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A NOTE ON MAXIMAL k-DEGENERATE GRAPHS

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ABSTRACT. A graph G is said to be k-degenerate whenever every subgraph of G has minimum degree at most k. A k-degenerate graph G is called maximal k-degenerate if there is no k-degenerate graph H of the same order, which properly contains G. In this paper, we investigate the structure of maximal k-degenerate graphs emphasing to various graph theoretical characteristics and degree constraints. The correspondence between the structure of maximal k-degenerate graphs and the structure of generalized α-critical graphs is characterized.

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and without multiple edges. For undefined concepts, we refer the reader to [4].

Let us denote by $\mathcal{I}$ the set of all mutually non-isomorphic graphs. If $\mathcal{P}$ is a non-empty subset of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$.

A property $\mathcal{P}$ is called *hereditary* if it follows from $G \in \mathcal{P}$, and H is a subgraph of G that $H \in \mathcal{P}$. The sets of graphs

$$\mathcal{O} = \{G \in \mathcal{I} \mid G \text{ is edgeless graph}\},$$
$$\mathcal{S}_k = \{G \in \mathcal{I} \mid \text{the maximum degree } \Delta(G) \le k\}$$

are examples of hereditary properties.

The set S of vertices of G is said to be $\mathcal{P}$*-independent* in G if the induced subgraph $\langle S\rangle_G$ belongs to $\mathcal{P}$. We shall use the notation $\alpha_{\mathcal{P}}(G)$ for the maximum size of a $\mathcal{P}$-independent set in G (for the vertex independence number $\alpha_{\mathcal{O}}(G)$ we prefer the notation $\alpha(G)$).

A graph G is called k*-degenerate* (we write $G \in \mathcal{D}_k$) for k, a non-negative integer, if for each subgraph H of G, the minimum degree of H does not exceed k.

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The following value plays a fundamental role in the theory of k-degenerate graphs:

$$s(G) = \max_{H \subseteq G} \min_{v \in V(H)} \deg_H(v) .$$

This number is called the *Szekeres-Wilf number*, and it is easy to see that G is k-degenerate if and only if $s(G) \le k$ (see [5], [7], [8], [9]). A survey of k-degenerate graphs was given in [10].

It is easy to check that the properties $\mathcal{O}$, $\mathcal{S}_k$ and $\mathcal{D}_k$ are hereditary. A hereditary property $\mathcal{P}$ can be uniquely characterized by the set of $\mathcal{P}$-maximal graphs. A graph $G \in \mathcal{P}$ is *$\mathcal{P}$-maximal* if for every edge e of the complement of G, the graph $G + e$ does not belong to $\mathcal{P}$.

It can be seen immediately that a graph G of order at most $k+1$ is $\mathcal{D}_k$-maximal (we shall prefer the term maximal k-degenerate) if and only if G is complete. Therefore, we restrict our attention mainly to maximal k-degenerate graphs with at least $k+2$ vertices.

The basic properties of maximal k-degenerate graphs have been studied in [2], [3], [6]. Let us recall those of them which we shall use in what follows.

PROPOSITION 1. ([5]) *A graph G of order $k+m$ is k-degenerate if and only if the vertex set $V(G)$ can be labelled $v_1, v_2, \dots, v_{k+m}$ in such a way that in the subgraph $\langle\{v_i, v_{i+1}, \dots, v_{k+m}\}\rangle$ of G, $\deg(v_i) \le k$ for each $i = 1, 2, \dots, m-1$.*

COROLLARY 1. *A graph G is k-degenerate if and only if the vertex set $V(G)$ can be labelled $v_1, v_2, \dots, v_{k+m}$ in such a way that in the subgraph $\langle\{v_i, v_{i+1}, \dots, v_{k+m}\}\rangle$ of $\overline{G}$, $\deg(v_i) \ge m - i$ for each $i = 1, 2, \dots, m-1$.*

PROPOSITION 2. ([5]) *Let G be a maximal k-degenerate graph of order p, $p \ge k+1$. Then*

(1) *the number of edges of G is equal to $kp - \binom{k+1}{2}$;*
(2) *the minimum degree of G is equal to k;*
(3) *G is k-connected.*

PROPOSITION 3. ([5]) *Let $G = (V, E)$ be a graph of order p, $p \ge k+1$, and let $v \in V$ be a vertex of degree k. Then G is a maximal k-degenerate graph if and only if $G - v$ is maximal k-degenerate.*

PROPOSITION 4. *Let $G = (V, E)$ be a maximal k-degenerate graph of order $k+m$, $m \ge 2$, $k \ge 0$, and $A = \{v \in V(G) \mid \deg_G(v) = k\}$. Then*

(1) *$\langle A\rangle_G$ is totally disconnected;*
(2) *$|A| \le m$.*

The following can be obtained by an easy observation.

PROPOSITION 5. *Let G be a graph of order p, $p \geq k+1$, and $A = \{v \in V(G) \mid \deg_G(v) < k\}$. Then a graph G of order $p \geq k+1$ is $\mathcal{S}_k$-maximal if and only if $\Delta(G) = k$, and either $\langle A \rangle_G$ is complete, or $A = \emptyset$.*

THEOREM 1. ([3]) *Let $n_k, n_{k+1}, \dots, n_r$, $r \geq k$, be non-negative integers and*

$$\sum_{\substack{i=k \\ i \neq 2k}}^{r} n_i = t.$$

Then the numbers n_i, $i \in \bigcup_{j=k}^{r} \{j\} \cup \{2k\}$ determine the numbers of vertices of degree d, $d \in \bigcup_{j=k}^{r} \{j\} \cup \{2k\}$, respectively of a maximal k-degenerate graph if and only if the following conditions hold

(i) *if $2k \leq r$, then $n_{2k} \geq r - t + 1$;*
(ii) *$n_k + n_{k+1} + \dots + n_{k+j} \geq j + 1$ for $j = 0, 1, \dots, r-k$;*
(iii) $\sum_{i=k}^{r} n_i(2k - i) = k^2 + k$.

2. The structure of maximal k-degenerate graphs

If we consider a maximal k-degenerate graph G of order $k+m$ for $m \geq 1$, but not too large, its complement $\overline{G}$ contains many isolated vertices. We show that under some conditions, the structure of non-trivial components of the complements of maximal k-degenerate graphs and maximal l-degenerate graphs of order $k+m$ and $l+m$ respectively is the same.

LEMMA 1. *Let $G = (V, E)$ be a maximal k-degenerate graph of order $k+m$, $m \geq 2$, $k \geq 0$, and $B = \{v \in V(G) \mid \deg_{\overline{G}}(v) \geq 1\}$. Then $|B| \leq \frac{m^2+m-2}{2}$.*

Proof. According to statement (1) of Proposition 2, the number of edges of $\overline{G}$ is equal to $\frac{m^2-m}{2}$. Since G is k-degenerate, by an application of Corollary 1, the vertices of G can be labelled $v_1, v_2, \dots, v_{k+m}$ such that in the induced subgraph $\langle \{v_i, v_{i+1}, \dots, v_{k+m}\} \rangle$ of $\overline{G}$, $\deg(v_i) \geq m - i$ for each $i = 1, 2, \dots, m-1$. Thus we have

$$|B| \leq m^2 - m - \sum_{i=1}^{m-1} (m - i - 1) = m^2 - m - \frac{(m-2)(m-1)}{2} = \frac{m^2 - m - 2}{2}.$$

□

LEMMA 2. *If* $m \geq 1$, *then* $G = K_m \cup \overline{K_k}$ *is a complement of some maximal* k*-degenerate graph of order* $k+m$.

The proof follows by the definition of maximal k-degenerate graphs.

Let $M(q,m)$ denote the set of the complements of all maximal q-degenerate graphs of order $q+m$, and $N(q,m) = \{G \mid G$ has no isolated vertices and there exists a p such that the graph $G \cup \overline{K_p}$ belongs to $M(q,m)\}$.

THEOREM 2. *Let* k, l, m *be non-negative integers such that* $\frac{m^2-m-2}{2} \leq k \leq l$. *Then* $N(k,m) = N(l,m)$.

Proof. We use induction on m.

(i) For $m = 1$, we have $N(k,1) = N(l,1) = \emptyset$.

(ii) Let us suppose that $N(k,q) = N(l,q)$ for $q \leq m$ and $\frac{q^2-q-2}{2} \leq k \leq l$. Let $\frac{(m+1)^2-(m+1)-2}{2} \leq k \leq l$. Since

$$\frac{(m+1)^2-(m+1)-2}{2} = \frac{m^2+m-2}{2} > \frac{m^2-m-2}{2},$$

we have $N(k,m) = N(l,m)$. Let $G^* \in N(k,m+1)$ is a graph of order p. Then, by Lemma 1, $p \leq \frac{(m+1)^2+(m+1)-2}{2}$, and we put

$$r = k+m+1-p \geq \frac{(m+1)^2-(m+1)-2}{2} + m+1 - \frac{(m+1)^2+(m+1)-2}{2} = 0.$$

Obviously, $G_1 = G^* \cup \overline{K_r} \in M(k,m+1)$, and there exists a vertex $v \in V(G)$ such that $\deg_{G_1}(v) = m$ (because the complement $\overline{G_1}$ of G_1 must have a vertex of degree k and $(k+(m+1)-1)-k = m$). If G' is the graph obtained from $G_1 - v$ by deleting its isolated vertices, then $G' \in N(k,m)$. But $N(k,m) = N(l,m)$, and therefore $G' \in N(l,m)$. For $s = l+m+1-p \geq 0$, $G_2 = (G^*-v) \cup \overline{K_s} \in M(l,m)$, and $\deg_{G^* \cup \overline{K_s}}(v) = m$. Hence $G^* \cup \overline{K_s} \in M(l,m+1)$, and this implies that $G^* \in N(l,m+1)$. Similarly, we can prove the opposite inclusion $N(l,m+1) \subseteq N(k,m+1)$. □

In the next, our method will aim to establish some graph theoretical characteristics of maximal k-degenerate graphs.

THEOREM 3. *Let* G *be a maximal* k*-degenerate graph of order* $p = k+m$, *where* $1 \leq m \leq \frac{1+\sqrt{1+8k}}{2}$. *Then* $\Delta(G) = p-1$.

Proof. Let us denote by B and C the following two sets:

$$B = \{v \in V(G) \mid \deg_{\overline{G}}(v) \geq 1\},$$
$$C = \{v \in V(G) \mid \deg_G(v) = p-1\}.$$

Clearly, $|C| = p - |B| = k + m - |B|$. By Lemma 1, we have

$$|C| = k + m - |B| \geq k + m - \frac{m^2 + m - 2}{2} = \frac{2k + m - m^2 + 2}{2}.$$

Since $m \leq \frac{1+\sqrt{1+8k}}{2}$, we get $\frac{2k+m-m^2+2}{2} \geq 1$; which implies that $|C| \neq \emptyset$. □

EXAMPLE 1. This example shows that the result of Theorem 3 is the best possible.

Let m be an integer, $m \geq 3$, and let $k = \binom{m}{2} - 1 - s$, where $s \in \{0, 1, \ldots, m-2\}$. Evidently, $\binom{m-1}{2} \leq k \leq \binom{m}{2} - 1$.

Let us consider the numbers

$$n_k = n_{k+1} = \cdots = n_{k+m-4} = 1,$$
$$n_{k+m-3} = s + 1,$$
$$n_{k+m-2} = k - s + 2.$$

It is not difficult to verify that the numbers $n_k, n_{k+1}, \ldots, n_{k+m-2}$ satisfy the conditions (i)–(iii) of Theorem 1, and therefore there exists a maximal k-degenerate graph G of order $k+m$ such that $n_k, n_{k+1}, \ldots, n_{k+m-2}$ determine the numbers of vertices of degrees $k, k+1, \ldots, k+m-2$, respectively.

Since $m \leq \frac{1+\sqrt{8k+1}}{2}$ if and only if $\binom{m}{2} \leq k$, and the graph G described above has order $p = k + m$, $\binom{m-1}{2} \leq k \leq \binom{m}{2} - 1$, and the maximum degree $\Delta(G) = k + m - 2 = p - 2$, the previous result cannot be improved.

The following two theorems state some graph theoretical invariants of maximal k-degenerate graphs.

THEOREM 4. *Let $k \geq 1$, $m \geq 2$ be integers, and let $G = (V, E)$ be a maximal k-degenerate graph of order $k + m$. Then*

(1) *the chromatic number $\chi_0(G)$ is equal to $k + 1$;*
(2) *the edge connectivity number $\lambda(G)$ is equal to k;*
(3) *the vertex independence number satisfies the inequality $\lceil \frac{k+m}{k+1} \rceil \leq \alpha(G) \leq m$.*

Proof.

(1) As G is k-degenerate, we have $\chi_0(G) \leq k + 1$. On the other hand, G contains a copy of a complete graph K_{k+1}. Hence $\chi_0(G) = k + 1$.

(2) It is known that for the minimum degree $\delta(G)$, the edge connectivity number $\lambda(G)$ and the vertex connectivity number $\kappa(G)$ satisfy $\delta(G) \geq \lambda(G) \geq \kappa(G)$. Since $\delta(G) = \kappa(G) = k$, we have $\lambda(G) = k$.

(3) As is well known, $\alpha(G)\chi_0(G) \geq |V(G)|$. As $|V(G)| = k + m$ and $\chi_0(G) = k$, we have $\lceil \frac{k+m}{k+1} \rceil \leq \alpha(G)$.

On the other hand, by Lemma 2, G contains a $(k+1)$-clique. Therefore, $\alpha(G) \leq |V(G)| - k = k + m - k = m$. □

EXAMPLE 2. We show that the bounds of Theorem 4 cannot be improved. We shall define two graphs G_1, G_2 of order $k+m$ for which the vertex independence numbers are m and $\lceil \frac{k+m}{k+1} \rceil$ respectively.

Let us consider a graph $G_1 = K_m \cup \overline{K_k}$. By Lemma 2, $\overline{G_1}$ is maximal k-degenerate graph, and it is obvious that $\alpha(\overline{G_1}) = m$.

Let the vertices of G_2 be denoted by $v_1, v_2, \dots, v_{k+m}$. For each vertex of G_2 we define its neighborhood as follows:

$$N(v_j) = \{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_{j+k}\} \quad \text{for } j = 1, 2, \dots, k;$$
$$N(v_{k+j}) = \{v_j, \dots, v_{k+j-1}, v_{k+j+1}, \dots, v_{2k+j}\} \quad \text{for } j = 1, 2, \dots, m-k;$$
$$N(v_{k+j}) = \{v_j, \dots, v_{k+j-1}, v_{k+j+1}, \dots, v_{k+m}\} \quad \text{for } j = m-k+1, m-k+2, \dots, m.$$

By an application of Theorem 1, we can verify that G_2 is maximal k-degenerate. Let S be a maximal independent set of vertices of G_2. Since the fact that $v_i \in S$, for some $i = 1, 2, \dots, m$, implies that $v_{i+1}, v_{i+2}, \dots, v_{i+k} \notin S$, we have $|S| \le \lceil \frac{k+m}{k+1} \rceil$. The opposite inequality follows from Theorem 4. Hence $\alpha(G_2) = \lceil \frac{k+m}{k+1} \rceil$.

THEOREM 5. *Let G be a maximal k-degenerate graph of order p. If $p \ge \binom{k+2}{2}$, then $\Delta(G) \ge 2k$.*

Proof. Suppose, on the contrary, that $\Delta(G) \le 2k-1$ for some G. Let us denote the vertices of G by $v_1, v_2, \dots, v_p$. Since G is maximal k-degenerate, we have $\sum_{i=1}^{p} \deg(v_i) = 2kp - k(k+1)$. On the other hand,

$$\sum_{i=1}^{p} \deg(v_i) \le \big[k + (k+1) + \dots + (2k-2)\big] + (p-k+1)(2k-1),$$

which implies $p \le \frac{k^2+3k}{2} < \binom{k+2}{2}$, what is a contradiction. Therefore $\Delta(G) \ge 2k$. □

EXAMPLE 3. We now demonstrate that the bound of Theorem 5 is the best possible. By an application of Theorem 1, there exists a maximal k-degenerate graph, which realizes the sequence

$$n_k = n_{k+1} = \dots = n_{2k-2} = 1, n_{2k-1} = \binom{k+1}{2} + 1.$$

The order of this graph G is $\sum_{i=k}^{2k-1} n_i = \binom{k+2}{2} - 1$ and $\Delta(G) = 2k-1$.

Remark. On the other hand, for arbitrary $p \geq \binom{k+2}{2}$, there exists a maximal k-degenerate graph of order $p > k+m$ with $\Delta(G) = 2k$. A graph G which realizes the sequence

$$n_k = n_{k+1} = \cdots = n_{2k-2} = 1,\ n_{2k-1} = \binom{k+1}{2} + 1,\ n_{2k} = p - m - k + 1,$$

where $m = \binom{k+1}{2} + 1$, shows this.

Our next theorem generalizes the result of Theorem 5.

THEOREM 6. *Let $k \geq 2$, $0 \leq s \leq k-2$, be integers, and let G be a maximal k-degenerate graph of order p. If $p > \frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}$, then $\Delta(G) \geq 2k - s$.*

The proof is as in Theorem 5.

We conclude this section with characterization of those maximal k-degenerate graphs which contain a hamiltonian cycle. The proof of our result is based on the following well-known result of Erdős and Chvátal (see, e.g., [1]).

LEMMA 3. *Let G be k-connected graph such that G does not contain $k+1$ independent vertices, $k \geq 2$. Then G has a hamiltonian cycle.*

THEOREM 7. *Let G be a maximal k-degenerate graph of order $k+m$, $k \geq 2$, $1 \leq m \leq k$. Then G has a hamiltonian cycle.*

Proof. Since G is maximal k-degenerate, by statement (3) of Proposition 2, G is k-connected. Theorem 4 states that $\alpha(G) \leq m \leq k$. Hence G does not contain any set of $k+1$ independent vertices.

Then, by Lemma 3, G has a hamiltonian cycle. □

Remark. We cannot guarantee the existence of hamiltonian cycle in a maximal k-degenerate graphs of order $k+m$, for $m > k$. By Lemma 2, $G = \overline{\overline{K_m} \cup \overline{K_k}}$ is maximal k-degenerate, and it is easy to verify that G contains no hamiltonian cycle.

3. $\alpha(n, \mathcal{D}_k)$-critical graphs

In this part, we investigate the correspondence between the structure of maximal k-degenerate graphs and $\alpha(n, \mathcal{D}_k)$-critical graphs.

The graph $G = (V, E)$ is said to be $\alpha(n, \mathcal{P})$-*critical* if G has no isolated vertices, and $\alpha_{\mathcal{P}}(G - e) > \alpha_{\mathcal{P}}(G) = n$ for every edge $e \in E(G)$. For the elementary properties of $\alpha(n, \mathcal{P})$-critical graphs, see [7]. In what follows, we shall concentrate to the structure of $\alpha(n, \mathcal{D}_k)$-critical graphs.

Our results are proved using the next two lemmas which follow straightforwardly from the definition of an $\alpha(n, \mathcal{P})$-critical graph.

LEMMA 4. *A graph G with no isolated vertices is $\alpha(n,\mathcal{P})$-critical if and only if*

(1) *the complement $\overline{G}$ of G does not contain the complement of any $\mathcal{P}$-maximal graph of order $n+1$;*
(2) *for each edge $e \in E(G)$ the graph $\overline{G}+e$ contains the complement of some $\mathcal{P}$-maximal graph of order $n+1$.*

LEMMA 5. *If G is $\alpha(n,\mathcal{P})$-critical, then for every $v \in V(G)$ there exists the set $U \subseteq V(G)$ with $|U| = n$ such that $\langle U\rangle_G$ is $\mathcal{P}$-independent and $v \notin U$.*

Using Theorem 2, we obtain:

THEOREM 8. *Let G be a $\alpha(k+m,\mathcal{D}_k)$-critical graph of order p and $\frac{m^2+m-2}{2} \le k$. Then for every l satisfying $\frac{m^2+m-2}{2} \le l \le p-m-1$ the graph G is $\alpha(l+m,\mathcal{D}_k)$-critical.*

Proof. As $k,l \ge \frac{m^2+m-2}{2}$ implies that $k,l \ge \frac{(m+1)^2-(m+1)-2}{2}$, by Theorem 2, we have $N(k,m+1) = N(l,m+1)$. Thus the assertion follows immediately using Lemma 4. □

The structure of $\alpha(k+m,\mathcal{D}_k)$-critical graphs for $m \le \frac{-1+\sqrt{9+8k}}{2}$ can be characterized in the following way.

THEOREM 9. *Let k and m satisfy $\frac{m^2+m-2}{2} \le k$, $m \ge 1$. Then a graph G of order $p > k+m$ is $\alpha(k+m,\mathcal{D}_k)$-critical if and only if the complement $\overline{G}$ of G is $\mathcal{S}_{m-1}$-maximal.*

Proof. Let us suppose that the graph G is $\alpha(k+m,\mathcal{D}_k)$-critical, and $U \subseteq V(G)$ is a maximal $\mathcal{D}_k$-independent set of vertices of G. Thus, $|U| = m+k$ and, according to Corollary 1, $\Delta(\overline{G}) \ge \Delta(\langle U\rangle_{\overline{G}}) \ge m-1$.

Firstly, we prove that $\Delta(\overline{G}) = m-1$. In order to obtain a contradiction, let us suppose that there exists a vertex w_0 of G with $\deg_{\overline{G}}(w_0) \ge m$. By Lemma 5, there exists a $\mathcal{D}_k$-independent set $W \subsetneqq V(G)$, $w_0 \notin W$ and $|W| = k+m$. Let the vertices $w_1, w_2, \dots, w_{k+m}$ of W be labelled according to Corollary 1 in such a way that in $\langle\{w_i, w_{i+1}, \dots, w_{k+m}\}\rangle$, $\deg(w_i) \ge m-i$ for each $i = 1,2,\dots,m-1$. Since

$$\sum_{i=0}^{m-1}(m-i+1) = \frac{m^2+3m}{2} = \frac{m^2+m-2}{2} + m + 1 \le k+m+1\,,$$

there exists a set $T \subsetneqq V(G)$ satisfying the following conditions:

(i) $w_0, w_1, \dots, w_{m-1} \in T$,
(ii) $|T| = k+m+1$,
(iii) $\deg(w_i) \ge m-i$ in $\langle T\rangle_{\overline{G}}$.

However, by Corollary 1, T is $\mathcal{D}_k$-independent in G, which is a contradiction. Now, let

$$A = \{v \in V(G) \mid \deg_{\overline{G}}(v) < m-1\} \neq \emptyset .$$

By Lemma 4, for each $e \in E(G)$ the graph $\overline{G} + e$ contains the complement of some $\mathcal{D}_k$-maximal graph of order $k+m+1$, implying $\Delta(\overline{G}+e) \geq m$. Thus, $\langle A \rangle_{\overline{G}}$ must be complete, and, by Proposition 5, $\overline{G}$ is $\mathcal{S}_{m-1}$-maximal.

Conversely, let G be a graph such that $\overline{G}$ is $\mathcal{S}_{m-1}$-maximal. Since $|V(G)| \geq k+m+1$, G does not possess any isolated vertices.

By Proposition 5, $\Delta(\overline{G}) = m-1$. Thus, according to Corollary 1, $\overline{G}$ does not contain the complement of any $\mathcal{D}_k$-maximal graph of order $m+k+1$. As for

$$A = \{v \in V(G) \mid \deg_{\overline{G}}(v) < m-1\}$$

the induced subgraph $\langle A \rangle_{\overline{G}}$ is complete, and $|V(G)| \geq k+m+1$, $\overline{G}$ contains at least

$$k+m+1-(m-1) = k+2 \geq \frac{m^2+m-2}{2} + 2 = \frac{m(m+1)}{2} + 1 \geq m+1$$

vertices of degree $m-1$.

Therefore, we can choose vertices $w_0, w_1, \dots, w_{m-1}$ so that $\deg(w_i) \geq m-i$ in $\langle V(G) \setminus \{w_0, w_1, \dots, w_{i-1}\} \rangle_{\overline{G}} + e$ is satisfied for $i = 0, 1, \dots, m-1$.

As described above, we construct the set $T \subseteq V(G)$. By Corollary 1, T is $\mathcal{D}_k$-independent, and therefore, by Lemma 4, G is $\alpha(k+m, \mathcal{D}_k)$-critical. □

REFERENCES

[1] BERGE, C.: *Graphs and Hypergraphs*, North-Holland, American Elsevier, 1973.

[2] BOROWIECKI, M.—MIHÓK, P.: *Hereditary properties of graphs.* In: Advances in Graph Theory (V. R. Kulli, ed.), Vishwa International Publication, Gulbarga, 1991, pp. 42–69.

[3] BOROWIECKI, M.—MIHÓK, P.—IVANČO, J.—SEMANIŠIN, G.: *Sequences realizable by maximal k-degenerate graphs*, J. Graph Theory **19** (1995), 117–124.

[4] CHARTRAND, G.—LESNIAK, L.: *Graphs and Digraphs*, Wadsworth & Brooks/Cole, Monterey California, 1986.

[5] LICK, D. R.—WHITE, A. R.: *k-degenerate graphs*, Canad. J. Math. **22** (1970), 1082–1096.

[6] MIHÓK, P.: *On graphs critical with respect to vertex partition numbers*, Discrete Math. **37** (1981), 123–126.

[7] MIHÓK, P.: *On Graphs Critical with respect to Generalized Independence Numbers.* In: Colloq. Math. Soc. János Bolyai 52. Combinatorics, Eger, 1987, pp. 417–421.

[8] MITCHEM, J.: *An extension of Brook's theorem to r-degenerate graphs*, Discrete Math. **17** (1977), 291–298.

[9] MITCHEM, J.: *Maximal k-degenerate graphs*, Utilitas Math. **11** (1977), 101–106.

[10] SIMÕES-PEREIRA, J. M. S.: *A survey of k-degenerate graphs*, Graph Theory Newsletter **5** (b) (1976), 1–7.

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