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Directoids with Sectionally Switching Involutions*

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Abstract

It is shown that every directoid equipped with sectionally switching mappings can be represented as a certain implication algebra. Moreover, if the directoid is also commutative, the corresponding implication algebra is defined by four simple identities.

Key words: Directoid; commutative directoid; semilattice; involution; implication algebra; sectionally switching mapping.

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The concept of directoid was introduced by J. Ježek and R. Quackenbush [4] in the sake to axiomatize algebraic structures defined on upward directed ordered sets. In certain sense, directoids generalize semilattices. For the reader convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A;\leq)$ is *upward directed* if $U(x,y)\neq\emptyset$ for every $x,y\in A$, where $U(x,y)=\{a\in A;\ x\leq a \text{ and } y\leq a\}$. Elements of $U(x,y)$ are referred to be common upper bounds of x,y. Of course, if $(A;\leq)$ has a greatest element then it is upward directed.

Let $(A;\leq)$ be an upward directed set and $\sqcup$ denots a binary operation on A. The pair $\mathcal{A}=(A;\sqcup)$ is called a *directoid* if

(i) $x\sqcup y\in U(x,y)$ for all $x,y\in A$;

(ii) if $x\leq y$ then $x\sqcup y=y$ and $y\sqcup x=y$.

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If, moreover, the operation $\sqcup$ is commutative, $\mathcal{A}$ is called a *commutative directoid.*

Example 1 Consider an ordered set $A = \{a, b, c, d, 1\}$ whose diagram is visualized in Fig. 1.

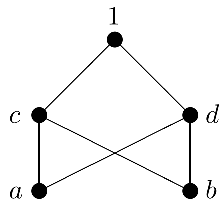

Fig. 1

Define $a \sqcup b = d$, $b \sqcup a = c$, $c \sqcup d = d \sqcup c = 1$ and for other couples $x, y \in A$ by the condition (ii). Then $\mathcal{A} = (A; \sqcup)$ is a directoid which is not commutative.

Of course, every $\vee$-semilattice is a commutative directoid. When we change in our Example 1 the definition of $\sqcup$ only in one instance, i.e. we put $b \sqcup a = d$, the resulting algebra is a commutative directoid which is not a semilattice.

The following axiomatization of directoids was involved in [4]:

Proposition 1 *A groupoid $\mathcal{A} = (A; \sqcup)$ is a directoid if and only if it satisfies the following identities*

(D1) $x \sqcup x = x$;

(D2) $(x \sqcup y) \sqcup x = x \sqcup y$;

(D3) $y \sqcup (x \sqcup y) = x \sqcup y$;

(D4) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

Then a binary relation $\leq$ defined on A by the rule

$$x \leq y \text{ if and only if } x \sqcup y = y \qquad (R)$$

is an order and $x \sqcup y \in U(x, y)$ for each $x, y \in A$.

A groupoid $\mathcal{A} = (A; \sqcup)$ is a commutative directoid if and only if it satisfies the identities (D1), (D4) *and*

(D5) $x \sqcup y = y \sqcup x$.

Let us note that if a directoid $\mathcal{A} = (A; \sqcup)$ is associative, i.e. if it satisfies the identity $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ then it is also commutative and hence a semilattice.

Of course, every upward directed set $(A; \leq)$ can be converted into a (comutative) directoid whenever one assignes to a couple $x, y \in A$ an element $\lambda(x, y) \in U(x, y)$ such that for $x \leq y$ we pick up $\lambda(x, y) = \lambda(y, x) = y$. Then for $x \sqcup y = \lambda(x, y)$, $(A; \sqcup)$ is a directoid; if, moreover, $\lambda(x, y) = \lambda(y, x)$ for every pair x, y of A, the directoid is commutative.

Let $(A; \leq, 1)$ be an ordered set with a greatest element 1. For $p \in A$, the interval $[p, 1]$ will be called a *section*. A mapping f of $[p, 1]$ into itself will be called a *sectional mapping*. To distinguish sectional mappings on different sections, we introduce the following notation: if f is a sectional mapping on $[p, 1]$ and $x \in [p, 1]$ then $f(x)$ will be denoted by x^p. A sectional mapping on $[p, 1]$ is called a *switching mapping* if $p^p = 1$ and $1^p = p$ and it is called an *involution* if $x^{pp} = x$ for each $x \in [p, 1]$. Of course, any involution is a bijection and if a sectional mapping on $[p, 1]$ is a switching involution then

$$x^p = 1 \text{ iff } x = p \quad \text{and} \quad x^p = p \text{ iff } x = 1.$$

$(A; \leq, 1)$ will be called *with sectionally switching involutions* if there is a sectional switching involution on the section $[p, 1]$ for each $p \in A$.

The concept of implication algebra was introduced by J. C. Abbott [1]. It is a groupoid $\mathcal{A} = (A; \circ)$ with a distinguished element 1 (which is an algebraic constant, namely $\mathcal{A}$ satisfies $x \circ x = 1$) in which an order $\leq$ can be induced by $x \leq y$ if and only if $x \circ y = 1$. It was shown [1] that $(A; \leq)$ is a semilattice with a greatest element 1 where $x \vee y = (x \circ y) \circ y$ and, moreover, every section $[p, 1]$ is equipped by a sectional antitone involution $x^p = x \circ p$ (which is in fact a complementation in this section). This concept was generalized in [2] and applied in [3] for axiomatization of logical connective implication in many-valued logics. Let us note the name implication algebra express the fact that $x \circ y$ is interpreted as a connective implication $x \Rightarrow y$.

Lemma 1 *Let $\mathcal{A} = (A; \circ, 1)$ be an algebra of type $(2, 0)$ satisfying the following conditions*

(A1) $x \circ x = 1$, $x \circ 1 = 1$*;*

(A2) $x \circ y = 1$ *implies* $y = (y \circ x) \circ x$*;*

(A3) $x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$.

Define a binary relation $\leq$ on A by the setting

$$x \leq y \textit{ if and only if } x \circ y = 1. \qquad (*)$$

Then $(A; \leq)$ is an ordered set with a greatest element 1 where for each $p \in A$ the mapping $x \mapsto x^p = x \circ p$ is a sectional switching involution on $[p, 1]$.

Proof By (A1) and (A2) we infer immediately

$$1 \circ x = (x \circ x) \circ x = x. \qquad (**)$$

Due to (A1), the relation $\leq$ is reflexive and $x \leq 1$ for each $x \in A$. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$, $y \circ x = 1$ and, by (A2), $y = (y \circ x) \circ x = 1 \circ x = x$ thus $\leq$ is antisymmetrical. Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = 1$, $y \circ z = 1$ and by (A1) and (A3) we have

$$x \circ z = x \circ (1 \circ z) = x \circ ((y \circ z) \circ z) = x \circ (((1 \circ y) \circ z) \circ z) = x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$$

thus $x \leq z$ proving transitivity of $\leq$.

Now, let $p \in A$ and $x \in [p, 1]$. Then $p \leq x$ and hence $p \circ x = 1$. Due to (A2) we conclude $x^{pp} = (x \circ p) \circ p = x$ thus every sectional mapping $x \mapsto x^p = x \circ p$ is an involution on $[p, 1]$. Applying (A1) and $(**)$ we infer that it is a switching mapping. □

Lemma 2 *Let* $\mathcal{A} = (A; \circ, 1)$ *satisfy* (A1), (A2), (A3) *and*

(A4) $y \circ (x \circ y) = 1$;

(A5) $x \circ ((x \circ y) \circ y) = 1$.

Then $(x \circ y) \circ y \in U(x, y)$ *for each* $x, y \in A$.

Proof By Lemma 1, $\leq$ defined by $(*)$ is an order on A. Replace x by $x \circ y$ in (A4) we obtain $y \circ ((x \circ y) \circ y) = 1$ thus $y \leq (x \circ y) \circ y$. By (A5) we have $x \leq (x \circ y) \circ y$ thus $(x \circ y) \circ y \in U(x, y)$. □

Since every implication algebra in the sense of [1] satisfies (A1)–(A5), it motivates us to introduce the following concept: An algebra $\mathcal{A} = (A; \circ, 1)$ satisfying (A1)–(A5) will be called a *weak d−implication algebra*. We can state

Theorem 1 *Let* $\mathcal{A} = (A; \circ, 1)$ *be a weak d-implication algebra. Define a binary operation* $\sqcup$ *on* A *by*

$$x \sqcup y = (x \circ y) \circ y$$

and for each $p \in A$ *define* $x^p = x \circ p$. *Then* $\mathcal{D}(A) = (A; \sqcup)$ *is a directoid with the greatest element* 1 *with sectionally switching involutions whose induced order coincides with that of* $\mathcal{A}$.

Proof Define $x \sqcup y = (x \circ y) \circ y$ and $x^p = x \circ p$, for $x \in [p, 1]$.
(a) Let $x \circ y = 1$. Then $x \sqcup y = (x \circ y) \circ y = 1 \circ y = y$.
(b) Let $\leq$ be the induced order on $\mathcal{A}$. By (A4) we have $x \circ y \in [y, 1]$. Suppose now $x \sqcup y = y$. Then, since the sectional mapping on $[y, 1]$ is an involution, we infer

$$x \circ y = (x \circ y)^{yy} = ((x \circ y) \circ y) \circ y = (x \sqcup y) \circ y = y \circ y = 1.$$

We have shown $x \circ y = 1$ if and only if $x \sqcup y = y$ thus the order on $\mathcal{A}$ defined by $(*)$ coincides with that of $(A; \sqcup)$ defined by (R). The fact that $(A; \sqcup)$ is a directoid follows directly by Lemma 2 and the fact that $x \leq y$ gets $x \sqcup y = (x \circ y) \circ y = 1 \circ y = y$ and, by (A2), also $y \sqcup x = (y \circ x) \circ x = y$. By Lemma 1, sectional mappings $x \mapsto x^p$ for $x \in [p, 1]$ are switching involutions. □

Example 2 Let $A = \{a, b, c, d, 1\}$ and the operation "$\circ$" on A is given by the table

$\circ$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	d	d	1	d	1
d	c	c	c	1	1
1	a	b	c	d	1

One can easily verify the conditions (A1) – (A5) thus $\mathcal{A} = (A; \circ, 1)$ is a weak d-implication algebra. For $\sqcup$ defined by $x \sqcup y = (x \circ y) \circ y$ we obtain just the directoid depicted in Example 1.

To show that directoids with sectional switching involutions can be represented by weak d-implication algebras, we need to prove the converse of Theorem 1.

Theorem 2 *Let $\mathcal{D} = (D; \sqcup, 1)$ be a directoid with a greatest element* 1, $\leq$ *its induced order. Let for each $p \in D$ there exists a sectional switching involution $x \mapsto x^p$ on $[p, 1]$. Define*

$$x \circ y = (x \sqcup y)^y.$$

Then $\mathcal{A}(D) = (D; \circ, 1)$ is a weak d-implication algebra.

Proof Since $y \leq x \sqcup y$ in $\mathcal{D}$, we have $x \sqcup y \in [y, 1]$ and hence the definition of the new operation "$\circ$" is sound. Moreover, $(x \circ y) \circ y = (x \sqcup y)^{yy} = x \sqcup y$.

We have to verify the conditions (A1)–(A5).

(A1): $x \circ x = (x \sqcup x)^x = x^x = 1$ and $x \circ 1 = (x \sqcup 1)^1 = 1^1 = 1$.

(A2): Suppose $x \circ y = 1$. Then $(x \sqcup y)^y = 1$ thus (since the sectional mapping is a switching bijection) also $x \sqcup y = y$. Conversely, if $x \sqcup y = y$ then $x \circ y = 1$, i.e. the order induced on $\mathcal{D}$ coincides with that given by $(*)$ in Theorem 1. Hence, if $x \circ y = 1$ then $x \leq y$ thus $y \in [x, 1]$, i.e. $(y \circ x) \circ x = y^{xx} = y$.

(A3): By (D4) we have $x \leq (x \sqcup y) \sqcup z$ thus

$$x \circ ((((x \circ y) \circ y) \circ z) \circ z) = x \circ ((x \sqcup y) \sqcup z) = 1.$$

(A4): Since $x \sqcup y \in [y, 1]$, we have $x \circ y = (x \sqcup y)^y \in [y, 1]$ thus $y \leq x \circ y$ whence $y \circ (x \circ y) = 1$.

(A5): Since $y \leq x \sqcup y$ we have

$$(x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y.$$

Thus $x \leq x \sqcup y = (x \circ y) \circ y$ proving $x \circ ((x \circ y) \circ y) = 1$. □

In what follows, we modify our results for commutative directoids. For this, define a one more concept.

An algebra $\mathcal{A} = (A; \circ, 1)$ of type (2,0) is called a *d-implication algebra* if it satisfies the identities (A1), (A3) and

(B1) $(x \circ y) \circ y = (y \circ x) \circ x$;

(B2) $((x \circ y) \circ y) \circ y = x \circ y$.

The fact that every d-implication algebra is also a weak d-implication algebra will be clear from the next theorems. Let us only mention that d-implication algebras are determined by identities and hence they form a variety.

Lemma 3 *Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra. Define a binary relation $\leq$ on A by the setting $x \leq y$ if and only if $x \circ y = 1$. Then $\leq$ is an order on A and 1 is a greatest element.*

Proof By (A1), $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$, $y \circ x = 1$ and, due to (B1), also $x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y$, i.e. $\leq$ is antisymmetrical. Transitivity of $\leq$ can be shown identically as in the proof of Lemma 1. By (A1), $x \leq 1$ for each $x \in A$. $\square$

Theorem 3 *Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra. Define*

$$x \sqcup y = (x \circ y) \circ y$$

and for $x \in [y, 1]$ let $x^y = x \circ y$. Then $\mathcal{C}(A) = (A; \sqcup)$ is a commutative directoid with a greatest element 1 and with sectionally switching involutions.

Proof By Lemma 3, $(A; \leq)$ is an ordered set where $x \leq y$ if and only if $x \circ y = 1$ and 1 is a greatest element of $(A; \leq)$. Due to (B1) we infer $x \sqcup y = y \sqcup x$.

By (B1) and (A3) we have

$$x \circ (x \sqcup y) = x \circ ((x \circ y) \circ y) = x \circ ((((x \circ y) \circ y) \circ y) \circ y) = 1$$

thus $x \leq x \sqcup y$. Analogously $y \leq y \sqcup x = x \sqcup y$ thus $x \sqcup y \in U(x, y)$. Further, if $x \leq y$ then

$$x \sqcup y = (x \circ y) \circ y = 1 \circ y = y.$$

We have shown that $(A; \sqcup)$ is a commutative directoid. Analogously as in the previous proofs, the induced order of $(A; \sqcup)$ coincides with $\leq$. Hence, 1 is a greatest element of $(A; \sqcup)$.

Now, let $y \in A$ and $x \in [y, 1]$. Then $y \leq x$ and hence $x^{yy} = (x \circ y) \circ y = x \sqcup y = x$. Further, $y^y = y \circ y = 1$ and $1^y = 1 \circ y = y$ thus for each $y \in A$ the mapping $x \mapsto x^y$ is a sectional switching involution on $[y, 1]$. $\square$

Theorem 4 *Let $\mathcal{C} = (C; \sqcup, 1)$ be a commutative directoid with a greatest element 1. Let $\leq$ be its induced order and for each $p \in C$ there exists a sectional switching involution $x \mapsto x^p$ on $[p, 1]$. Define*

$$x \circ y = (x \sqcup y)^y.$$

Then $\mathcal{A}(C) = (C; \circ, 1)$ is a d-implication algebra.

Proof It was shown in Theorem 2 that "$\circ$" is correctly defined operation on C satisfying (A1) and (A3), and that $(x \circ y) \circ y = x \sqcup y$. Since $x \sqcup y = y \sqcup x$, (B1) is evident. It remains to prove (B2). Since $y \leq x \sqcup y$, we derive

$$((x \circ y) \circ y) \circ y = (x \sqcup y) \circ y = (x \sqcup y)^y = x \circ y. \qquad \square$$

Remark 1 Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra, $\mathcal{C}(A)$ the induced commutative directoid and $\mathcal{A}(\mathcal{C}(A))$ the induced d-implication algebra. Denote by $\bullet$ the binary operation in $\mathcal{A}(\mathcal{C}(A))$. Then

$$x \bullet y = (x \sqcup y)^y = ((x \circ y) \circ y) \circ y = x \circ y$$

by (B2) thus $\mathcal{A}(\mathcal{C}(A)) = \mathcal{A}$.

Remark 2 Let $\mathcal{C} = (C; \sqcup, 1)$ be a commutative directoid with 1 and with sectionally switching involutions. Let $\mathcal{A}(C)$ be the induced d-implication algebra and $\mathcal{C}(\mathcal{A}(C))$ the induced directoid. Denote by $\cup$ the binary operation in $\mathcal{C}(\mathcal{A}(C))$. Since $x \sqcup y \in [y, 1]$, we derive

$$x \cup y = (x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y$$

thus also $\mathcal{C}(\mathcal{A}(C)) = \mathcal{C}$.

Remark 3 Hence, the mutual correspondence between commutative directoids with 1 and with sectional switching involutions and d-algebras is one-to-one and hence every such $\mathcal{C}$ can be identify with $\mathcal{A}(\mathcal{C})$. However, d-implication algebras form a variety thus also the induced commutative directoids.

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