_1, a_2, a_3\}$ is preserved. $\square$

We proved in [9] that for a connected compatible ordering $\leq$ of a lattice $\mathbf{L} = \langle P; \leq^* \rangle$ there corresponds a pair (θ_1, θ_2) of complementary congruence relations of $\mathbf{L}$. Thus, if $a \leq^* b$ in $\mathbf{L}$, then there are elements $a = a_1 \prec^* a_2 \prec^* \cdots \prec^* a_n = b$ in $\mathbf{L}$ such that either $a_i \, \theta_1 \, a_{i+1}$ or $a_i \, \theta_2 \, a_{i+1}$ for all $0 \leq i < n$. Hence, if $a \prec^* b$, then $a \, \theta_1 \, b$ or $a \, \theta_2 \, b$ which together with Corollary 3 and [9; Lemma 1] yield $[a,b]^* = [a,b]$ or $[a,b]^* = [b,a]$; thus $a \prec b$ or $b \prec a$. We have the following Condition (A):

(A) $G(\mathbf{P}) = G(\mathbf{L})$.

Although Condition (A) is necessary, it is not sufficient for $\mathbf{P}$ to be a compatible ordered set of $\mathbf{L}$ even when $\mathbf{P}$ itself is a lattice.

Let $C = \{u \prec^* x_1 \prec^* \cdots \prec^* x_m \prec^* v \succ^* y_n \succ^* \cdots \succ^* y_1 \succ^* u\}$ be a proper cell of $\mathbf{L}$ where $m \geq 1$ and $n > 1$ and let us suppose that $x_k \geq u \geq y_t$ for some $1 \leq k \leq m$ and $1 \leq t \leq n$. For the case $x_1 \vee y_1 = v$, we have that $v = x_1 \vee y_t \leq x_1 \vee u = x_1$ and $y_1 = u \vee y_1 \leq x_k \vee y_1 = v$, so $[y_1, v] = [y_1, v]^*$, that is, $y_1 \leq y_2 \leq \cdots \leq y_n$. Since $v \leq x_1$ implies that $y_n = y_n \wedge v \leq y_n \wedge x_1 = u$, we have $[y_n, u] = [u, y_n]^*$, which yields $y_n \leq y_{n-1} \leq \cdots \leq y_1$. This shows that $y_1 = y_2 = \cdots = y_n$, which contradicts $n > 1$. In the case $x_m \wedge y_n = u$, we have $v = x_m \vee y_t \leq x_m \vee u = x_m$ and $y_n = u \vee y_n \leq x_k \vee y_n = v$, which yield $y_n = y_n \wedge v \leq y_n \wedge x_m = u$ and $u = x_m \wedge y_n \leq x_m \wedge v = x_m$. Since $u \leq x_m$ implies that $y_1 = y_1 \vee u \leq y_1 \vee x_m = v$, we also have $[y_n, u] = [u, y_n]^*$ and $[y_1, v] = [y_1, v]^*$, which lead to the same contradiction as above. We shall get a similar contradiction if we suppose other cases. This means that a proper cell of $\mathbf{L}$ cannot be "bent" in $\mathbf{P}$. That is:

(B) All proper cells of $\mathbf{L}$ are preserved or reversed in $\mathbf{P}$.

We say that an ordered set $\mathbf{P}$ satisfies the *lower bound property* (*LBP*) if any pairs of elements of $\mathbf{P}$ which have a lower bound have a greatest lower bound. Dually, $\mathbf{P}$ satisfies the *upper bound property* (*UBP*) if any pairs of elements of $\mathbf{P}$ which have an upper bound have a least upper bound.

In [9], we proved that if $\mathbf{P}$ is a compatible ordered set of a lattice $\mathbf{L}$, then $\mathbf{P}$ satisfies both LBP and UBP, and hence, the following Condition (C) holds:

(C) $\mathbf{P}$ contains no crisscross as a c-subset.

We shall now prove that Conditions (A), (B) and (C) altogether are equivalent to the following Condition (D):

(D) $\mathbf{P}$ is a compatible ordered set of $\mathbf{L}$.

For a pair of discrete lattices $\mathbf{L}$ and $\mathbf{M}$, Condition (B) is equivalent to the following Condition (B'):

(B') All proper cells of $\mathbf{L}$ and all proper cells of $\mathbf{M}$ are preserved or reversed.

Thus, we answer a question raised by Jakubík.

LEMMA 2. *Let* $\mathbf{P} = (P; \leq)$ *be a connected compatible ordering of a lattice* $\mathbf{L} = (P; \leq^*)$. *Then* $G(\mathbf{P}) = G(\mathbf{L})$ *and all proper cells of* $\mathbf{P}$ *and all proper cells of* $\mathbf{L}$ *are preserved or reversed.*

Proof. By [9], $\mathbf{P}$ satisfies LBP and UBP. Let $a \wedge b$ and $a \vee b$ denote the greatest lower bound and the least upper bound of any a and b in $\mathbf{P}$ if they are bounded below or bounded above, respectively.

Let $C = \{u \prec x_1 \prec \cdots \prec x_m \prec v \succ y_n \succ \cdots \succ y_1 \succ u\}$ be a proper cell of $\mathbf{P}$, that is, $m > 1$ or $n > 1$ and $x_1 \vee y_1 = v$ or $x_m \wedge y_n = u$. We may assume that $x_1 \vee y_1 = v$ (if $x_m \wedge y_n = u$ we can argue analogously). Let $w = x_m \wedge y_n$. Since $u \leq x_1 \wedge w \leq x_1$, $u \prec x_1$ and $x_1 \vee y_1 = v \neq y_n$, we have $x_1 \wedge w = u$. Similarly, $y_1 \wedge w = u$. Hence, $\mathbf{A} = \big(A = \{u, v, x_1, x_m, y_1, y_n, w\}; \vee, \wedge, \leq\big)$ is a lattice and $\leq^*$ is a compatible ordering of $\mathbf{A}$.

Suppose $x_1 \geq^* u \geq^* y_1$. Then $y_1 \leq^* v \leq^* x_1$. Since $\leq$ is a compatible ordering of $\mathbf{L}$, we have $[x_1, v] = [v, x_1]^*$ and $[y_1, v] = [y_1, v]^*$; hence, $y_1 \leq^* y_n \leq^* v$ and $v \leq^* x_m \leq^* x_1$. Since $\leq^*$ is a compatible ordering of $\mathbf{A}$, we have $(x_m \leq^* x_1 \implies w = x_m \wedge w \leq^* x_1 \wedge w = u)$ & $(y_1 \leq^* y_n \implies u = w \wedge y_1 \leq^* w \wedge y_n = w)$, which yield $w = u$. Now, $(v \leq^* x_m \implies y_n \leq^* u)$ yields $[u, y_n] = [y_n, u]^*$, that is, $y_1 = y_2 = \cdots = y_n$. Similarly, we have $x_1 = x_2 = \cdots = x_m$. Thus, $m = 1$ and $n = 1$, which is a contradiction. We will get a similar contradiction if $x_1 \leq^* u \leq^* y_1$. Hence, $x_1 \geq^* u \leq^* y_1$ or $x_1 \leq^* u \geq^* y_1$. In either cases, C is preserved or reversed.

The argument before Condition (B) prove that all proper cells of $\mathbf{L}$ are preserved or reversed in $\mathbf{P}$. □

In the following three lemmata, we assume that an ordered set $\mathbf{P}$ satisfies Condition (C).

LEMMA 3. *Let* $\mathbf{L} = (P; \vee, \wedge, \leq^*)$ *be a discrete lattice and* $\mathbf{P} = (P; \leq)$ *be a discrete ordered set with* $G(\mathbf{P}) = G(\mathbf{L})$. *Assume that all proper cells of* $\mathbf{L}$ *are preserved or reversed in* $\mathbf{P}$.

(i) $(a \succ^* c \prec^* b) \implies \big((c \prec a \implies b \leq a \vee b)$ & $(a \prec c \implies a \vee b \leq b)\big)$.
(ii) $(a \prec^* c \succ^* b) \implies \big((c \prec a \implies b \leq a \wedge b)$ & $(a \prec c \implies a \wedge b \leq b)\big)$.

Proof.

(i) If $a \prec^* a \vee b \succ^* b$, then the lemma follows by Lemma 1. We may assume that $a = x_1 \prec^* x_2 \prec^* \cdots \prec^* x_m \prec^* a \vee b$ and $b = y_1 \prec^* y_2 \prec^* \cdots \prec^* y_m \prec^* a \vee b$ for some $x_2, \ldots, x_m, y_2, \ldots, y_n \in P$. Then the set $\{c, a \vee b, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is a proper cell of $\mathbf{L}$. Hence, the interval $[b, a \vee b]$ is preserved (reversed) if the interval $[c, a]$ is preserved (reversed).

We can prove (ii) analogously. □

LEMMA 4. *Let* $\mathbf{L} = (P; \vee, \wedge, \leq^*)$ *be a discrete lattice and* $\mathbf{P} = (P; \leq)$ *be a discrete ordered set with* $G(\mathbf{P}) = G(\mathbf{L})$. *Assume that all proper cells of* $\mathbf{L}$ *are preserved or reversed in* $\mathbf{P}$. *If* $a \prec^* b$, *then for all* $c \in P$

(i) $a \prec b$ *implies* $a \vee c \leq b \vee c$ *and* $a \wedge c \leq b \wedge c$,
(ii) $b \prec a$ *implies* $b \vee c \leq a \vee c$ *and* $b \wedge c \leq a \wedge c$.

Moreover, if $a, b, c \in P$, *then* $a \leq b$ *implies* $a \vee c \leq b \vee c$ *and* $a \wedge c \leq b \wedge c$.

Proof. (i) and (ii) follow directly from Lemma 3. Let $a \leq b$ and $c \in P$. We may assume that $a = a_0 \prec a_1 \prec \cdots \prec a_n = b$ for some $a_1, a_2, \ldots, a_n \in P$. Since $G(\mathbf{P}) = G(\mathbf{L})$, we have $a_i \prec^* a_{i+1}$ or $a_{i+1} \prec^* a_i$ for all $0 \leq i < n$. So, we obtain either $(a_i \prec^* a_{i+1} \ \& \ a_i \prec a_{i+1})$ or $(a_{i+1} \prec^* a_i \ \& \ a_i \prec a_{i+1})$ for each $0 \leq i < n$. In either cases, it follows by (i) and (ii) respectively that $a_i \vee c \leq a_{i+1} \vee c$ and $a_i \wedge c \leq a_{i+1} \wedge c$ for all $0 \leq i < n$. Hence, by induction, $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$. □

THEOREM 1. *Let* $\mathbf{L}$ *be a discrete lattice and* $\mathbf{P}$ *be a discrete connected ordered set having no crisscross as a c-subset. Then the following statements are equivalent:*

(i) $\mathbf{P}$ *is a compatible ordered set of* $\mathbf{L}$.
(ii) $G(\mathbf{P}) = G(\mathbf{L})$ *and all proper cells of* $\mathbf{L}$ *and all proper cells of* $\mathbf{P}$ *are preserved or reversed.*
(iii) $G(\mathbf{P}) = G(\mathbf{L})$ *and all proper cells of* $\mathbf{L}$ *are preserved or reversed in* $\mathbf{P}$.

Proof. (i) $\Longrightarrow$ (ii) is a consequence of Lemma 2, (ii) $\Longrightarrow$ (iii) is immediate and we can prove (iii) $\Longrightarrow$ (i) by using Lemma 4 to show that $\leq$ is a sublattice of $\mathbf{L}^2$. □

In [9] we proved that if $\mathbf{P}$ is also a lattice, then condition (i) of Theorem 1 is equivalent to Condition (a). We obtain the following corollary, which answer in the affirmative a question posed by Jakubík [3].

COROLLARY 3. *Let* $\mathbf{L}$ *and* $\mathbf{L}_1$ *be discrete lattices. Then the following statements are equivalent:*

(i) $G(\mathbf{L}) = G(\mathbf{L}_1)$ *and all proper cells of* $\mathbf{L}$ *and all proper cells of* $\mathbf{L}_1$ *are preserved or reversed.*
(ii) $G(\mathbf{L}) = G(\mathbf{L}_1)$ *and all proper cells of* $\mathbf{L}$ *are preserved or reversed in* $\mathbf{L}_1$.

If $\mathbf{P}$ is a compatible ordered set of both lattices $\mathbf{L}$ and $\mathbf{L}_1$, then $G(\mathbf{L}) = G(\mathbf{P}) = G(\mathbf{L}_1)$ and all proper cells of $\mathbf{L}$ are preserved or reversed in $\mathbf{P}$; hence, by the equivalence of conditions (i) and (ii) of Theorem 1, they are preserved or reversed in $\mathbf{L}_1$. Therefore, $\mathbf{L}$ is a compatible lattice order of $\mathbf{L}_1$ and the converse also holds (see [7]). Hence, we obtain the following theorem.

THEOREM 2. *Let* $\mathbf{L}$ *and* $\mathbf{L}_1$ *be discrete lattices and* $\mathbf{P}$ *be a discrete connected ordered set. If* $\mathbf{P}$ *is a compatible ordered set of* $\mathbf{L}$, *then* $\mathbf{P}$ *is a compatible ordered set of* $\mathbf{L}_1$ *if and only if* $\mathbf{L}$ *is a compatible lattice order of* $\mathbf{L}_1$.

An ordered set $\mathbf{P}$ is said to be *upper semimodular* if $\mathbf{P}$ satisfies the following *Upper Covering Condition* (UCC):

(UCC) If a and b cover c with $a \neq b$ and a least upper bound of a and b (denoted by $a \vee b$) exists in $\mathbf{P}$, then both a and b are covered by $a \vee b$.

Dually, $\mathbf{P}$ is said to be *lower semimodular* if $\mathbf{P}$ satisfies the dual of (UCC) which is called the *Lower Covering Condition* (LCC). If $\mathbf{P}$ satisfies both (UCC) and (LCC), then $\mathbf{P}$ is said to be *modular.*

Let $\mathbf{L}$ be a discrete modular lattice; then $\mathbf{L}$ contains no proper cells. Hence, if $\mathbf{P}$ is a discrete ordered set having the same graph as $\mathbf{L}$, then conditions (iii) of Theorem 1 holds. We obtain the following corollaries.

COROLLARY 4. *Let* $\mathbf{L}$ *be a discrete modular lattice and* $\mathbf{P}$ *be a discrete ordered set satisfying Condition* (C). *Then* $G(\mathbf{P}) = G(\mathbf{L})$ *if and only if* $\mathbf{P}$ *is a connected compatible ordered set of* $\mathbf{L}$.

COROLLARY 5. *Let* $\mathbf{L}$ *be a discrete lattice and* $\mathbf{P} = (P; \leq)$ *be a discrete ordered set having the same graph as* $\mathbf{L}$ *and satisfying Condition* (C). *If* $\mathbf{L}$ *is modular, then so is* $\mathbf{P}$.

Proof. It follows from [9] that $\mathbf{P}$ is a compatible ordered set of $\mathbf{L}$. Suppose that $\mathbf{P}$ is not modular. Then $\mathbf{P}$ fails either (UCC) or (LCC), that is, there exist a, b, c with $a \neq b$ such that either

(i) $a \prec c \succ b$ but $a \not\succ a \wedge b$ or $b \not\succ a \wedge b$,

or

(ii) $a \succ c \prec b$ but $a \not\prec a \vee b$ or $b \not\prec a \vee b$,

where $a \wedge b$ and $a \vee b$ denote the greatest lower bound and the least upper bound of a and b in $\mathbf{P}$ respectively. Hence, $\mathbf{P}$ contains a proper cell $C = \{a \wedge b \prec x_1 \prec \cdots \prec x_m = a \prec c \succ b = y_n \succ \cdots \succ y_1 \succ a \wedge b\}$ or $D = \{c \prec a = x_1 \prec \ldots \prec x_m \prec a \vee b \succ y_n \succ \cdots \succ y_1 = b \succ c\}$ for some $x_1, \ldots, x_m, y_1, \ldots, y_n \in P$. So C, C^∂, D or D^∂ is a proper cell in $\mathbf{L}$, which is a contradiction. □

COROLLARY 6. ([3]) *Let* $\mathbf{L}$ *and* $\mathbf{L}_1$ *be discrete lattices whose graphs are isomorphic. If* $\mathbf{L}$ *is modular* (*distributive*), *then so is* $\mathbf{L}_1$.

As Jakubík has observed in [5], the modularity condition in Corollary 4 cannot be replaced by semimodularity. In fact, we have examples (see Figure 2 (a) and (b)) of semimodular lattices whose graphs are isomorphic, but one is not a compatible ordering of the other.

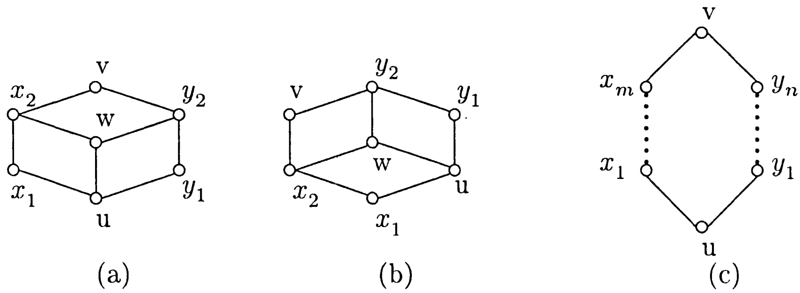

Figure 2.

In [4], J a k u b í k has shown that a discrete lattice **L** is modular if and only if **L** does not contain a c-sublattice isomorphic to one of the lattices in Figure 2. In fact, all c-sublattices of a lattice **L** which are isomorphic to one of the lattice in Figure 2 are proper cells of **L**. It is interesting to ask whether for a discrete lattice **L** and a discrete connected ordered set **P**, **P** is a compatible ordered set of **L** if and only if **L** and **P** have the same graphs and the isomorphism preserves the order on all c-sublattices of **L** which are isomorphic to one of the lattices in Figure 2. Unfortunately, Figure 3 and Figure 4 show that this is not the story.

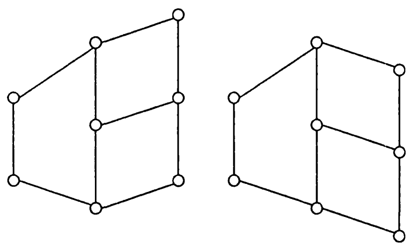

Figure 3.

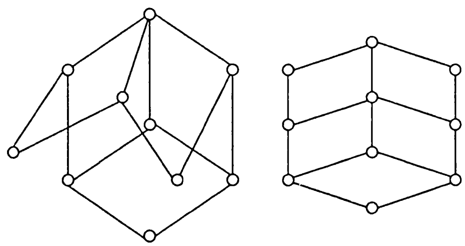

Figure 4.

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