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Variance Components and Nonlinearity*

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Abstract

Unknown parameters of the covariance matrix (variance components) of the observation vector in regression models are an unpleasant obstacle in a construction of the best estimator of the unknown parameters of the mean value of the observation vector. Estimators of variance componets must be utilized and then it is difficult to obtain the distribution of the estimators of the mean value parameters. The situation is more complicated in the case of nonlinearity of the regression model. The aim of the paper is to contribute to a solution of the mentioned problem.

Key words: Variance components; nonlinear regression model; linearization region; insensitiveness region.

2000 Mathematics Subject Classification: 62F10, 62J05

1 Introduction

The regression model is assumed to be of the form

$$\mathbf{Y} \sim_n \left(\mathbf{f}(\boldsymbol{\beta}), \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \right),$$

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where $\mathbf{Y}$ is an n-dimensional random vector (observation vector) with the mean value equal to $\mathbf{f}(\boldsymbol{\beta}), \boldsymbol{\beta} \in R^k$ (k-dimensional Euclidean space) and the covariance matrix equal to $\sum_{i=1}^p \vartheta_i \mathbf{V}_i$. Here $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$ is a p-dimensional vector of variance components and $\boldsymbol{\vartheta} \in \underline{\vartheta} \subset R^p$; $\underline{\vartheta}$ is an open set in R^p. The symmetric and positive semidefinite (p.s.d.) matrices $\mathbf{V}_1, \dots, \mathbf{V}_p$ are given and all variance components are positive.

The problem is to find a decision whether the model can be linearized (with respect to $\boldsymbol{\beta}$) and estimators of the variance components ($\boldsymbol{\vartheta}$) can be used instead of the true values in estimation of $\boldsymbol{\beta}$. One of the possible approaches is demonstrated in the case of the bias of the estimator of $\boldsymbol{\beta}$.

2 Preliminaries

In the following text it will be assumed that the model considered can be characterized with sufficient accuracy as

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n\left(\mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \sum_{i=1}^p \vartheta_i \mathbf{V}_i\right) \tag{1}$$

where

$$\mathbf{f}_0 = \mathbf{f}(\boldsymbol{\beta}_0), \quad \mathbf{F} = \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}'}\Big|_{u=\beta_0}, \quad \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = [\kappa_1(\delta\boldsymbol{\beta}), \dots, \kappa_n(\delta\boldsymbol{\beta})]',$$

$$\kappa_i(\delta\boldsymbol{\beta}) = \delta\boldsymbol{\beta}' \frac{\partial^2 f_i(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}'}\Big|_{u=\beta_0} \delta\boldsymbol{\beta}, \quad i = 1, \dots, n,$$

and the vector $\boldsymbol{\beta}_0$ is as near as possible to the true value $\boldsymbol{\beta}^*$ of the parameter $\boldsymbol{\beta}$.

The linear version of the model considered is

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n\left(\mathbf{F}\delta\boldsymbol{\beta}, \sum_{i=1}^p \vartheta_i \mathbf{V}_i\right), \delta\boldsymbol{\beta} \in R^k, \boldsymbol{\vartheta} \in \underline{\vartheta}. \tag{2}$$

The regularity of the model will be assumed in the following consideration, i.e., the rank of the matrix $\mathbf{F}$ is $r(\mathbf{F}) = k < n$, and $\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^p \vartheta_i \mathbf{V}_i$ is positive definite (p.d.).

Lemma 2.1 *In the model (2) the $\boldsymbol{\vartheta}_0$-LBLUE (locally best linear unbiased estimator) of the parameter $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}_0 + \widehat{\delta\boldsymbol{\beta}}$, where*

$$\widehat{\delta\boldsymbol{\beta}} = [\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)(\mathbf{Y} - \mathbf{f}_0)$$
$$\sim N_k\left(\delta\boldsymbol{\beta}, [\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)[\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\right).$$

Here $\boldsymbol{\vartheta}^*$ is the actual value of the vector parameter $\boldsymbol{\vartheta}$.

Proof is well known and therefore it is omitted.

The notation $\mathbf{S}_A$ ($\mathbf{A}$ is any $n \times n$ matrix) means the matrix with the (i,j)-th entry equal to

$$\{\mathbf{S}_A\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i \mathbf{A} \mathbf{V}_j \mathbf{A}), \quad i,j = 1,\dots,p.$$

Further $(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+$ is the Moore–Penrose generalized inverse of the matrix $\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F$, $\mathbf{M}_F = \mathbf{I} - \mathbf{P}_F = \mathbf{F}\mathbf{F}^+$ (in more detail cf. [7]).

Lemma 2.2 *Let in the model (2) the matrix* $\mathbf{S}_{(M_F \Sigma_0 M_F)^+}$ *be regular* ($\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^{(0)} \mathbf{V}_i$, $\boldsymbol{\vartheta}^{(0)}$ *is the value of the parameter* $\boldsymbol{\vartheta}$ *as near as possible to the actual value* $\boldsymbol{\vartheta}^*$). *Then the* $\boldsymbol{\vartheta}_0$*-MINQUE (minimin norm quadratic unbiased estimator; in more detail cf. [8]) of the vector* $\boldsymbol{\vartheta}$ *is*

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_F \Sigma_0 M_F)^+} \begin{pmatrix} (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_1 (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0) \\ \vdots \\ (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_p (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0) \end{pmatrix}.$$

In the case of normality the variance matrix of this estimator is $\mathrm{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}_{(M_F \Sigma_0 M_F)^+}$.

Proof Cf. [8].

3 Influence of nonlinearity on the estimator of $\boldsymbol{\vartheta}$

Lemma 3.1 *In the model (1) the bias of the estimator from Lemma 2.2 at the point* $\boldsymbol{\beta}_0$ *is*

$$E_{\beta_0,\vartheta}(\hat{\boldsymbol{\vartheta}}) - \boldsymbol{\vartheta} = \frac{1}{4}\mathbf{S}^{-1}_{(M_X \Sigma_0 M_F)^+} \begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_1 (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_p (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix}.$$

Proof It is valid

$$\begin{aligned} &E_{\beta_0,\vartheta}\Big[(\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0)\Big] \\ &= E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0) \\ &\quad + \mathrm{Tr}\left[(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] \end{aligned}$$

Now it is sufficient to use the equalities

$$E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0) = \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})$$

and

$$\mathrm{Tr}\left[(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] = \left\{\mathbf{S}_{(M_F \Sigma_0 M_F)^+}\right\}_{j,\cdot} \boldsymbol{\vartheta}. \qquad \square$$

Let the Bates and Watts intrinsic measure of nonlinearity [1] at the point $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$ be denoted as $K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)$,

$$K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0) = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}_0^{-1}\mathbf{M}_F^{\Sigma_0^{-1}}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta} \in R^k \right\}.$$

Theorem 3.2 *Let* $\mathbf{C}_0 = \mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}$. *If*

$$\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \leq \frac{2\varepsilon}{K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)},$$

then

$$\forall\{i = 1, \ldots, p\}\ |E_{\beta_0,\vartheta}(\hat{\vartheta}_i) - \vartheta_i| \leq \sum_{i=1}^{p} |k_{i,j}|\varepsilon^2,$$

where

$$\mathbf{k}'_i = (k_{i,1}, \ldots, k_{i,p}) = \left\{\mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\right\}_{i,\cdot} [\mathrm{Diag}(\boldsymbol{\vartheta}_0)]^{-1}, \quad i = 1, \ldots, p.$$

Proof Let $\hat{\boldsymbol{\zeta}} = (\hat{\zeta}_1, \ldots, \hat{\zeta}_p)'$, where

$$\hat{\zeta}_i = (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_i(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+(\mathbf{Y} - \mathbf{f}_0), \quad i = 1, \ldots, p.$$

Then, with respect to Lemma 3.1

$$\begin{aligned}
&E_{\beta_0,\vartheta}(\hat{\boldsymbol{\vartheta}}) - \boldsymbol{\vartheta} = \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\left[E_{\beta_0,\vartheta}(\hat{\boldsymbol{\zeta}}) - \mathbf{S}_{(M_F\Sigma_0 M_F)^+}\boldsymbol{\vartheta}\right] \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix} \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\left[\mathrm{Diag}(\boldsymbol{\vartheta}_0)\right]^{-1}\mathrm{Diag}(\boldsymbol{\vartheta}_0) \\
&\quad\times\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{k}'_1 \\ \vdots \\ \mathbf{k}'_p \end{pmatrix}\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_1^{(0)}\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_p^{(0)}\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix}.
\end{aligned}$$

The inclusions $\mathcal{M}(\mathbf{V}_i) = \{\mathbf{V}_i\mathbf{u} : \mathbf{u} \in R^n\} \subset \mathcal{M}(\boldsymbol{\Sigma}_0)$, $i = 1, \ldots, p$, are a consequnce of the assumption that the matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are p.s.d. and $\vartheta_i > 0$, $i = 1, \ldots, p$. These inclusions imply

$$\begin{gathered}
\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_i^{(0)}\mathbf{V}_i(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\
\leq \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\Sigma}_0(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).
\end{gathered}$$

Thus we obtain

$$|E_{\beta_0,\vartheta}(\hat{\vartheta}_i)-\vartheta_i| = \frac{1}{4}\sum_{j=1}^{p} k_{i,j}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\vartheta_j^{(0)}\mathbf{V}_j(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})$$
$$\leq \frac{1}{4}\sum_{j=1}^{p}|k_{i,j}|\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).$$

Now the definition of $K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)$ can be used and thus

$$\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq \left(K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)\right)^2(\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta})^2.$$

If

$$\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \leq \frac{2\varepsilon}{K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)},$$

then

$$\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq 4\varepsilon^2$$

and also

$$|E_{\beta_0,\vartheta}(\hat{\vartheta}_i)-\vartheta_i| \leq \frac{1}{4}\sum_{j=1}^{p}|k_{i,j}|\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq \sum_{j=1}^{p}|k_{i,j}|\varepsilon^2. \quad \square$$

4 Linearization region

In the case of the model (2) when variance components are known, then the BLUE of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = [\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{Y}.$$

This estimator is biased in the model (1) and

$$\mathbf{b} = E_\beta(\hat{\boldsymbol{\beta}})-\boldsymbol{\beta} = \frac{1}{2}[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).$$

Let the Bates and Watts parametric curvature at the point $(\boldsymbol{\beta}_0,\boldsymbol{\vartheta}_0)$ be denoted as $K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)$

$$K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0) = \sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}_0^{-1}\mathbf{P}_F^{\Sigma_0^{-1}}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta}\in R^k\right\}.$$

Lemma 4.1 *Let in the model (1)*

$$\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \leq \frac{2\varepsilon}{K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)}.$$

Then

$$\forall\{\mathbf{h}\in R^k\}|\mathbf{h}'\mathbf{b}| \leq \varepsilon\sqrt{\mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{h}}.$$

Proof Cf. in [4] and [6].

Remark 4.2 Theorem 3.2 and Lemma 4.1 show that the regions of linearization for $\boldsymbol{\vartheta}$

$$\mathcal{L}_{\vartheta} = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon}{K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)} \right\}$$

and for the bias $\mathbf{b}$

$$\mathcal{L}_{b} = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon}{K^{(par)}_{\vartheta_0}(\boldsymbol{\beta}_0)} \right\}$$

have the same shape, i.e. we have to use the smaller of them. Usually $\mathcal{L}_b \subset \mathcal{L}_{\vartheta}$.

The necessary condition for efficient utilization of Theorem 3.2. and Lemma 4.1 is $\delta\boldsymbol{\beta}^* \in \mathcal{L}_b \cap \mathcal{L}_{\vartheta}$ and at the same time the difference $\boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}_0$ must be in so called nonsensitivenes region which is in more detail described in the following section.

5 Nonsensitiveness region

How the small shift $\delta\boldsymbol{\vartheta}$ of the parameter $\boldsymbol{\vartheta}$ can change the statistical properties of the estimator $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})$ is given in the following statement.

Lemma 5.1 *Let*

$$\begin{aligned}
\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) &= \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{Y},\\
\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0) &= \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Y},\\
\mathbf{v} &= \mathbf{Y} - \mathbf{F}\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0).
\end{aligned}$$

Then

(i) $\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) = \mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0) - \mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}$,
where $\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \delta\vartheta_i \mathbf{V}_i$ *and* $\mathbf{L}_h' = \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)$.

(ii) $E_{\beta}(\mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}) = \mathbf{0}$.

(iii) $\operatorname{cov}_{\vartheta_0}\left(\mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}, \hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0)\right) = \mathbf{0}$.

Proof Cf. [2] and [3].

Corollary 5.2 *Let*

$$\mathbf{W}_h = \begin{pmatrix} \mathbf{L}_h'\mathbf{V}_1 \\ \vdots \\ \mathbf{L}_h'\mathbf{V}_p \end{pmatrix} [\mathbf{M}_F(\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_F]^{+}(\mathbf{V}_1\mathbf{L}_h, \dots, \mathbf{V}_p\mathbf{L}_h).$$

Then

$$\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta})] = \mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}_0)] + \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta}.$$

If an experimenter can admit

$$\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}_0)] + \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta}} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}_0)]},$$

then $\delta\boldsymbol{\vartheta}^*$ *must be in the region*

$$\mathcal{N}_h = \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta} \leq 2\varepsilon\mathbf{h}'\left[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\right]^{-1}\mathbf{h}\right\}.$$

In order to recognize whether $\delta\boldsymbol{\beta}^*$ and $\delta\boldsymbol{\vartheta}^*$ are in the regions $\mathcal{L}_b \cap \mathcal{L}_\vartheta$ and $\mathcal{N}_h$, respectively, we must have some information an an accuracy of the estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\vartheta}}$.

The first orientation on the confidence region of the parameter $\boldsymbol{\beta}$ is the set

$$\mathcal{E}_\beta = \left\{\delta\boldsymbol{\beta} : (\delta\boldsymbol{\beta} - \widehat{\delta\boldsymbol{\beta}})'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}(\delta\boldsymbol{\beta} - \widehat{\delta\boldsymbol{\beta}}) \leq \chi^2_k(0,1-\alpha)\right\},$$

where $\chi^2_k(0,1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with k degrees of freedom.

Unfortunately the confidence region for the parameter $\boldsymbol{\vartheta}$ is not known, however some information on it we can obtain by the help of the following lemma.

Lemma 5.3 *Let*

$$\mathbf{Y} \sim N_n(\mathbf{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i).$$

Then

$$\delta\boldsymbol{\vartheta}'(2\mathbf{S}^{-1}_{(M_F\Sigma_0M_F)^+})^{-1}\delta\boldsymbol{\vartheta} < t^2$$

$$\Rightarrow \forall\{i=1,\dots,p\}\ |\vartheta_i| \leq t\sqrt{2\{\mathbf{S}^{-1}_{(M_F\Sigma_0M_F)^+}\}_{i,i}}.$$

Proof It is a direct consequence of Theorem 2.2. in [5]

Remark 5.4 If the real number $t > 0$ is sufficiently large such that

$$|\hat{\vartheta}_i - \vartheta^*| < t\sqrt{2\{\mathbf{S}^{-1}_{(M_F\Sigma_0M_F)^+}\}_{i,i}}, \quad i=1,\dots,p,$$

occur with certainty (with sufficiently high probability), then we can be practically sure that the actual value $\boldsymbol{\vartheta}^*$ of the vector $\boldsymbol{\vartheta}$ is in the domain

$$\mathcal{K}_\vartheta = \left\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}})'\mathbf{S}_{(M_F\Sigma_0M_F)^+}(\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}}) < 2t^2\right\}.$$

6 Inference on linearization

A comparison of the sets $\mathcal{E}_\beta, \mathcal{K}_\vartheta, \mathcal{N}_h, \mathcal{L}_b, \mathcal{L}_\vartheta$ leads to a decision whether the considered model with unknown variance components can be linearized. In the first step we shall take into account the following lemma.

Lemma 6.1 *Let $\boldsymbol{\vartheta}_0$ be given. Then for any $\tau > 0$ the notation $\boldsymbol{\vartheta}_\tau$ means $\tau\boldsymbol{\vartheta}_0$.*

(i) $\mathbf{k}'_i(\tau\boldsymbol{\vartheta}_0) = \left\{\mathbf{S}^{-1}_{(M_F\Sigma_{\tau\vartheta_0}M_F)^+}\right\}_{i,\cdot}[\mathrm{Diag}(\tau\boldsymbol{\vartheta}_0)]^{-1} = \tau\mathbf{k}'_i(\boldsymbol{\vartheta}_0)$.

(ii) $\mathbf{S}_{(M_F\Sigma_{\tau\vartheta_0}M_F)^+} = \tau^2\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}$.

(iii) $K^{(int)}_{\tau\vartheta_0}(\boldsymbol{\beta}_0) = \sqrt{\tau}K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)$.

(iv) $\mathbf{W}_h(\tau\boldsymbol{\vartheta}_0) = \frac{1}{\tau}\mathbf{W}_h(\boldsymbol{\vartheta}_0)$.

Proof The statements are direct consequences of definitions.

Corollary 6.2 *If*

$$\sum_{j=1}^n |k_{i,j}(\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \leq \varepsilon_2\sqrt{\left\{2\mathbf{S}^{-1}_{M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{i,i}},$$

then

$$\forall\{\tau > 0\}\ \sum_{j=1}^n |k_{i,j}(\tau\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \leq \varepsilon_2\sqrt{\left\{2\mathbf{S}^{-1}_{M_F\Sigma_{\tau\vartheta_0}M_F)^+}\right\}_{i,i}}$$

(consequence of Lemma 6.1 (i) and (ii)).

$$\frac{1}{\tau}\delta\boldsymbol{\vartheta}'\mathbf{W}_h(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} = \delta\boldsymbol{\vartheta}'\mathbf{W}_h(\tau\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \leq 2\varepsilon_3\,\mathrm{Var}_{\tau\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\tau\boldsymbol{\vartheta}_0)]$$
$$= \tau 2\varepsilon_3\,\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]$$

(it is to be remarked that $\hat{\boldsymbol{\beta}}(\tau\boldsymbol{\vartheta}_0) = \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)$). The last inequality can be interpreted as follows. If the value $\boldsymbol{\vartheta}_0$ is changed into $\tau\boldsymbol{\vartheta}_0$, then the admissible shift $\delta\boldsymbol{\vartheta}$ is changed into the shif $\sqrt{\tau}\delta\boldsymbol{\vartheta}$.

Now the sequence of the steps necessary to make a decision can be described.

(i) When the values $\boldsymbol{\vartheta}_0$ and ε_2 are chosen the value ε_1 is determined in such a way that

$$\sum_{j=1}^n |k_{i,j}(\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \leq \varepsilon_2\sqrt{\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\}_{i,i}}, \quad i = 1, \dots, p,$$

(it implies $|E_{\vartheta_0}(\hat{\vartheta}_i) - \vartheta_i| \leq \varepsilon_2\sqrt{\mathrm{Var}_{\vartheta_0}(\hat{\vartheta}_i)}$, $i = 1, \dots, p$, i.e. biases caused by nonlinearity can be neglected). Thus we determined the region $\mathcal{L}_{\vartheta_0}$, i.e.

$$\mathcal{L}_{\vartheta_0} = \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}_{\vartheta_0}\mathbf{F}\delta\boldsymbol{\beta} \leq \frac{2\varepsilon_1}{K^{(int)}_{\vartheta_0}(\beta_0)}\right\}.$$

(ii) To choose the value ε_3 and to determine the set

$$\mathcal{N}_h = \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_h(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \leq 2\varepsilon_3 \operatorname{Var}_{\vartheta_0}\left[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)\right]\right\}.$$

Shifts $\delta\boldsymbol{\vartheta}$ inside the set $\mathcal{N}_h$ does not enlarge the standard deviation of the estimator $\mathbf{h}'\widehat{\delta\boldsymbol{\beta}}$ more than $\varepsilon_3\sqrt{\operatorname{Var}_{\vartheta_0}(\mathbf{h}'\widehat{\boldsymbol{\beta}})}$.

(iii) To check the inclusions $\mathcal{E}_\beta \subset \mathcal{L}_b \cap \mathcal{L}_\vartheta$ and

$$\mathcal{K}_\vartheta = \left\{\mathbf{u} : \mathbf{u}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\mathbf{u}/2 \leq t^2\right\} \subset \mathcal{N}_h.$$

If these inclusions are satisfied (the actual value $\delta\boldsymbol{\beta}^*$ of $\delta\boldsymbol{\beta}$ is suffiently small for the bias of the estimator $\hat{\boldsymbol{\vartheta}}$ and the actual $\delta\boldsymbol{\vartheta}^*$ is with high probability in the nonsensitiveness region), then the model with the estimated variance components can be linearized and the estimates $\hat{\boldsymbol{\vartheta}}$ can be used for the estimation of $\boldsymbol{\beta}$ without any essential deterioration of the statistical properties.

However if the last inclusion is not satisfied, then the model with unknown variance components cannot be linearized and it would be necessary to prepare another experiment in order to make the estimators of $\boldsymbol{\vartheta}$ more precise. In more detail it is shown in the next section.

7 Numerical example

Two points A and B with coordinates $(0,0)$ and $(0,800)$ are located in a plane. Third point P is determined by measurement of the angles $\angle BAP$ and $\angle PBA$, respectively, and distances AP and BP. The coordinates and distances are given in meters, angles are given in sexagesimal system. Measurement are stochastically independent, the variance in measurement of angles is $\sigma_\omega^2 = (10'')^2 = (10/206264.806\,\text{rad})^2 = (4.848 \times 10^{-5}\,\text{rad})^2$ and the variance in measurement distances is $\sigma_D^2 = (0.05\,\text{m})^2$. Each angle is measured $M(=2)$-times and each distance is measured $N(=2)$-times. The approximate value of the parameter $\boldsymbol{\beta}$ is

$$\boldsymbol{\beta}^{(0)} = \begin{pmatrix} 107.180 \\ 400.000 \end{pmatrix}.$$

Thus the quadratized version of the model can be written as

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \mathbf{Y}_4 \end{pmatrix} \sim N_{2M+2N}\left[\left[\begin{pmatrix} \mathbf{f}_1^{(0)} \\ \mathbf{f}_2^{(0)} \\ \mathbf{f}_3^{(0)} \\ \mathbf{f}_4^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{pmatrix}\delta\boldsymbol{\beta} + \frac{1}{2}\begin{pmatrix} \boldsymbol{\kappa}_1(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_2(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_3(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_4(\delta\boldsymbol{\beta}) \end{pmatrix}\right], \sigma_\omega^2\mathbf{V}_1 + \sigma_D^2\mathbf{V}_2\right],$$

where

$$
\begin{aligned}
\mathbf{f}_1 &= \arctan \frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1}, \quad \mathbf{f}_2 = \arctan \frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1 - 800},\\
\mathbf{f}_3 &= \sqrt{(\boldsymbol{\beta}_1 - 800)^2 + \boldsymbol{\beta}_2^2}, \quad \mathbf{f}_4 = \sqrt{\boldsymbol{\beta}_1^2 + \boldsymbol{\beta}_2^2},\\
\mathbf{f}_1^{(0)} &= \mathbf{1}_M \alpha_{AP}^{(0)}, \quad \mathbf{f}_2^{(0)} = \mathbf{1}_M \alpha_{BP}^{(0)}, \quad \mathbf{f}_3^{(0)} = \mathbf{1}_N D_{BP}^{(0)}, \quad \mathbf{f}_4^{(0)} = \mathbf{1}_N D_{AP}^{(0)},\\
\alpha_{AP}^{(0)} &= 75^0, \quad \alpha_{BP}^{(0)} = 150^0, \quad D_{BP}^{(0)} = 800.000 \quad D_{AP}^{(0)} = 414.110,\\
\mathbf{F}_1 &= \mathbf{1}_M \times \mathbf{f}_1' = \mathbf{1}_M \otimes \left(\frac{-\sin(\alpha_{AP}^{(0)})}{D_{AP}^{(0)}}, \frac{\cos(\alpha_{AP}^{(0)})}{D_{AP}^{(0)}} \right),\\
\mathbf{F}_2 &= \mathbf{1}_M \otimes \mathbf{f}_2' = \mathbf{1}_M \otimes \left(-\frac{\sin(\alpha_{BP}^{(0)})}{D_{BP}^{(0)}}, \frac{\cos(\alpha_{BP}^{(0)})}{D_{BP}^{(0)}} \right),\\
\mathbf{F}_3 &= \mathbf{1}_N \otimes \mathbf{f}_3' = \mathbf{1}_N \otimes \left(\cos(\alpha_{BP}^{(0)}), \sin(\alpha_{BP}^{(0)}) \right),\\
\mathbf{F}_4 &= \mathbf{1}_N \otimes \mathbf{f}_4' = \mathbf{1}_N \otimes \left(\cos(\alpha_{AP}^{(0)}), \sin(\alpha_{AP}^{(0)}) \right),\\
\mathbf{f}_1' &= \left(-\frac{0.9659258}{414.110}, \frac{0.2588190}{414.110} \right), \quad \mathbf{f}_2' = \left(-\frac{0.5000000}{800.000}, -\frac{0.8660254}{800.000} \right),\\
\mathbf{f}_3' &= (-0.8660254, 0.5000000), \quad \mathbf{f}_4' = (0.2588190, 0.9659258),\\
\boldsymbol{\kappa}_1(\delta\boldsymbol{\beta}) &= \mathbf{1}_M \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{2\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2}, & \frac{(\beta_2^{(0)})^2-(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2} \\ \frac{(\beta_2^{(0)})^2-(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2}, & -\frac{2\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_2(\delta\boldsymbol{\beta}) &= \mathbf{1}_M \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{2(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2}, & \frac{(\beta_1^{(0)}-800)^2-(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2} \\ \frac{(\beta_1^{(0)}-800)^2-(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2}, & \frac{2(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_3(\delta\boldsymbol{\beta}) &= \mathbf{1}_N \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}}, & -\frac{(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}} \\ -\frac{(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}}, & \frac{(\beta_1^{(0)}-800)^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_4(\delta\boldsymbol{\beta}) &= \mathbf{1}_N \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{(\beta_2^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}}, & -\frac{\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}} \\ -\frac{\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}}, & \frac{(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}} \end{pmatrix} \delta\boldsymbol{\beta}',\\
\mathbf{V}_1 &= \begin{pmatrix} \mathbf{I}_{2M,2M}, & \mathbf{0}_{2M,2N} \\ \mathbf{0}_{2N,2M}, & \mathbf{0}_{2N,2N} \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} \mathbf{0}_{2M,2M}, & \mathbf{0}_{2M,2N} \\ \mathbf{0}_{2N,2M}, & \mathbf{I}_{2N,2N} \end{pmatrix}.
\end{aligned}
$$

Let

$$c_{i,j} = \mathbf{f}_i' \left[\frac{M(\mathbf{f}_1\mathbf{f}_1' + \mathbf{f}_2\mathbf{f}_2')}{(\sigma^{(0)})_\omega^2} + \frac{N(\mathbf{f}_3\mathbf{f}_3' + \mathbf{f}_4\mathbf{f}_4')}{(\sigma^{(0)})_D^2} \right]^{-1} \mathbf{f}_j, \quad i,j = 1,2.$$

Then

$$\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix} = \begin{pmatrix} 3.97\times 10^{17}, & 5.9\times 10^{10} \\ 5.9\times 10^{10}, & 4.98\times 10^{5} \end{pmatrix},$$

where

$$\begin{aligned} \boxed{11} &= \frac{2M}{(\sigma_\omega^{(0)})^4} - 2M\frac{c_{1,1}+c_{2,2}}{(\sigma_\omega^{(0)})^6} + M^2\frac{c_{1,1}^2+2c_{1,2}^2+c_{2,2}^2}{(\sigma_\omega^{(0)})^8}, \\ \boxed{12} &= MN\frac{c_{1,3}^2+c_{2,3}^2+c_{1,4}^2+c_{2,4}^2}{(\sigma_\omega^{(0)})^4(\sigma_D^{(0)})^4} = \boxed{2,1}, \\ \boxed{22} &= \frac{2N}{(\sigma_D^{(0)})^4} - 2N\frac{c_{3,3}+c_{4,4}}{(\sigma_D^{(0)})^6} + N^2\frac{c_{3,3}^2+2c_{3,4}^2+c_{4,4}^2}{(\sigma_D^{(0)})^8}. \end{aligned}$$

Further

$$\begin{aligned} \begin{pmatrix} \mathbf{k}'_{1,\cdot} \\ \mathbf{k}'_{1,\cdot} \end{pmatrix} &= \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}^{-1} \left[\mathrm{Diag}\left\{\left(\frac{10}{206264.806}\right)^2, 0.05^2\right\}\right]^{-1} \\ &= \begin{pmatrix} 1.09\times 10^{-9}, & -1.21\times 10^{-10} \\ -1.29\times 10^{-4}, & 8.18\times 10^{-4} \end{pmatrix}. \end{aligned}$$

Let $\varepsilon_2 = 0.05$. Then

$$\begin{aligned} \varepsilon_1 &= \sqrt{0.05}\min\left\{\sqrt{\frac{\sqrt{\left\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{1,1}}}{\sum_{j=1}^p |k_{1,j}|}}, \sqrt{\frac{\sqrt{\left\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{2,2}}}{\sum_{j=1}^p |k_{2,j}|}}\right\} \\ &= 0.522576 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_b &= \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon_1}{K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)}\right\} \\ &= \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le 21280.6\right\}, \\ \mathcal{L}_\vartheta &= \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon_1}{K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)}\right\} \\ &= \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F}\delta\boldsymbol{\beta} \le 32735.1\right\}. \end{aligned}$$

Now we can check whether $\mathcal{E}_\beta \subset \mathcal{L}_\vartheta$. At least it must be satisfied the inequality $\chi_k^2(0; 1-\alpha) = \chi_2^2(0; 0.95) = 5.99 \ll 2\varepsilon_1/K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0) = 32735.1$.

Now it is necessary to check the inclusion $\mathcal{E}_\beta \subset \mathcal{L}_b \cap \mathcal{L}_\vartheta$. If $1-\alpha = 0.95$, $\varepsilon_1 = 0.522576$, then the situation is given in Fig. 1.

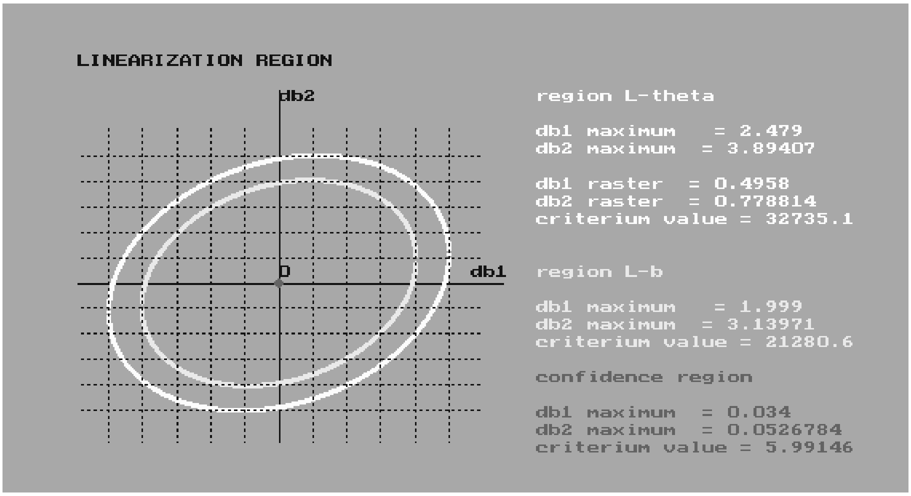

Fig. 1: Regions $\mathcal{L}_b$, $\mathcal{L}_\vartheta$ (with the same ε_1) and $\mathcal{E}_\beta$ (for $1-\alpha=0.95$).

As far as the linearization is concerned, there is no problem, since the region $\mathcal{L}_b$ and $\mathcal{L}_\vartheta$ are very large in a comparison with the confidence ellipse $\mathcal{E}_\beta$.

Now it is to be checked the inclusions $\mathcal{K}_\vartheta \subset \mathcal{N}_{h_i}$, $i=1,2$. We need the matrices

$$\mathbf{W}_{h_i} = \begin{pmatrix} \mathbf{L}'_{h_i}\mathbf{V}_1 \\ \mathbf{L}'_{h_i}\mathbf{V}_2 \end{pmatrix} [\mathbf{M}_F(\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_F]^+(\mathbf{V}_1\mathbf{L}_{h_i}, \mathbf{V}_p\mathbf{L}_{h_i}),$$

where

$$\mathbf{L}_{h_i} = \mathbf{h}'_i[\mathbf{F}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)^{-1}\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0), \quad \mathbf{h}_1 = (1,0)', \quad \mathbf{h}_2 = (0,1).$$

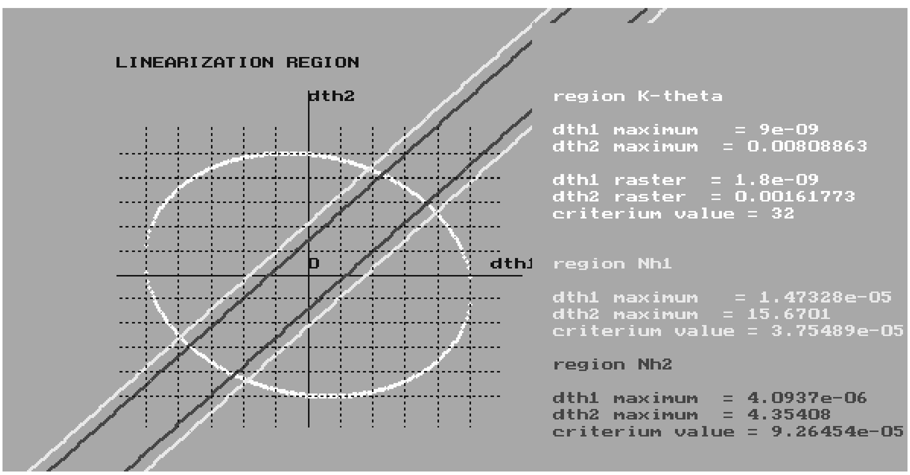

Fig. 2: Regions $\mathcal{K}_\vartheta$ for $t=4$, $\mathcal{N}_{h_1}$ and $\mathcal{N}_{h_2}$ for $\varepsilon_3 = 0.1$.

Further

$$\begin{aligned}
\mathcal{K}_{\vartheta} &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\delta\boldsymbol{\vartheta} \le 2t^2\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\delta\boldsymbol{\vartheta} \le 32\right\}, \\
\mathcal{N}_{h_1} &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_1}\delta\boldsymbol{\vartheta} \le 2\varepsilon_3\mathbf{h}_1'(\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F})^{-1}\mathbf{h}_1'\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_1}\delta\boldsymbol{\vartheta} \le 3.75489 \times 10^{-5}\right\}, \\
\mathcal{N}_{h_2} &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_2}\delta\boldsymbol{\vartheta} \le 2\varepsilon_3\mathbf{h}_2'(\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F})^{-1}\mathbf{h}_2'\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_2}\delta\boldsymbol{\vartheta} \le 9.26454 \times 10^{-5}\right\}.
\end{aligned}$$

For $\varepsilon_3 = 0.1$ and $t = 4$, see Fig. 2.

As far as the sensitiveness is concerned, the situation is more complicated. Fig. 2 shows that an accuracy of the estimators $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ based on the measurement results only is not sufficient. It is necessary to realize an additional experiment for the more accurate estimation of the parameters ϑ_1 and ϑ_2.

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