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Linear Model with Nuisance Parameters and with Constraints on Useful and Nuisance Parameters*

Pavla KUNDEROVÁ[1], Jaroslav MAREK[2]

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: [1]*kunderov@inf.upol.cz*
[2]*marek@inf.upol.cz*

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Abstract

The properties of the regular linear model are well known (see [1], Chapter 1). In this paper the situation where the vector of the first order parameters is divided into two parts (to the vector of the useful parameters and to the vector of the nuisance parameters) is considered. It will be shown how the BLUEs of these parameters will be changed by constraints given on them. The theory will be illustrated by an example from the practice.

Key words: Regular linear regression model; nuisance parameters; BLUE; constraints.

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1 Introduction, notations

The following notation will be used throughout the paper:

R^n	the space of all n-dimensional real vectors;
$\boldsymbol{u}_p$, $\boldsymbol{A}_{m,n}$	the real column p-dimensional vector, the real $m \times n$ matrix;

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$\boldsymbol{A}', r(\boldsymbol{A})$	the transpose, the rank of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}_{[r,s]}$	rs-th element of matrix $\boldsymbol{A}$;
$\mathscr{M}(\boldsymbol{A}), \mathrm{Ker}(\boldsymbol{A})$	the column space, the null space of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}^-$	a generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$);
$\boldsymbol{A}^+$	the Moore–Penrose generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$, $(\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$, $(\boldsymbol{A}^+\boldsymbol{A})' = \boldsymbol{A}^+\boldsymbol{A}$);
$\boldsymbol{P}_A$	the orthogonal projector in the Euclidean norm onto $\mathscr{M}(\boldsymbol{A})$;
$\boldsymbol{M}_A = \boldsymbol{I} - \boldsymbol{P}_A$	the orthogonal projector in the Euclidean norm onto $\mathscr{M}^{\perp}(\boldsymbol{A}) = \mathrm{Ker}(\boldsymbol{A}')$;
$\boldsymbol{I}_k$	the $k \times k$ identity matrix;
$0_{m,n}$	the $m \times n$ null matrix;
$\boldsymbol{o}$	the null vector;
$1_k = (1, \ldots, 1)' \in R^k$.	

If $\mathscr{M}(\boldsymbol{A}) \subset \mathscr{M}(\boldsymbol{S})$, $\boldsymbol{S}$ p.s.d., then the symbol $\boldsymbol{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{M}(\boldsymbol{S})$ onto $\mathscr{M}(\boldsymbol{A})$ along $\mathscr{M}(\boldsymbol{S}\boldsymbol{A}^{\perp})$. A general representation of all such projectors $\boldsymbol{P}_A^{S^-}$ is given by $\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{S}^-\boldsymbol{A})^-\boldsymbol{A}'\boldsymbol{S}^- + \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{S}^-)$, where $\boldsymbol{B}$ is arbitrary, (see [3], (2.14)). $\boldsymbol{M}_A^{S^-} = \boldsymbol{I} - \boldsymbol{P}_A^{S^-}$.

Assertion 1 (see [1], Lemma 10.1.35) *Let $\boldsymbol{X}$ be any $n \times k$ matrix and Σ an $n \times n$ p.s.d. matrix.*
(i) If Σ is p.d., then

$$(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ = \Sigma^{-1} - \Sigma^{-1}\boldsymbol{X}(\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X})^-\boldsymbol{X}'\Sigma^{-1} = \Sigma^{-1}\boldsymbol{M}_X^{\Sigma^{-1}}.$$

(ii)
$$\begin{aligned}(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ &= \boldsymbol{M}_X(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ = (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+\boldsymbol{M}_X \\ &= \boldsymbol{M}_X(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+\boldsymbol{M}_X.\end{aligned}$$

2 Best linear unbiased estimators

Let us consider the following linear model

$$\boldsymbol{Y} = (\boldsymbol{X}, \boldsymbol{S})\begin{pmatrix}\beta \\ \kappa\end{pmatrix} + \varepsilon, \tag{1}$$

where $\boldsymbol{Y} = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)'$ is a random observation vector; $\beta \in R^k$ is a vector of the useful parameters; $\kappa \in R^l$ is a vector of the nuisance parameters; $\boldsymbol{X}_{n,k}$ is a design matrix belonging to the vector β; $\boldsymbol{S}_{n,l}$ is a design matrix belonging to the vector κ.

We suppose that

1. $E(\boldsymbol{Y}) = \boldsymbol{X}\beta + \boldsymbol{S}\kappa$, $\forall \beta \in R^k$, $\forall \kappa \in R^l$,
2. $\mathrm{var}(\boldsymbol{Y}) = \Sigma$ is a known matrix,
3. matrix Σ is not a function of the vector $(\beta', \kappa')'$.

If matrix Σ is positive definite and $r(\boldsymbol{X}, \boldsymbol{S}) = k + l < n$, the model is said to be *regular*, (see [1], p. 13).

Theorem 1 *In the regular model (1) the BLUEs of the parameters are given as*

$$\begin{aligned}\hat{\beta} &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{Y}\\ &\quad - \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{X}\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\}\boldsymbol{Y}\\ &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\}\boldsymbol{Y}, \qquad (2)\end{aligned}$$

$$\begin{aligned}\hat{\kappa} &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{X}\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\}\boldsymbol{Y}\\ &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{Y}, \qquad (3)\end{aligned}$$

where $\boldsymbol{C} = \boldsymbol{X}'\Sigma^{-1}\boldsymbol{X}$*.*

Proof According to the Theorem 1.1.1 in [1] and using the following Rohde's formula for inverse of partitioned p.d. matrix (see [1], Lemma 10.1.40)

$$\begin{pmatrix}\boldsymbol{F}, & \boldsymbol{G}\\ \boldsymbol{G}', & \boldsymbol{H}\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}\boldsymbol{F}^{-1}+\boldsymbol{F}^{-1}\boldsymbol{G}(\boldsymbol{H}-\boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{F}^{-1}, & -\boldsymbol{F}^{-1}\boldsymbol{G}(\boldsymbol{H}-\boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\\ -(\boldsymbol{H}-\boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{F}^{-1}, & (\boldsymbol{H}-\boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\end{pmatrix} \qquad (4)$$

the BLUE of the vector parameter $(\beta', \kappa')'$ is given by

$$\begin{pmatrix}\hat{\beta}\\ \hat{\kappa}\end{pmatrix} = \left[\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{S}'\end{pmatrix}\Sigma^{-1}(\boldsymbol{X},\boldsymbol{S})\right]^{-1}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y}$$

$$= \begin{bmatrix}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X}, & \boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\\ \boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}, & \boldsymbol{S}'\Sigma^{-1}\boldsymbol{S}\end{bmatrix}^{-1}\begin{pmatrix}\boldsymbol{X}'\Sigma^{-1}\\ \boldsymbol{S}'\Sigma^{-1}\end{pmatrix}\boldsymbol{Y} = \begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{Y}\\ \boldsymbol{S}'\Sigma^{-1}\boldsymbol{Y}\end{pmatrix},$$

where

$\boxed{11} = \boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}$,

$\boxed{12} = -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}$,

$\boxed{21} = -[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}$,

$\boxed{22} = [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}$.

As Σ is supposed to be positive definite, we utilized Assertion 1, (i). The rest of the proof is obvious. □

Theorem 2 *For the estimators* $\hat{\beta}$*,* $\hat{\kappa}$ *is valid*

$$\begin{aligned}\mathrm{var}(\hat{\beta}) &= \boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}, \qquad (5)\\ \mathrm{var}(\hat{\kappa}) &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}, \qquad (6)\\ \mathrm{cov}(\hat{\beta},\hat{\kappa}) &= -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}. \qquad (7)\end{aligned}$$

Proof

$$\begin{aligned}
\operatorname{var}(\hat\beta) &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\}\Sigma\\
&\quad\times\{\boldsymbol{I}-(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\}\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}\\
&= \boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1},\\
\operatorname{var}(\hat\kappa) &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{M}_X\Sigma\boldsymbol{M}_X(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\\
&\quad\times\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}=[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1},\\
\operatorname{cov}(\hat\beta,\hat\kappa) &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\}\\
&\quad\times\Sigma(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}\\
&= -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}]^{-1}.
\end{aligned}$$

In the course of the proof the Assertion 1, (ii) was used. □

Let us consider model (1) with constrains given on both parameters, i.e. the model

$$\boldsymbol{Y}=(\boldsymbol{X},\boldsymbol{S})\begin{pmatrix}\beta\\ \kappa\end{pmatrix}+\varepsilon,\quad \boldsymbol{b}+\boldsymbol{B}_1\beta+\boldsymbol{B}_2\kappa=\boldsymbol{o},\tag{8}$$

where we suppose for the $q\times k$ matrix $\boldsymbol{B}_1$ and $q\times l$ matrix $\boldsymbol{B}_2$ that

$$r(\boldsymbol{B}_2)=l<q,\quad r(\boldsymbol{B}_1,\boldsymbol{B}_2)=q<k+l.$$

Theorem 3 *The BLUEs $\hat{\hat\beta}$, $\hat{\hat\kappa}$ of the parameters β,κ under the model (8) are given by*

$$\begin{aligned}
\hat{\hat\beta} &= \hat\beta-(\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')\\
&\quad\times[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}(\boldsymbol{B}_1\hat\beta+\boldsymbol{B}_2\hat\kappa+\boldsymbol{b}),
\end{aligned}\tag{9}$$

$$\hat{\hat\kappa} = \hat\kappa+\boldsymbol{Z}^{-1}\boldsymbol{U}'\left[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}'\right]^{-1}(\boldsymbol{B}_1\hat\beta+\boldsymbol{B}_2\hat\kappa+\boldsymbol{b}),\tag{10}$$

where $\boldsymbol{U}=\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}-\boldsymbol{B}_2$, $\boldsymbol{Z}=\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^+\boldsymbol{S}$ and where $\hat\beta,\hat\kappa$ are given in Theorem 1.

Proof In the following regular model with constraints

$$\begin{gathered}
\boldsymbol{Y}\sim_n(\boldsymbol{A}\theta,\Sigma),\quad \boldsymbol{b}+\boldsymbol{B}\theta=\boldsymbol{o},\\
r(\boldsymbol{A}_{n,k})=k<n,\quad r(\boldsymbol{B}_{q,k})=q<k,\quad \Sigma\ \text{p.d.},
\end{gathered}$$

there is (according [2], theorem 4.3.1) for the BLUE of the parameter θ

$$\begin{aligned}
\hat{\hat\theta} &= \{\boldsymbol{I}-(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}'[\boldsymbol{B}(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}']^{-1}\boldsymbol{B}\}\hat\theta\\
&\quad-(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}'[\boldsymbol{B}(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}']^{-1}\boldsymbol{b},
\end{aligned}$$

where $\hat\theta=(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\Sigma^{-1}\boldsymbol{Y}$, is the BLUE of θ without constraints.

In the model (8) we have

$$\boldsymbol{A} \to (\boldsymbol{X}, \boldsymbol{S}), \quad \theta \to \begin{pmatrix} \beta \\ \kappa \end{pmatrix}, \quad \boldsymbol{B} \to (\boldsymbol{B}_1, \boldsymbol{B}_2).$$

Thus analogously

$$\begin{pmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\kappa}} \end{pmatrix} = \left\{ \begin{pmatrix} \boldsymbol{I}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I} \end{pmatrix} - \left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right.$$

$$\times \left. \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1} (\boldsymbol{B}_1, \boldsymbol{B}_2) \right\} \begin{pmatrix} \hat{\beta} \\ \hat{\kappa} \end{pmatrix}$$

$$- \left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1} \boldsymbol{b},$$

where $\hat{\beta}, \hat{\kappa}$ are given in Theorem 1.

Let us calculate first

$$\left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1}$$

$$= \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1}$$

$$= \left(\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} [\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \right)^{-1},$$

where $\boldsymbol{U} = \boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} - \boldsymbol{B}_2$ and where $\boxed{11}$, $\boxed{12}$, $\boxed{21}$, $\boxed{22}$ are given in the proof of Theorem 1. Further

$$\left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} = \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} [\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \\ -[\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \end{pmatrix}.$$

Let us (for the sake of simplicity) use the notation $\boldsymbol{Z} = \boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}$, then

$$\begin{pmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\kappa}} \end{pmatrix} = \left\{ \begin{pmatrix} \boldsymbol{I}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I} \end{pmatrix} - \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} \boldsymbol{Z}^{-1} \boldsymbol{U}' \\ -\boldsymbol{Z}^{-1} \boldsymbol{U}' \end{pmatrix} \right.$$

$$\times \left. [\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} \boldsymbol{Z}^{-1} \boldsymbol{U}']^{-1} (\boldsymbol{B}_1, \boldsymbol{B}_2) \right\} \begin{pmatrix} \hat{\beta} \\ \hat{\kappa} \end{pmatrix}$$

$$- \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} \boldsymbol{Z}^{-1} \boldsymbol{U}' \\ -\boldsymbol{Z}^{-1} \boldsymbol{U}' \end{pmatrix} [\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} \boldsymbol{Z}^{-1} \boldsymbol{U}']^{-1} \boldsymbol{b}.$$

Thus

$$\begin{aligned}\hat{\hat{\beta}} &= \left\{\boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1\right\}\hat{\beta}\\ &\quad - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2\hat{\kappa}\\ &\quad - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{b}.\end{aligned}$$

$$\begin{aligned}\hat{\hat{\kappa}} &= \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1\hat{\beta}\\ &\quad + \left[\boldsymbol{I} + \boldsymbol{Z}^{-1}\boldsymbol{U}'(\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}')^{-1}\boldsymbol{B}_2\right]\hat{\kappa}\\ &\quad + \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{b}.\end{aligned}$$

The statement of the Theorem 3 is now obvious. □

Theorem 4 *For the BLUEs $\hat{\hat{\beta}}$, $\hat{\hat{\kappa}}$ it is valid*

$$\begin{aligned}\mathrm{var}(\hat{\hat{\beta}}) &= \mathrm{var}(\hat{\beta}) - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\\ &\qquad \times (\boldsymbol{B}_1\boldsymbol{C}^{-1} + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}), \qquad (11)\end{aligned}$$

$$\mathrm{var}(\hat{\hat{\kappa}}) = \mathrm{var}(\hat{\kappa}) - \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{U}\boldsymbol{Z}^{-1}. \qquad (12)$$

Proof We have

$$\mathrm{var}(\hat{\hat{\beta}}) = \mathrm{var}[\boldsymbol{A}\hat{\beta} - \boldsymbol{B}\hat{\kappa}],$$

where

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1,\\ \boldsymbol{B} &= (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2.\end{aligned}$$

Analologously

$$\mathrm{var}(\hat{\hat{\kappa}}) = \mathrm{var}[\boldsymbol{F}\hat{\beta} + \boldsymbol{G}\hat{\kappa}],$$

where

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1,\\ \boldsymbol{G} &= \boldsymbol{I} + \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2.\end{aligned}$$

We get the expressions for $\mathrm{var}(\hat{\hat{\beta}})$ and $\mathrm{var}(\hat{\hat{\kappa}})$ after longer but easy calculations. □

Example 1 Consider the following situation. Let's have points F_1, F_2 and F_3 of existing local network and points P_1 and P_2, for which it is necessary to estimate their coordinates (see Figure 1). We have the measured values Y_1, Y_2 of coordinates of the point $F_1 = (\beta_1, \beta_2)$, the measured values Y_3, Y_4 of coordinates of the point $F_2 = (\beta_3, \beta_4)$ and the measured values Y_5, Y_6 of coordinates of the point $F_3 = (\beta_5, \beta_6)$. Moreover, we have the measured values Y_7, Y_8, Y_9, Y_{10} and Y_{11} of angles β_7 and β_8 and distances β_9, β_{10} and β_{11}. Finally, we know the measured values Y_{12} and Y_{13} of angles κ_1 and κ_2. The values β and κ are in meters and in radians, respectively.

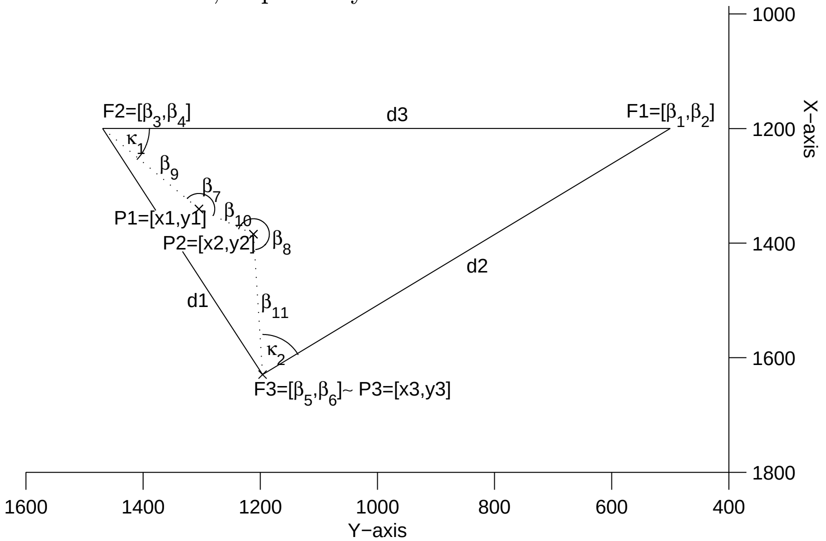

Figure 1: Layout of the situation in Example 1

We have the model (1), where $(X, S) = I_{13}$.
Assume the results of measurements to be (see [4])

$$\mathbf{Y} = \begin{pmatrix} 1200.003 \text{ m} \\ 499.999 \text{ m} \\ 1200.001 \text{ m} \\ 1469.113 \text{ m} \\ 1629.649 \text{ m} \\ 1196.073 \text{ m} \\ 2.876604026 \text{ rad} \\ 4.207717253 \text{ rad} \\ 216.347 \text{ m} \\ 103.095 \text{ m} \\ 245.478 \text{ m} \\ 0.707031134 \text{ rad} \\ 1.080434554 \text{ rad} \end{pmatrix}.$$

We take the covariance matrix Σ from the model (1) in the form

$$\Sigma = \begin{pmatrix} \Sigma^F & 0_{6,5} & 0_{6,2} \\ 0_{5,6} & \Sigma^{d,a} & 0_{5,2} \\ 0_{2,6} & 0_{2,5} & \Sigma^a \end{pmatrix}.$$

We assume the coordinate accuracy of the points F_1, F_2 and F_3 of existing local network to be approximately the same as the accuracy of measured parameters β_j, $j = 7, \ldots, 11$, and as the accuracy of measured parameters κ_1 and κ_2.

The accuracy of coordinates Y_i, $i = 1 \dots 6$, of the points F_1, F_2 and F_3 is given by the covariance matrix Σ^F:

$$\Sigma^F = 0.001^2 \times \begin{pmatrix} 1.6987 & 1.5583 & 0.1928 & 1.0711 & -1.8915 & -2.6295 \\ 1.5583 & 7.3592 & -1.4785 & -3.895 & -0.0798 & -3.4642 \\ 0.1928 & -1.4785 & 5.0406 & -1.4122 & -5.2334 & 2.8907 \\ 1.0711 & -3.895 & -1.4122 & 6.5277 & 0.341 & -2.6328 \\ -1.8915 & -0.0798 & -5.2334 & 0.341 & 7.125 & -0.2613 \\ -2.6295 & -3.4642 & 2.8907 & -2.6328 & -0.2613 & 6.097 \end{pmatrix}.$$

The accuracy of measured distances was 3 mm and the accuracy of measured angles was 5 cc $= 5\pi/(200 \cdot 100 \cdot 100) = 5/636620$, (the standard deviation of the theodolite is $\sigma_t = 5$ cc, i.e. that which corresponds to 5 centesimal seconds). We thus suppose that the covariance matrix for $(Y_7, \dots, Y_{11})$ is

$$\begin{aligned} \Sigma^{d,a} &= \begin{pmatrix} 0.003^2 \times \boldsymbol{I}_{3,3} & \boldsymbol{0}_{3,2} \\ \boldsymbol{0}_{2,3} & (\frac{5\pi}{200\cdot 100\cdot 100})^2 \times \boldsymbol{I}_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} 0.003^2 \times \boldsymbol{I}_{3,3} & \boldsymbol{0}_{3,2} \\ \boldsymbol{0}_{2,3} & 6.17 \times 10^{-11} \times \boldsymbol{I}_{2,2} \end{pmatrix}. \end{aligned}$$

Accordingly, we suppose that the covariance matrix of measured angles $(\boldsymbol{Y}_{12}, \boldsymbol{Y}_{13})$ is

$$\Sigma^a = \left(\frac{5\pi}{200 \cdot 100 \cdot 100}\right)^2 \times \boldsymbol{I}_{2,2} = \left(\frac{5}{636620}\right)^2 \times \boldsymbol{I}_{2,2} = 6.17 \times 10^{-11} \times \boldsymbol{I}_{2,2}\,.$$

The aim is to find conditions for parameters β and κ.

To that end, we first determine (see Figure 1) the coordinates of points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$:

$$\begin{aligned} x_1 &= \beta_3 + \beta_9 \cos\left(\tfrac{\pi}{2} + \kappa_1\right), \\ y_1 &= \beta_4 + \beta_9 \cos\left(\tfrac{\pi}{2} + \kappa_1\right), \end{aligned}$$

(it follows from the fact that the point P_1 shall be situated on a circle with circumference β_9 and with center in point F_2, and from the fact that the point P_1 is reached from the point F_2 via the angle $\angle F_1, F_2, P_1 = \kappa_1$);

$$\begin{aligned} x_2 &= x_1 + \beta_{10} \cos\left(\left(\arctan \tfrac{\beta_4 - y_1}{\beta_3 - x_1} + 0 \cdot \pi\right) + \pi + \beta_7\right), \\ y_2 &= y_1 + \beta_{10} \sin\left(\left(\arctan \tfrac{\beta_4 - y_1}{\beta_3 - x_1} + 0 \cdot \pi\right) + \pi + \beta_7\right), \end{aligned}$$

(it follows from the fact that the point P_2 shall be situated on a circle with circumference β_{10} and with center in point P_1, and from the fact that the point P_2 is reached from the point P_1 via the angle $\angle F_2, P_1, P_2 = \beta_7$);

$$\begin{aligned} x_3 &= x_2 + \beta_{11} \cos\left(\left(\arctan \tfrac{y_1 - y_2}{x_1 - x_2} + 0 \cdot \pi\right) + \pi + \beta_8\right), \\ y_3 &= y_2 + \beta_{11} \sin\left(\left(\arctan \tfrac{y_1 - y_2}{x_1 - x_2} + 0 \cdot \pi\right) + \pi + \beta_8\right), \end{aligned}$$

(it follows from the fact that the point P_3 shall be situated on a circle with circumference β_{11} and with center in point P_2, and from the fact that the point P_3 is reached from the point P_2 via the angle $\angle P_1, P_2, P_3 = \beta_8$).

It can be seen from Figure 1 that the conditions $\mathbf{g}(\beta, \kappa) = \boldsymbol{o}$ for parameters β and κ are (involving the conditions given above)

$$\begin{aligned} g_1 &= (x_3 - \beta_5)^2 + (y_3 - \beta_6)^2 = 0, \\ g_2 &= \left(\pi + \arctan \frac{y_3 - \beta_2}{x_3 - \beta_1}\right) - \left(\pi + \arctan \frac{y_2 - y_3}{x_2 - x_3}\right) - \kappa_2 = 0. \end{aligned}$$

The first constraint says that the point P_3 is equivalent to F_3.

The second constraint reflects the fact that $\angle P_2, P_3, F_1 = \kappa_2$.

Now we use the Taylor expansion—the linear version of the condition

$$\mathbf{g}(\beta, \kappa) = \begin{pmatrix} g_1(\beta, \kappa) \\ g_2(\beta, \kappa) \end{pmatrix} = \boldsymbol{o}$$

is $\boldsymbol{B}_1 \delta\beta + \boldsymbol{B}_2 \delta\kappa + b = \boldsymbol{o}$, where the matrix $\boldsymbol{B}_1 = \frac{\partial \mathbf{g}(\beta^0, \kappa^0)}{\partial \beta'}$, $\boldsymbol{B}_2 = \frac{\partial \mathbf{g}(\beta^0, \kappa^0)}{\partial \kappa'}$, and $b = \mathbf{g}(\beta^0, \kappa^0)$ at the approximate point.

So we can consider the model (8).

In the linearized model we determine numerically the estimates and the covariance matrices according to Theorem 3 and Theorem 4

$$\hat{\hat{\beta}} = \begin{pmatrix} 1200.000 \text{ m} \\ 500.000 \text{ m} \\ 1200.000 \text{ m} \\ 1469.112 \text{ m} \\ 1629.651 \text{ m} \\ 1196.073 \text{ m} \\ 2.876605771 \text{ rad} \\ 4.207720046 \text{ rad} \\ 216.347 \text{ m} \\ 103.096 \text{ m} \\ 245.475 \text{ m} \end{pmatrix} \quad \text{and} \quad \hat{\hat{\kappa}} = \begin{pmatrix} 0.707030785 \text{ rad} \\ 1.080438743 \text{ rad} \end{pmatrix}.$$

$$\operatorname{var}(\hat{\hat{\beta}}) = (\, Q_1 \,,\, Q_2 \,) \quad \text{and} \quad \operatorname{var}(\hat{\hat{\kappa}}) = \begin{pmatrix} 4.90 \cdot 10^{-11} & -9.24 \cdot 10^{-12} \\ -9.24 \cdot 10^{-12} & 4.30 \cdot 10^{-11} \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} 1.43 \cdot 10^{-6} & 1.79 \cdot 10^{-6} & -4.45 \cdot 10^{-7} & 8.08 \cdot 10^{-7} & -9.83 \cdot 10^{-7} & -2.60 \cdot 10^{-6} \\ 1.79 \cdot 10^{-6} & 7.14 \cdot 10^{-6} & -1.01 \cdot 10^{-6} & -3.60 \cdot 10^{-6} & -7.73 \cdot 10^{-7} & -3.54 \cdot 10^{-6} \\ -4.45 \cdot 10^{-7} & -1.01 \cdot 10^{-6} & 3.34 \cdot 10^{-6} & -1.83 \cdot 10^{-6} & -2.90 \cdot 10^{-6} & 2.85 \cdot 10^{-6} \\ 8.08 \cdot 10^{-7} & -3.60 \cdot 10^{-6} & -1.83 \cdot 10^{-6} & 6.07 \cdot 10^{-6} & 1.02 \cdot 10^{-6} & -2.47 \cdot 10^{-6} \\ -9.83 \cdot 10^{-7} & -7.73 \cdot 10^{-7} & -2.90 \cdot 10^{-6} & 1.02 \cdot 10^{-6} & 3.88 \cdot 10^{-6} & -2.49 \cdot 10^{-7} \\ -2.60 \cdot 10^{-6} & -3.54 \cdot 10^{-6} & 2.85 \cdot 10^{-6} & -2.47 \cdot 10^{-6} & -2.49 \cdot 10^{-7} & 6.01 \cdot 10^{-6} \\ 6.18 \cdot 10^{-10} & -1.03 \cdot 10^{-9} & 1.19 \cdot 10^{-10} & 1.96 \cdot 10^{-9} & -7.29 \cdot 10^{-10} & -9.31 \cdot 10^{-10} \\ 1.03 \cdot 10^{-9} & -1.32 \cdot 10^{-9} & 1.23 \cdot 10^{-9} & 2.21 \cdot 10^{-9} & -2.26 \cdot 10^{-9} & -8.86 \cdot 10^{-10} \\ -2.36 \cdot 10^{-7} & -8.35 \cdot 10^{-8} & -1.30 \cdot 10^{-6} & 5.29 \cdot 10^{-7} & 1.54 \cdot 10^{-6} & -4.45 \cdot 10^{-7} \\ 1.95 \cdot 10^{-7} & -3.96 \cdot 10^{-7} & -1.52 \cdot 10^{-7} & 8.10 \cdot 10^{-7} & -4.31 \cdot 10^{-8} & -4.13 \cdot 10^{-7} \\ -1.32 \cdot 10^{-6} & 8.08 \cdot 10^{-7} & -3.93 \cdot 10^{-6} & -4.53 \cdot 10^{-7} & 5.26 \cdot 10^{-6} & -3.56 \cdot 10^{-7} \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 6.18\cdot10^{-10} & 1.03\cdot10^{-9} & -2.36\cdot10^{-7} & 1.95\cdot10^{-7} & -1.32\cdot10^{-6} \\ -1.03\cdot10^{-9} & -1.32\cdot10^{-9} & -8.35\cdot10^{-8} & -3.96\cdot10^{-7} & 8.08\cdot10^{-7} \\ 1.11\cdot10^{-10} & 1.23\cdot10^{-9} & -1.30\cdot10^{-6} & -1.52\cdot10^{-7} & -3.93\cdot10^{-6} \\ 1.96\cdot10^{-9} & 2.21\cdot10^{-9} & 5.29\cdot10^{-7} & 8.10\cdot10^{-7} & -4.53\cdot10^{-7} \\ -7.29\cdot10^{-10} & -2.26\cdot10^{-9} & 1.54\cdot10^{-6} & -4.31\cdot10^{-8} & 5.26\cdot10^{-6} \\ -9.31\cdot10^{-10} & -8.86\cdot10^{-10} & -4.45\cdot10^{-7} & -4.13\cdot10^{-7} & -3.56\cdot10^{-7} \\ 5.11\cdot10^{-11} & -1.05\cdot10^{-11} & -4.57\cdot10^{-9} & -4.63\cdot10^{-9} & -2.57\cdot10^{-9} \\ -1.05\cdot10^{-11} & 5.05\cdot10^{-11} & -3.63\cdot10^{-9} & -4.45\cdot10^{-9} & 6.83\cdot10^{-11} \\ -4.57\cdot10^{-9} & -3.63\cdot10^{-9} & 2.20\cdot10^{-5} & -2.15\cdot10^{-6} & -4.26\cdot10^{-6} \\ -4.63\cdot10^{-9} & -4.45\cdot10^{-9} & -2.15\cdot10^{-6} & 2.30\cdot10^{-5} & -1.59\cdot10^{-6} \\ -2.57\cdot10^{-9} & 6.83\cdot10^{-11} & -4.26\cdot10^{-6} & -1.59\cdot10^{-6} & 1.51\cdot10^{-5} \end{pmatrix}.$$

All computations in the example were performed in Matlab.

To make comparisons easier, the following table shows the results.

$\boldsymbol{Y}_{[i]}$	$\sqrt{\mathrm{var}(\boldsymbol{Y})}_{[i,i]}$	$\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}_{[i]}$	$\sqrt{\left[\mathrm{var}\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}\right]_{[i,i]}}$	$\boldsymbol{Y}_{[i]}-\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}_{[i]}$
1200.003 m	4.12 mm	1200.000 m	1.13 mm	3 mm
499.999 m	8.58 mm	500.000 m	2.67 mm	−1 mm
1200.001 m	7.10 mm	1200.000 m	1.83 mm	1 mm
1469.113 m	8.08 mm	1469.112 m	2.46 mm	1 mm
1629.649 m	8.44 mm	1629.651 m	1.97 mm	−2 mm
1196.073 m	7.81 mm	1196.073 m	2.45 mm	0 mm
2.876604026 rad	5.00 cc	2.87605771 rad	4.55 cc	−1.111 cc
4.207717253 rad	5.00 cc	4.20772005 rad	4.53 cc	−1.778 cc
216.347 m	3.00 mm	216.347 m	4.69 mm	0 mm
103.095 m	3.00 mm	103.096 m	4.79 mm	−1 mm
245.478 m	3.00 mm	245.475 m	3.89 mm	3 mm
0.707031134 rad	5.00 cc	0.707030785 rad	4.46 cc	0.222 cc
1.080434554 rad	5.00 cc	1.080438743 rad	4.17 cc	−2.667 cc

The second column shows that the dispersions of elements of the measured vector $\boldsymbol{Y}$ are different. We can see in the table that dispersions of some elements of estimators $\hat{\hat{\beta}}$ and $\hat{\hat{\kappa}}$ have decreased and some have increased in the process of estimation, which is due to the tendency to distribute the uncertainty of measurements equally.

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