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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 44 (2005), No. 1, 19--23

Persistent URL: http://dml.cz/dmlcz/133381

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **44** (2005) 19–23

Sheffer Operation in Ortholattices *

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(Received March 4, 2005)

Abstract

We introduce the concept of Sheffer operation in ortholattices and, more generally, in lattices with antitone involution. By using this, all the fundamental operations of an ortholattice or a lattice with antitone involution are term functions built up from the Sheffer operation. We list axioms characterizing the Sheffer operation in these lattices.

Key words: Ortholattice, orthocomplementation, lattice with antitone involution, Sheffer operation.

2000 Mathematics Subject Classification: 06C15, 06E30

The concept of Sheffer operation (the so-called Sheffer stroke in [1]) was introduced by H. M. Sheffer in 1913. H. M. Sheffer [3] showed that all Boolean functions could be obtained from a single binary operation as term operations. In what follows, we are going to show that this works also in ortholattices and, more generally, in lattices with antitone involution and we will set up an equational axiomatization of this Sheffer operation.

Our basic concepts are taken from [1] and [2]. By a *bounded lattice* we mean a lattice with least element **0** and greatest element **1**. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. A mapping $x \mapsto x^\perp$ is called an antitone involution on $\mathcal{L}$ if

$x \le y$ implies $y^\perp \le x^\perp$ (antitone)

$x^{\perp\perp} = x$ (involution).

*Supported by the Research Project MSM 6198959214.

The fact that an antitone involution $^\perp$ is a unary operation of $\mathcal{L}$ will be expressed by the notation $\mathcal{L} = (L; \vee, \wedge, ^\perp)$. If $\mathcal{L}$ is a bounded lattice with an antitone involution which is, moreover, a complementation on $\mathcal{L}$, i.e. it satisfies

$$x \vee x^\perp = \mathbf{1} \quad \text{and} \quad x \wedge x^\perp = \mathbf{0},$$

then $x^\perp$ is called an *orthocomplement* of x and $\mathcal{L} = (L; \vee, \wedge, ^\perp, \mathbf{0}, \mathbf{1})$ an *ortholattice*.

It is worth noticing that if $^\perp$ is an antitone involution on $\mathcal{L}$, then $\mathcal{L} = (L; \vee, \wedge, ^\perp)$ satisfies the *De Morgan laws*

$$x^\perp \vee y^\perp = (x \wedge y)^\perp \quad \text{and} \quad x^\perp \wedge y^\perp = (x \vee y)^\perp.$$

Our basic concept is the following.

Definition 1 Let $\mathcal{A} = (A; \circ)$ be a groupoid. The operation $\circ$ is called *Sheffer operation* if it satisfies the following identities:

(S1) $x \circ y = y \circ x$ (commutativity)

(S2) $(x \circ x) \circ (x \circ y) = x$ (absorption)

(S3) $x \circ ((y \circ z) \circ (y \circ z)) = ((x \circ y) \circ (x \circ y)) \circ z$

(S4) $(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = x$ (absorption)

If, moreover, it satisfies

(S5) $y \circ (x \circ (x \circ x)) = y \circ y,$

it is called an *ortho-Sheffer operation.*

Remark 1 (S2) implies also weak idempotency $(x \circ x) \circ (x \circ x) = x$.

Lemma 1 *Let $\mathcal{A} = (A; \circ)$ be a groupoid with a Sheffer operation. Define a binary relation $\leq$ on A as follows*

$$x \leq y \quad \textit{if and only if} \quad x \circ y = x \circ x.$$

Then $\leq$ is an order on A.

Proof Reflexivity of $\leq$ is evident. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = x \circ x$ and $x \circ y = y \circ x = y \circ y$, i.e. $x \circ x = y \circ y$ and hence by (S2) also $x = (x \circ x) \circ (x \circ x) = (y \circ y) \circ (y \circ y) = y$. Thus $\leq$ is antisymmetric. Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = x \circ x$, $y \circ z = y \circ y$ and hence $(x \circ y) \circ (x \circ y) = (x \circ x) \circ (x \circ x) = x$, i.e. by (S3) and (S2) also

$$\begin{aligned}(x \circ z) &= ((x \circ y) \circ (x \circ y)) \circ z = x \circ ((y \circ z) \circ (y \circ z)) \\ &= x \circ ((y \circ y) \circ (y \circ y)) = x \circ y = x \circ x\end{aligned}$$

proving $x \leq z$. Thus $\leq$ is also transitive and hence it is an order on A. □

Because of Lemma 1, $\leq$ will be called the *induced order* of $\mathcal{A} = (A; \circ)$.

Lemma 2 *Let $\circ$ be a Sheffer operation on A and $\leq$ the induced order of $\mathcal{A} = (A;\circ)$. Then*

(a) $x \leq y$ *if and only if* $y \circ y \leq x \circ x$*;*

(b) $x \circ (y \circ (x \circ x)) = x \circ x$ *is the identity of* $\mathcal{A}$*;*

(c) $x \leq y$ *implies* $y \circ z \leq x \circ z$*;*

(d) $a \leq x$ *and* $a \leq y$ *imply* $x \circ y \leq a \circ a$*.*

Proof (a) If $x \leq y$ then $x \circ y = x \circ x$ and, by (S2),

$$(x \circ x) \circ (y \circ y) = (x \circ y) \circ (y \circ y) = y = (y \circ y) \circ (y \circ y)$$

thus $y \circ y \leq x \circ x$.

Conversely, if $y \circ y \leq x \circ x$ then, analogously, we can prove

$$(x \circ x) \circ (x \circ x) \leq (y \circ y) \circ (y \circ y)$$

which, by (S2), yields $x \leq y$.

(b) This identity follows directly by (S2) if $x \circ x$ is considered instead of x:

$$x \circ x = ((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ y) = x \circ ((x \circ x) \circ y) = x \circ (y \circ (x \circ x)).$$

(c) Let $x \leq y$. Then $x \circ y = x \circ x$, i.e.

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (x \circ x) = x$$

and hence, by (S3),

$$(y \circ z) \circ (x \circ z) = (y \circ z) \circ (((x \circ y) \circ (x \circ y)) \circ z)$$
$$(y \circ z) \circ (x \circ ((y \circ z) \circ (y \circ z))) = (y \circ z) \circ (y \circ z)$$

by the previous identity (b). Thus $y \circ z \leq x \circ z$.

(d) Suppose $a \leq x$ and $a \leq y$. Then by (c),

$$a \circ a \geq x \circ a \quad \text{and} \quad x \circ a = a \circ x \geq y \circ x.$$

Using transitivity of $\leq$, we conclude $a \circ a \geq y \circ x = x \circ y$. □

Theorem 1 *Let $\circ$ be a Sheffer operation on A and $\leq$ the induced order on $\mathcal{A} = (A, \circ)$. Define*

$$x \vee y = (x \circ x) \circ (y \circ y), \quad x^{\perp} = x \circ x \quad \textit{and} \quad x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}.$$

Then $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, ^{\perp})$ is a lattice with antitone involution.

Proof By (S2) and (S4) we obtain

$$x \circ ((x \circ x) \circ (y \circ y)) = ((x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))))$$
$$\circ ((x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y)))) = x \circ x$$

and, analogously $y \circ ((x \circ x) \circ (y \circ y)) = y \circ y$, thus $x \leq (x \circ x) \circ (y \circ y)$ and $y \leq (x \circ x) \circ (y \circ y)$. Suppose now $x \leq c$ and $y \leq c$. Then $x \circ c = x \circ x$, $y \circ c = y \circ y$ and, by Lemma 2 (c) and (d), $c = (c \circ c) \circ (c \circ c) \geq (x \circ c) \circ (y \circ c) = (x \circ x) \circ (y \circ y)$. Hence, $(x \circ x) \circ (y \circ y)$ is the least common upper bound of x, y, i.e. $x \vee y = (x \circ x) \circ (y \circ y)$.

By (S2), $x^{\perp\perp} = (x \circ x) \circ (x \circ x) = x$ and, by Lemma 2(c), the mapping $x \mapsto x^{\perp} = x \circ x$ is antitone, i.e. it is an antitone involution on $(A, \leq)$. Applying the De Morgan laws we conclude $x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}$. Hence, $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, ^{\perp})$ is a lattice with antitone involution. □

Because of Theorem 1, we call $\mathcal{L}(\mathcal{A})$ the *induced lattice* of $\mathcal{A} = (A, \circ)$.

Theorem 2 *Let $\circ$ be an ortho-Sheffer operation on A and $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, ^{\perp})$ the induced lattice. Then $\mathcal{L}(\mathcal{A})$ is an ortholattice $(A; \vee, \wedge, ^{\perp}, \mathbf{0}, \mathbf{1})$ where $\mathbf{1} = x \circ (x \circ x)$ and $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$.*

Proof Due to Theorem 1, we only need to verify that $x \circ (x \circ x)$ is the greatest element $\mathbf{1}$ of $(A, \leq)$, $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$ is the least element of $(A; \leq)$ and $x^{\perp}$ is a complement of x.

By (S5) we obtain immediately $y \leq x \circ (x \circ x)$ for all $x, y \in A$. Hence $x \circ (x \circ x) = z \circ (z \circ z)$ for all $x, z \in A$, i.e. it is a constant of $(A, \circ)$ which is greater than each element $y \in A$. Denote this constant by $\mathbf{1}$. Hence, $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$ is an algebraic constant of $(A; \circ)$ and, due to Lemma 2(a), $\mathbf{0} = \mathbf{1} \circ \mathbf{1} \leq y \circ y$. Taking $y = x \circ x$, we have $\mathbf{0} \leq (x \circ x) \circ (x \circ x) = x$ for each $x \in A$, i.e. $\mathbf{0}$ is the least element of $(A; \leq)$.

Applying the operations $\vee, \wedge, ^{\perp}$ introduced in Theorem 1 we have immediately

$$x^{\perp} \vee x = ((x \circ x) \circ (x \circ x)) \circ (x \circ x) = x \circ (x \circ x) = \mathbf{1}.$$

By the De Morgan law also $x \wedge x^{\perp} = \mathbf{0}$, i.e. $x^{\perp}$ is a complement and hence an orthocomplement of x. □

Theorem 3 *Let $\mathcal{L} = (L; \vee, \wedge, ^{\perp})$ be a lattice with antitone involution. Define*

$$x \circ y = x^{\perp} \vee y^{\perp}.$$

Then $\circ$ is Sheffer operation on L. If $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, \mathbf{0}, \mathbf{1})$ is an ortholattice then this Sheffer operation $\circ$ satisfies also (S5).

Proof (S1) is evident. We prove (S2):

$$x = x \vee (x \wedge y) = x \vee (x^{\perp} \vee y^{\perp})^{\perp} = (x \circ x) \circ (x \circ y).$$

For (S3) we compute

$$x \circ ((y \circ z) \circ (y \circ z)) = x^{\perp} \vee (y^{\perp} \vee z^{\perp})^{\perp\perp} = x^{\perp} \vee (y^{\perp} \vee z^{\perp}) = (x^{\perp} \vee y^{\perp}) \vee z^{\perp}$$
$$= (x^{\perp} \vee y^{\perp})^{\perp\perp} \vee z^{\perp} = ((x \circ y) \circ (x \circ y)) \circ z.$$

We prove (S4):

$$(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = (x^{\perp} \vee (x \vee y)^{\perp})^{\perp}$$
$$= x \wedge (x \vee y) = x.$$

Suppose now that $x^{\perp}$ is an orthocomplement of x, then

$$y \circ (x \circ (x \circ x)) = y^{\perp} \vee (x^{\perp} \vee x)^{\perp} = y^{\perp} \vee (x \wedge x^{\perp}) = y^{\perp} \vee \mathbf{0} = y^{\perp} = y \circ y$$

thus $\circ$ satisfies also (S5). □

Let $\mathcal{A} = (A; \circ)$ be a groupoid with ortho-Sheffer operation. We denoted by $\mathcal{L}(\mathcal{A})$ the ortholattice induced by $\mathcal{A}$ as considered in Theorems 1 and 2. Analogously, when given an ortholattice $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ denote by $\mathcal{A}(\mathcal{L})$ the groupoid $(L, \circ)$ where $\circ$ is the ortho-Sheffer operation defined as in Theorem 3. Using Theorems 1, 2, 3 and easy computations, one can prove the following correspondence theorem.

Theorem 4 *Let* $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ *be an ortholattice and* $\mathcal{A} = (A, \circ)$ *a groupoid with ortho-Sheffer operation. Then*

$$\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A} \quad \textit{and} \quad \mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}.$$

Proof The proof is an easy exercise left to the reader. □

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