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MATHEMATICAL MODELLING OF ROCK BOLT SYSTEMS II*

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Abstract. The main goal of the paper is to describe a reinforcement consisting of fully grouted bolts, which is applied to stabilizing underground openings and tunnels. After a variational formulation is given, the existence and uniqueness is proved. Some asymptotic results that make it possible to replace the real system with a continuous one more suitable for discretization are presented. Some other types of reinforcements and properties are studied.

Keywords: boundary value problems for differential equations, variational methods, existence and uniqueness theorems, dependence of solutions of PDE on parameters, rock bolt systems

MSC 2000: 35A05, 35A15, 35B30

1. Introduction

Stabilizing of underground openings such as tunnels excavated in rock masses remains a major concern of geotechnical engineers dealing with this kind of structures. It is mainly focused on limiting the consequences of a pressure relief of the surrounding ground due to the excavation process and more specifically on maintaining the tunnel closure within an admissible value compatible with the appropriate subsequent conditions of the structure.

A new support system based on the use of metallic inclusions (bolts) seems to be a very good way how to maintain such a pressure relief within reasonable limits. In this article we will continue the mathematical modelling of rock bolt systems begun in the paper [1] where we dealt with special bolts fixed to the rock mass at their end points. Now we are going to deal with bolts which are fully grouted (Fig. 1).

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In this case after bore holes are made bolts (steel bars) are inserted and glued up with a special cement.

The bolting process described for different bolts in [1] remains the same in the case of fully grouted bolts and we refer the reader to this paper to familiarize oneself with it. After describing a variational formulation of the model with isolated bolts we will deal with the model the bolts of which are described as a "continuous" system.

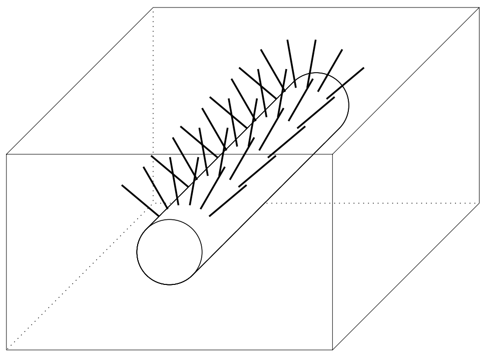

Figure 1a. The three-dimensional model of a reinforced tunnel.

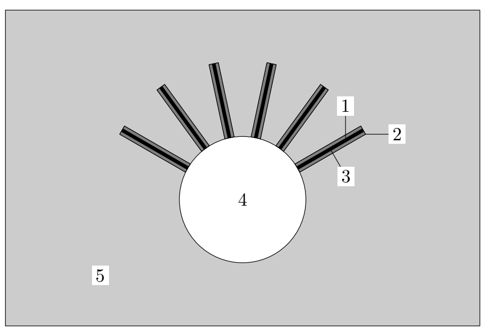

Figure 1b. 1 – special cement, 2 – bore hole, 3 – bolt, 4 – tunnel or underground opening, 5 – rock mass.

The existence and uniqueness of those problems will be established and some assertions showing the model with the "continuous" bolt system to correspond in an asymptotic way to the one with isolated bolts will be proved. Owing to the composite nature of the rock bolted mass, numerical simulation of the behaviour of such structures turns out to be a difficult task, referring for example to the finite element method, the size of the elements in the reinforced zone should be smaller than the dimensions of the bolts, which are from two to three metres long and only from two to three centimetres thick, to say nothing of the number of bolts. Thus it would lead to a numerical problem of unreasonable size even by the standards of modern computer capabilities, so a homogenization method seems to be necessary for solving it.

In the subsequent chapters this paper is concerned with other so called hybrid bolts which can be described in the following way. Every bolt is partly grouted and one end of the bolt is provided with a bearing plate, leaning against the wall of the underground opening, and a nut. By screwing the nuts the bolts can be prestressed and the stress-strain field in the reinforced zone can be influenced. A similar homogenization method making it possible to discretize these problems with a finite element technique, will be developed.

2. Variational formulation of the problem with individual bolts

Let us start with a variational formulation of the problem which is essential for the solution to the whole problem arising in the models of tunnels as it was described in Chapter 4 of the paper [1]. First of all let us introduce the assumptions which we are going to deal with in this and the subsequent chapters.

1. Linear elastic behaviour of the rock mass.
2. Linear elastic behaviour of the bolts.
3. The bolts are fully connected with the rock mass and the influence of a special cement is neglected.
4. The volume of bore holes is small in comparison with the underground opening dimensions so that it can be neglected.

Let an elastic body occupy a bounded region Ω with a Lipschitz boundary and let $x = (x_1, x_2, x_3)$ be Cartesian coordinates of the point x. Let us denote the displacement vector field by $u = (u_1, u_2, u_3)$ and the associated strain tensor field by

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3. \tag{2.1}$$

The stress tensor is related to the strain tensor by means of the generalized Hooke Law

$$\tau_{ij} = c_{ijkl}\, e_{kl}, \qquad i, j = 1, 2, 3. \tag{2.2}$$

We use the following summation convention: whereas a subscript is repeated in a term, summation is required to be taken over that subscript from 1 to 3.

Assume that c_{ijkl} are bounded and measurable functions in Ω satisfying the conditions

$$c_{ijkl} = c_{jikl} = c_{klij}. \tag{2.3}$$

Moreover, let there exist a positive constant K such that the inequality

$$c_{ijkl}(x)\,\xi_{ij}\,\xi_{kl} \geqslant K\,\xi_{ij}\,\xi_{ij} \tag{2.4}$$

holds almost everywhere in Ω for any symmetric ξ_{ij}. Let us have the following decomposition of the boundary $\partial\Omega$:

$$\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_0 \cup \mathscr{R},$$

where Γ_u, Γ_τ, Γ_0 are mutually disjoint open parts and the surface measure of $\mathscr{R}$ is zero. Let the body Ω be fixed on Γ_u:

$$u(x) = 0, \quad x \in \Gamma_u$$

and let the tractions be prescribed on Γ_τ:

$$T_i(u)(x) = \tau_{ij}(x)\nu_j(x) = P_i(x), \qquad i = 1, 2, 3, \quad x \in \Gamma_\tau, \tag{2.5}$$

where $\nu(x) = \big(\nu_1(x), \nu_2(x), \nu_3(x)\big)$ is the unit outward normal to $\partial\Omega$ at x. Define the normal and tangential components of the displacement and stress vectors by

$$\begin{aligned} &u_\nu = u_j\nu_j, (u_t)_i = u_i - u_\nu\nu_i,\\ &T_\nu = \tau_{jk}\nu_j\nu_k, \quad (T_t)_i = \tau_{ij}\nu_j - T_\nu\nu_i, \qquad i = 1, 2, 3 \end{aligned} \tag{2.6}$$

and impose the so called contact conditions on Γ_0:

$$u_\nu = 0, \qquad (T_t)_i = 0, \quad i = 1, 2, 3.$$

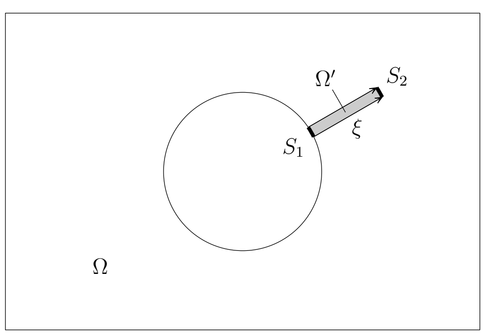

Figure 2. The cross section of a tunnel reinforced by a single bolt.

Let us consider the situation in Fig. 2 which schematically corresponds to the cross section of the body depicted in Figs. 1a–b. The bolt is described by a one-to-one transformation $\xi\colon\ S_1 \mapsto S_2$, where S_1, S_2 are Lipschitz two-dimensional surfaces. Moreover, there exist two positive constants K_1, K_2 such that the inequalities

$$K_1 \|x_1 - x_2\| \leqslant \bigl\|\xi(x_1) - \xi(x_2)\bigr\| \leqslant K_2 \|x_1 - x_2\|, \qquad x_1, x_2 \in S_1 \tag{2.7}$$

are satisfied, where $\|.\|$ is the Euclidean norm in $\mathbb{R}^3$. Let us define other two transformations $\gamma\colon\ S_1 \mapsto \mathbb{R}^3$, $\delta\colon\ \langle 0,1\rangle \times S_1 \mapsto \Omega$ by

$$\gamma(x) = \frac{\xi(x) - x}{\bigl\|\xi(x) - x\bigr\|}, \qquad \delta(t,x) = x + t\bigl(\xi(x) - x\bigr).$$

Moreover, let us assume the transformation $\delta(t,x)$ is a one-to-one transformation from $\langle 0,1\rangle \times S_1$ to Ω'. Let us extend $\gamma(x)$ from S_1 to Ω' as follows: if $y \in \Omega'$ and $y = \delta(t,x)$ then $\gamma(y) = \gamma(x)$. From the definition it is clear that the vector field $\gamma(x)$ defined in Ω' is constant on every segment between the points $x, \xi(x), x \in S_1$. We can say that Ω' corresponds to the area occupied by the bolt and the vector field $\gamma(x)$ defined in Ω' is parallel to the bolt direction. Because we deal with the small displacements the longitudinal bolt deformation (the deformation in the direction $\gamma(x)$) can be approximated by the term $\bigl\langle D_{\gamma(x)} u(x),\ \gamma(x)\bigr\rangle$, where the vector $u(x)$ is the displacement at the point x, $x \in \Omega'$, $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$ and $D_{\gamma(x)}$ is the symbol of derivation in the $\gamma(x)$ direction. In our model transversal deformations of the bolts are neglected.

Denote by

$$V = \bigl\{u \in \bigl[H^1(\Omega)\bigr]^3 \mid u = 0 \text{ on } \Gamma_u,\ u_\nu = 0 \text{ on } \Gamma_0\bigr\}$$

the space of virtual displacements and assume $F \in \bigl[L_2(\Omega)\bigr]^3$ and $P \in \bigl[L_2(\Gamma_\tau)\bigr]^3$ are the prescribed body forces and the surface loads. Then let us introduce the forms

$$\begin{aligned} A(u,v) &= \int_\Omega c_{ijkl}\, e_{ij}(u)\, e_{kl}(v)\, \mathrm{d}x, \\ a(u,v) &= \int_{\Omega'} E\bigl\langle D_{\gamma(x)} u(x),\ \gamma(x)\bigr\rangle \bigl\langle D_{\gamma(x)} v(x),\ \gamma(x)\bigr\rangle\, \mathrm{d}x, \\ L(v) &= \int_\Omega F_i\, v_i\, \mathrm{d}x + \int_{\Gamma_\tau} P_i\, v_i\, \mathrm{d}\Gamma, \end{aligned}$$

where E is Young's modulus of the bolt material. Let us notice the definitions of the form $A(\cdot,\cdot)$ and $a(\cdot,\cdot)$. The first corresponds to the deformation energy of the rock mass without the bore hole, which is neglected due to the condition discussed above. The second form corresponds to the energy of longitudinal bolt deformations while the energy of transversal bolt deformations is neglected. We can define the functional of the total potential energy by

$$\mathscr{L}(u) = \frac{1}{2}A(u,u) + \frac{1}{2}a(u,u) - L(u).$$

Definition 2.1. An element $u \in V$ is called a solution to the given bolt problem if $\mathscr{L}(u) \leqslant \mathscr{L}(v)$ for all $v \in V$.

Let us consider another subspace $R \subset \left[H^1(\Omega)\right]^3$. Define

$$R = \left\{v \in \left[H^1(\Omega)\right]^3 \mid v(x) = a + b \times x\right\},$$

where a, b are vectors from $\mathbb{R}^3$ and $\times$ is the vector product in $\mathbb{R}^3$. This subspace corresponds to the rigid-body translations and the rigid-body rotations.

Theorem 2.1. *Let $F \in \left[L_2(\Omega)\right]^3$, $P \in \left[L_2(\Gamma_\tau)\right]^3$ and $\Gamma_u \neq \emptyset$ or $\Gamma_u = \emptyset$ and $R \cap V = \{0\}$. Moreover, let c_{ijkl} and $\xi\colon S_1 \mapsto S_2$ satisfy the conditions mentioned above. There exists one and only one solution of the bolt problem and the inequality*

$$\|u\|_{[H^1(\Omega)]^3} \leqslant K\left(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3}\right) \tag{2.8}$$

is fulfilled, where K is a positive constant.

Proof. Let us assume that $a(u,u) \geqslant 0$ for every $u \in V$. Then there exists a positive constant K such that the following inequality is fulfilled:

$$K\|u\|_V^2 \leqslant a(u,u) + A(u,u) \qquad \forall u \in V.$$

This inequality is a consequence of Korn's inequality [2] and the existence of a solution can be established in the same way as in the case of the theory of elasticity [3]. The uniqueness and the validity of the inequality (2.8) are consequences of Korn's inequality, too. $\square$

Theorem 2.2. *Let $F \in \left[L_2(\Omega)\right]^3$, $P \in \left[L_2(\partial\Omega)\right]^3$, $\Gamma_u = \emptyset$, $\Gamma_0 = \emptyset$ and let the conditions of total equilibrium*

$$\int_\Omega F_i \,\mathrm{d}x + \int_{\partial\Omega} P_i \,\mathrm{d}x = 0, \quad i = 1,2,3, \tag{2.9}$$

$$\int_\Omega (x \times F)_i \,\mathrm{d}x + \int_{\partial\Omega} (x \times P)_i \,\mathrm{d}x = 0, \quad i = 1,2,3 \tag{2.10}$$

be fulfilled. Then there exists a solution u of the bolt problem and if u' is another solution to that problem then $u - u' \in R$.

P r o o f. Let Q be the orthogonal complement of R in $\left[H^1(\Omega)\right]^3$ with respect to the usual scalar product. As a consequence of Korn's inequality it is clear that the functional $\mathscr{L}(\cdot)$ is coercive on Q ([2], [3]) and because of the fact that this functional is convex, the existence of a minimum u is guaranteed [2].

In our case the existence of the minimum u of the bolt problem is equivalent to the validity of the equality

$$A(u, v) + a(u, v) = L(v), \quad \forall v \in V, \tag{2.11}$$

which is clear from some results of variational calculus. We refer the reader to [4], [6]. Now it is sufficient to check the equality (2.11) for all $v \in R$. Because of the conditions (2.9) and (2.10) we have $L(v) = 0$ for all $v \in R$ and it is enough to prove that $A(u, v) = a(u, v) = 0$ for all $v \in R$. The first equality holds because of the very well known fact that $e_{ij}(v) = 0$ if and only if $v \in R$ [2]. The second equality will be proved if the equality

$$\left\langle D_{\gamma(x)} v(x),\, \gamma(x)\right\rangle = 0 \tag{2.12}$$

is fulfilled in Ω' for all $v \in R$. It is easy to see that $v \in R$ if and only if $v = Hx + c$, where H is a 3×3 skew-symmetric matrix and c is a vector from $\mathbb{R}^3$. Then

$$\begin{aligned}\left\langle D_{\gamma(x)} v(x),\, \gamma(x)\right\rangle &= \left\langle H\gamma(x),\, \gamma(x)\right\rangle = \left\langle \gamma(x),\, H^T\gamma(x)\right\rangle \\ &= -\left\langle \gamma(x),\, H\gamma(x)\right\rangle = -\left\langle \gamma(x),\, D_{\gamma(x)} v(x)\right\rangle \\ &= -\left\langle D_{\gamma(x)} v(x),\, \gamma(x)\right\rangle,\end{aligned}$$

which results in the equality (2.12). The equalities $e_{ij}(v) = 0$ and (2.12), which hold for each $v \in R$, imply that $u + v$ is another solution of the bolt problem.

Now if u' is another solution then u, u' satisfy the equations

$$\begin{aligned}A(u, u - u') + a(u, u - u') &= L(u - u'), \\ A(u', u - u') + a(u', u - u') &= L(u - u').\end{aligned}$$

Subtracting them we obtain

$$A(u - u', u - u') + a(u - u', u - u') = 0,$$

which implies $e_{ij}(u - u') = 0$ and consequently, because of the fact mentioned above, $u - u' \in R$. $\square$

R e m a r k 2.1. It is easy to see that there exists a unique solution to the problem with respect to the subspace Q defined in the proof. For such solutions we have

$$\|u\|_V \leqslant K \left(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\partial\Omega)]^3}\right),$$

where K is a positive constant.

3. Continuous approximations of bolt systems

In the previous chapter we have dealt with the single bolt model and it is evident that this approach can be extended to several bolts. But in real cases the number of bolts, applied to reinforce the underground opening, obviously reaches a few hundred so it is very difficult to approximate such a problem by any numerical method in spite of the standards of modern computer capabilities. The way how to overcome such difficulties is to replace the system with distinct bolts by a continuous system. The main goal of this chapter is to suggest such a method and to prove that this model of bolt systems corresponds to the model with distinct bolts in an asymptotic way.

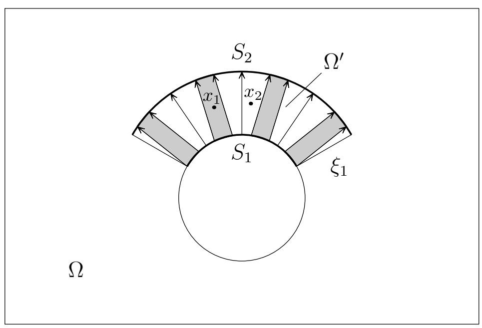

Figure 3a. $c_1(x_1) = E$, $c_1(x_2) = 0$.

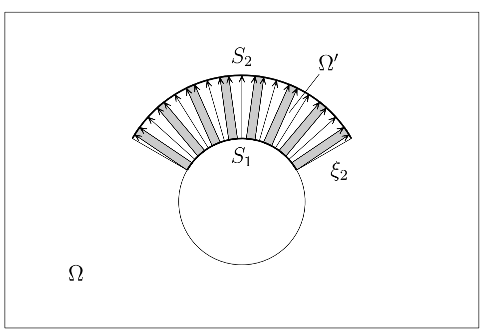

Figure 3b. The model of a cross section of a tunnel reinforced by a greater number of thinner bolts.

Let us have a look at Figs. 3a–b. There are two different sets of bolts inserted in the subarea Ω'. The situation depicted in Figs. 3a–b can be viewed as the first two steps of a more general process, in which the bolts are being spread over the subarea Ω'. In this case the formulation of the bolt problem differs from the one used in the previous chapter. The surfaces S_1, S_2 do not correspond to the surfaces describing the end points of the bolts but they are wider and remain the same for the whole process defined below. The position of the bolts is defined by the sequence of functions $c_n\colon \Omega' \mapsto \mathbb{R}$ which characterize both the position and the mechanical properties of the bolts. Let us define this process in a more precise way.

Let $\xi_n\colon S_1 \mapsto S_2$ be a sequence of one-to-one transformations satisfying the condition

$$\begin{gathered}\exists K_1, K_2 > 0,\ \forall x, y \in S_1,\ \forall n\colon \\ K_1\|x-y\| \leqslant \|\xi_n(x)-\xi_n(y)\| \leqslant K_2\|x-y\|.\end{gathered} \tag{3.1}$$

Define other two sequences $\gamma_n\colon S_1 \mapsto \mathbb{R}^3$, $\delta_n\colon \langle 0,1\rangle \times S_1 \mapsto \Omega'$ in a similar way as in Chapter 2,

$$\gamma_n(x) = \frac{\xi_n(x)-x}{\|\xi_n(x)-x\|}, \tag{3.2}$$

$$\delta_n(t,x) = x + t\left(\xi_n(x)-x\right), \tag{3.3}$$

where the functions δ_n are one-to-one transformations from $\langle 0,1\rangle \times S_1$ to Ω' and the vector field γ_n defined on S_1 can be extended to the whole subarea Ω' by

$$\gamma_n(y) = \gamma_n(x), \tag{3.4}$$

where $y = \delta_n(t,x)$. It is clear that the vector fields $\gamma_n(x)$ are constant on the intervals $\langle x, \xi_n(x)\rangle$.

Let $c_n\colon S_1 \mapsto \mathbb{R}$ be another sequence of functions, which equal E (Young's modulus of steel from which the bolts are made) at the points of S_1 where the bolt runs through and which vanish in the rest of S_1. The functions $c_n\colon S_1 \mapsto \mathbb{R}$ can be extended to the subarea Ω' by

$$c_n(y) = c_n(x), \tag{3.5}$$

where $y = \delta_n(t,x)$. We can see that these functions correspond to the characteristic functions of the sets occupied by the bolts. Now let us define the process mentioned above.

Definition 3.1. We say the sequences $\xi_n\colon S_1 \mapsto S_2$, $c_n\colon \Omega' \mapsto \mathbb{R}$ d-converge to $\xi\colon S_1 \mapsto S_2$, $c\colon \Omega' \mapsto \mathbb{R}$ if the following conditions are fulfilled:

1. The sequence ξ_n uniformly converges to ξ on S_1, which implies that even the sequence δ_n uniformly converges to δ on $\langle 0,1\rangle \times S_1$.
2. The functions c_n, c are measurable, c is non-negative and there exists $K > 0$ such that
$$\|c(x)\| < K.$$
(Due to the definition of c_n we have $\|c_n(x)\| \leqslant E$.)
3. $\forall f \in C\bigl(\overline{\Omega'}\bigr)$:
$$\int_{\Omega'} c_n \cdot f \mathrm{d}x \to \int_{\Omega'} c \cdot f \mathrm{d}x.$$

To demonstrate the substance of this convergence we present a simple example.

Example 3.1. Let us define S_1, S_2, ξ_n, ξ: $S_1 = \langle 0,1\rangle \times \langle 0,1\rangle \times \{0\}$, $S_2 = \langle 0,1\rangle \times \langle 0,1\rangle \times \{1\}$, $\xi_n = \xi$ and $\xi(x,y,0) = (x,y,1)$. Let us define $\Omega' = \langle 0,1\rangle \times \langle 0,1\rangle$ and $c_n\colon \Omega' \mapsto \mathbb{R}$ by

$$c_n(x,y,t) = \begin{cases} 1 \text{ on } \left\langle \dfrac{j+\frac14}{n}, \dfrac{j+\frac34}{n} \right\rangle \times \left\langle \dfrac{k+\frac14}{n}, \dfrac{k+\frac34}{n} \right\rangle \times \langle 0,1), \\ \qquad\qquad j = 0,\dots,n-1,\ k = 0,\dots,n-1, \\ \\ 0 \text{ on the rest of the surface } S_1. \end{cases} \tag{3.6}$$

By virtue of Definition 3.1 it is evident that ξ_n, c_n d-converge to ξ, c, where $c = \frac14$ on the whole S_1. One step of this process corresponds to Fig. 4a.

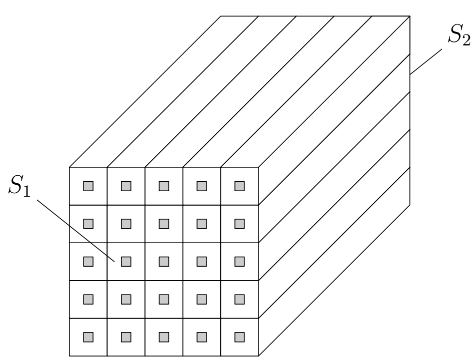

Figure 4a. A prism reinforced by bolts.

E x a m p l e 3.2. Consider a circular cylindrical tube depicted in Fig. 4b. This cylinder can be gradually divided in finer regular parts in a similar way as in Example 3.1. Then the limit function c in cylindrical coordinates will be

$$c(r,\theta,z) = K\frac{1}{r},$$
$$\xi(r_1,\theta,z) = (r_2,\theta,z),$$

where $r \in \langle r_1, r_2\rangle$, $\theta \in (0, 2\pi)$, $z \in \langle 0, l\rangle$; r_1, r_2 are the outer and inner radii, l is the length of the tube.

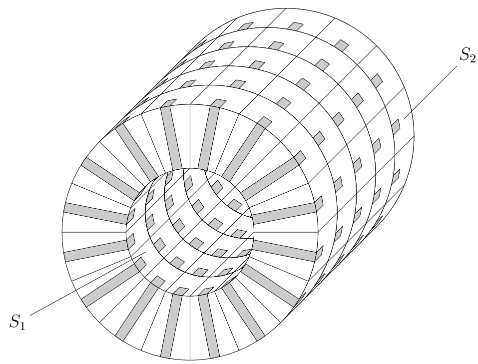

Figure 4b. A tube reinforced by bolts.

R e m a r k 3.3. The above examples encourage us to consider $c = E \cdot \varrho$, where E is Young's modulus of the steel from which the bolts are made and ϱ is the density of the steel material per unit.

In the examples mentioned above we can see that the limit functions of the convergent process are continuous and that is why it is easier to approximate them numerically. So there is a natural question whether the solutions u_n which correspond to ξ_n, c_n converge to the solution u which corresponds to ξ, c. Before formulating the main result of this chapter, let us prove an auxiliary lemma and let us introduce the forms

$$a_n(u,v) = \int_{\Omega'} c_n(x)\big\langle D_{\gamma_n(x)}u(x),\, \gamma_n(x)\big\rangle\big\langle D_{\gamma_n(x)}v(x),\, \gamma_n(x)\big\rangle\,\mathrm{d}x,$$
$$a(u,v) = \int_{\Omega'} c(x)\big\langle D_{\gamma(x)}u(x),\, \gamma(x)\big\rangle\big\langle D_{\gamma(x)}v(x),\, \gamma(x)\big\rangle\,\mathrm{d}x.$$

Lemma 3.1. *Let ξ_n, c_n d-converge to ξ, c and moreover, let $u_n \in \big[H^1(\Omega)\big]^3$ be a sequence which weakly converges to $u \in [H^1(\Omega)]^3$. Then for each $v \in \big[H^1(\Omega)\big]^3$ we* *have*

$$a_n(u_n, v) \to a(u, v). \tag{3.7}$$

P r o o f. If we consider the definitions of $\gamma_n(x)$, $c_n(x)$ and Ω' (see (3.2)–(3.5)) we can rewrite the classical Green Formula into

$$\begin{aligned}\int\limits_{\Omega'} \big(D_{\gamma_n(x)} w(x)\big) c_n(x) z(x)\,\mathrm{d}x = \int\limits_{S_1 \cup S_2} & w(x) c_n(x) z(x) \langle \gamma_{n(x)}, \nu \rangle \,\mathrm{d}\Gamma \\ - \int\limits_{\Omega'} & w(x)\, c_n(x) \big(D_{\gamma_n(x)} z(x)\big) \mathrm{d}x,\end{aligned} \tag{3.8}$$

where ν is the unit outward normal to the boundary of Ω' and this formula is valid for all $w, z \in H^1(\Omega')$. Due to the definition of $\gamma_n(x)$ the equality

$$\big\langle D_{\gamma_n(x)} u(x),\, \gamma_n(x) \big\rangle = D_{\gamma_n(x)} \big\langle u(x),\, \gamma_n(x) \big\rangle \tag{3.9}$$

is true for each $u \in \big[H^1(\Omega)\big]^3$.

We first prove the convergence assertion (3.6) for $v \in \big[C^2\big(\overline{\Omega}\big)\big]^3$. We have

$$\begin{aligned} & a_n(u_n, v) - a(u, v) \\ &\quad = \int_{\Omega'} \big\langle D_{\gamma_n(x)}(u_n - u),\, \gamma_n(x) \big\rangle\, c_n(x) \big\langle D_{\gamma_n(x)} v,\, \gamma_n(x) \big\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega'} \big\langle D_{\gamma_n(x)} u - D_{\gamma(x)} u,\, \gamma_n(x) \big\rangle\, c_n(x) \big\langle D_{\gamma_n(x)} v,\, \gamma_n(x) \big\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega'} \big\langle D_{\gamma(x)} u,\, \gamma_n(x) - \gamma(x) \big\rangle\, c_n(x) \big\langle D_{\gamma_n(x)} v,\, \gamma_n(x) \big\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega'} \big\langle D_{\gamma(x)} u,\, \gamma(x) \big\rangle\, c_n(x) \big\langle D_{\gamma_n(x)} v - D_{\gamma(x)} v,\, \gamma_n(x) \big\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega'} \big\langle D_{\gamma(x)} u,\, \gamma(x) \big\rangle\, c_n(x) \big\langle D_{\gamma(x)} v,\, \gamma_n(x) - \gamma(x) \big\rangle \,\mathrm{d}x \\ &\quad + \int_{\Omega'} \big\langle D_{\gamma(x)} u,\, \gamma(x) \big\rangle \big(c_n(x) - c(x)\big) \big\langle D_{\gamma(x)} v,\, \gamma(x) \big\rangle \,\mathrm{d}x. \end{aligned} \tag{3.10}$$

So we have begun by splitting $a_n(u_n, v) - a(u, v)$ into six integrals and now let us continue with estimating the first integral. Considering the equality (3.8), the formula (3.7), and the fact $v \in \left[C^2(\overline{\Omega})\right]^3$, we have

$$\int_{\Omega'} \left\langle D_{\gamma_n(x)}(u_n - u),\, \gamma_n(x)\right\rangle c_n(x)\left\langle D_{\gamma_n(x)}v,\, \gamma_n(x)\right\rangle \mathrm{d}x$$

$$= -\int_{\Omega'} \left\langle u_n - u,\, \gamma_n(x)\right\rangle c_n(x) D_{\gamma_n(x)}\left(\left\langle D_{\gamma_n(x)}v,\, \gamma_n(x)\right\rangle\right)\mathrm{d}x$$

$$+ \int_{S_1\cup S_2} \left\langle u_n - u,\, \gamma_n(x)\right\rangle c_n(x)\left\langle D_{\gamma_n(x)}v,\, \gamma_n(x)\right\rangle\left\langle \gamma_n(x),\, \nu\right\rangle \mathrm{d}\Gamma.$$

Due to the fact that the sequence u_n weakly converges to u in $[H^1(\Omega)]^3$, the same sequence converges to u in the norm of the space $[L_2(\Omega)]^3$ and the traces of u_n on $S_1 \cup S_2$ converge to u in the space $[L_2(S_1 \cup S_2)]^3$. This is a very well known consequence of the compact embedding theorem and the Kondrachov theorem in the Sobolev spaces [4], [5]. If we consider condition 2 from Definition 3.1 and the Cauchy-Schwarz inequality then the first integral converges to zero. Let us estimate the second integral. Consider the equality

$$\left\langle D_{\gamma_n(x)}u - D_{\gamma(x)}u,\, \gamma_n(x)\right\rangle = \frac{\partial u_i}{\partial x_j}\left(\gamma_n - \gamma\right)_j \left(\gamma_n\right)_i .$$

Due to the uniform convergence of γ_n to γ, the sequence

$$\left\langle D_{\gamma_n(x)}u - D_{\gamma(x)}u,\, \gamma_n(x)\right\rangle$$

converges to zero in $L_2(\Omega')$.

The Cauchy-Schwarz inequality together with condition 2 of Definition 3.1 guarantee the convergence from the second integral to zero. The estimate of the third integral is similar to the second and is based on the equality

$$\left\langle D_{\gamma(x)}u,\, \gamma_n(x) - \gamma(x)\right\rangle = \frac{\partial u_i}{\partial x_j}\gamma_i \left(\gamma_n - \gamma\right)_j .$$

The rest of the proof is the same as the proof of convergence of the second integral.

Due to the same reasons the fourth and the fifth integrals converge to zero, too.

Let us finish by estimating the sixth integral. The convergence of the last integral is a consequence of conditions 2 and 3 from Definition 3.1 and the fact that $C\left(\overline{\Omega'}\right)$ is dense in the space $L_2(\Omega')$ [4]. Then the Cauchy-Schwarz inequality guarantees the convergence of the integral in question.

We have proved the convergence of the sequence $a_n(u_n, v)$ to $a(u, v)$ for all $v \in \left[C^2(\overline{\Omega})\right]^3$. Let us notice that the set $\left[C^2(\overline{\Omega})\right]^3$ is dense in the space $\left[H^1(\Omega)\right]^3$, and let us consider condition 2 from Definition 3.1. Then the convergence of $a_n(u_n, v)$ to $a(u, v)$ for all $v \in \left[H^1(\Omega)\right]^3$ is a simple consequence of the Cauchy-Schwarz inequality again. $\square$

Theorem 3.1. *Let the assumptions of Theorem 2.1 be fulfilled and let ξ_n, c_n d-converge to ξ, c. Then the sequence u_n of the bolt problem solutions which corresponds to ξ_n, c_n converges to the bolt problem solution which corresponds to ξ, c in the space $\left[H^1(\Omega)\right]^3$.*

P r o o f. First let us prove that the sequence u_n of solutions to the bolt problems which correspond to ξ_n, c_n is bounded in the norm of $\left[H^1(\Omega)\right]^3$. We have the following sequence of variational equations in V defined in Chapter 2:

$$A(u_n, v) + a_n(u_n, v) = L(v), \quad \forall v \in V. \tag{3.11}$$

After replacing v with u_n and applying Korn's inequality we have

$$\begin{aligned} K\|u_n\|^2_{[H^1(\Omega)]^3} &\leqslant A(u_n, u_n) \leqslant A(u_n, u_n) + a_n(u_n, u_n) = L(u_n) \\ &\leqslant \|L\|_{V^*}\, \|u_n\|_{[H^1(\Omega)]^3}, \end{aligned}$$

where K is a positive constant independent of u_n. Now it is clear that the sequence u_n is bounded in $\left[H^1(\Omega)\right]^3$ and so there is a subsequence u_{n_k} which weakly converges to u^*. If we denote this subsequence by u_n then we can consider (3.10) as the subsequence of equations which corresponds to the subsequence u_n. Let us recall the weak convergence of u_n to u^* and Lemma 3.1. Then the equality

$$A(u^*, v) + a(u^*, v) = L(v)$$

is fulfilled, which means that u^* is a solution to the bolt problem corresponding to ξ, c. Because the problem in question has a unique solution the initial sequence u_n weakly converges to u^* which is a solution to that problem. Denote that solution by u.

It is sufficient to prove that the sequence u_n strongly converges to u. Because u_n, u are solutions to the above problems we have

$$\begin{aligned} A(u_n, u_n - u) + a_n(u_n, u_n - u) &= L(u_n - u), \\ A(u, u_n - u) + a(u, u_n - u) &= L(u_n - u). \end{aligned}$$

Subtracting these equations and adding the term $-a_n(u, u_n - u)$ to both sides of the resulting equation we have

$$A(u_n - u,\, u_n - u) + a_n(u_n - u,\, u_n - u) = a(u, u_n - u) - a_n(u, u_n - u).$$

After applying Korn's inequality we have

$$\begin{aligned} K\|u_n - u\|^2_{[H^1(\Omega)]^3} &\leqslant A(u_n - u,\, u_n - u) \\ &\leqslant A(u_n - u,\, u_n - u) + a_n(u_n - u,\, u_n - u) \\ &= a(u, u_n - u) - a_n(u,\, u_n - u), \end{aligned}$$

where K is a positive constant independent of u_n. Applying Lemma 3.1 to the term $a(u, u_n - u) - a_n(u,\, u_n - u)$ we obtain the convergence of the sequence u_n to u. $\square$

Theorem 3.2. *Let the assumptions of Theorem* 2.2 *be fulfilled, let* ξ_n, c_n *d-converge to* ξ, c *and let* $Q \subset \left[H^1(\Omega)\right]^3$ *be a subspace satisfying* $Q \cap R = \emptyset$ *where the subspace* R *corresponds to the rigid-body translations and the rigid-body rotations. Then the sequence* $u_n \in Q$ *of the bolt problem solutions which corresponds to* ξ_n, c_n *converges to the bolt problem solution* $u \in Q$ *which corresponds to* ξ, c *in the space* $\left[H^1(\Omega)\right]^3$.

Proof. We can prove this result in the same way as we did Theorem 3.1. We only have to restrict ourselves to the subspace Q where the existence and uniqueness are guaranteed as they were in Theorem 2.2. $\square$

Remark 3.4. The conditions when only the loads and body forces are prescribed are natural in many geomechanical problems.

4. Some other properties of solutions

In this chapter we are going to deal with similar problems to those we did in Chapter 6 in [1]. So far we have been dealing with the existence, uniqueness, and continuous dependence on the data (F, P). Now we are going to pay our attention to the continuous dependence on the data which characterize the bolt systems.

Theorem 4.1. *Let the assumptions of Theorem* 2.1 *be fulfilled. Let* $\xi\colon\ S_1 \mapsto S_2$, $c\colon\ \Omega' \mapsto \mathbb{R}$, $c'\colon\ \Omega' \mapsto \mathbb{R}$ *be functions, where* c, c' *characterize two different bolt systems, and let these three functions satisfy the same conditions as the corresponding functions did in Chapter* 3. *Let* u, u' *be the two solutions to the bolt problems which correspond to the data* (F, P, ξ, c) *and* (F, P, ξ, c'). *Then there exists a positive constant* K *independent of* (F, P, ξ, c, c') *such that the inequality*

$$\|u - u'\|_{[H^1(\Omega)]^2} \leqslant K\|c(x) - c'(x)\|_{L_\infty(\Omega')}\big(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3}\big)$$

holds.

P r o o f. The proof of this theorem is parallel to the one of Theorem 6.1 [1] and we refer the reader to this source. □

Let V be the subspace of $\left[H^1(\Omega)\right]^3$ defined in Chapter 2 and let V_γ be the subspace of V defined by

$$V_\gamma = \left\{u \in V \mid \left\langle D_\gamma u,\, \gamma\right\rangle = 0 \text{ on } \Omega'\right\},$$

where γ is the vector field defined by (3.2), (3.4).

Theorem 4.2. *Let all assumptions of Theorem* 4.1 *be fulfilled and let* $\xi\colon\ S_1 \mapsto S_2$, $c\colon\ \Omega' \mapsto \mathbb{R}$ *be given. Moreover, let* $c(x) \geqslant K$, *where* K *is a positive constant. Let* λ_n *be a sequence of positive numbers which converges to infinity and let* u_n *be the sequence of bolt problem solutions which correspond to* $\xi(x)$, $\lambda_n c(x)$. *Then* u_n *converges to* u, *which is a minimum of the functional*

$$\mathscr{L}_0 = \frac{1}{2}\, A(u,u) - L(u), \quad u \in V_\gamma.$$

P r o o f. The proof is similar to that of Theorem 6.2 [1] and we refer the reader to this source again. □

R e m a r k 4.1. The last theorem demonstrates that if we gradually replace the material of bolts with a harder one the solution to these bolt problems converge to the solution to the elasticity problem with constrains.

5. Model of the prestressed hybrid bolt reinforcement

In this chapter we are going to deal with bolts which are partially grouted and which are partially inserted in the surrounded rock mass without any contacts (see Fig. 5). The bolts are furnished with bearing plates and nuts, which can be prestressed by screwing. The main goal of this chapter is to give a variational formulation of the problems with these hybrid bolts together with the boundary condition which corresponds to the prestressed bolts.

First of all we formulate the assumptions which are necessary for our formulations. These assumptions are similar to those in Chapter 2, namely:

1. Linear elastic behaviour of the rock mass.
2. Linear elastic behaviour of the bolts.
3. The bolts are partially grouted into the rock mass and the influence of the cement is neglected.
4. There are no contacts between some parts of the bolts and the rock mass.

5. The volume of the bore holes is small in comparison with the dimensions of the underground opening and can be neglected.

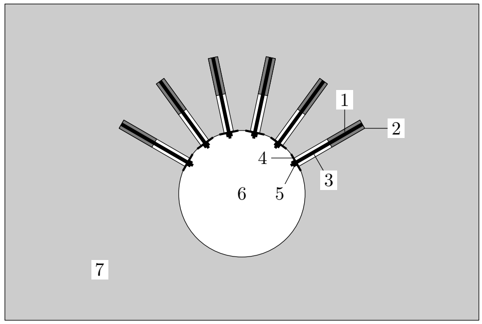

Figure 5. 1 – special cement, 2 – bore hole, 3 – bolt, 4 – bearing plate, 5 – nut, 6 – tunnel or underground opening, 7 – rock mass.

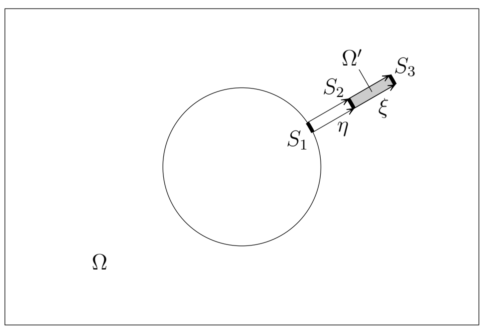

Figure 6. The cross section of a tunnel reinforced by a single bolt.

Let us formulate the problem for a single bolt and then continue with an asymptotic process which is similar to the one formulated in Chapter 3. Let us keep the notation from the previous chapter and consider the situation in Fig. 6. We have two one-to-one transformations $\eta\colon\ S_1 \mapsto S_2$, $\xi\colon\ S_2 \mapsto S_3$, where S_1, S_2 are Lipschitz two-dimensional surfaces. Moreover, there exist two positive constants K_1, K_2 such that

$$\begin{aligned} K_1\|x_1 - x_2\| &\leqslant \|\xi(x_1) - \xi(x_2)\| \leqslant K_2\|x_1 - x_2\|, \quad x_1, x_2 \in S_2, \\ K_1\|y_1 - y_2\| &\leqslant \|\eta(y_1) - \eta(y_2)\| \leqslant K_2\|y_1 - y_2\|, \quad y_1, y_2 \in S_1. \end{aligned}$$

Let us define

$$w\colon\ S_1 \mapsto \mathbb{R}^3, \quad w(x) = \frac{\eta(x) - x}{\|\eta(x) - x\|}$$

and introduce a form $b(\cdot,\cdot)$ by

$$b(u,v) = \int\limits_{S_1} d(x)\big\langle u\big(\eta(x)\big) - u(x),\, w(x)\big\rangle\big\langle v\big(\eta(x)\big) - v(x),\, w(x)\big\rangle \,\mathrm{d}\Gamma,$$

where $d(x) = E/\|\eta(x) - x\|$, E is Young's modulus of the material from which the bolts are made, the term $\|\eta(x) - x\|$ corresponds to the length of the bolt and $u, v \in V$. The space V is defined in Chapter 2. The bilinear form $b(u,v)$ corresponds to the deformation energy of the part of the bolt which is not grouted. This procedure is studied in detail in [1].

Let us define the functional of the total potential energy by

$$\mathscr{L} = \frac{1}{2}A(u,u) + \frac{1}{2}\,a(u,u) + \frac{1}{2}\,b(u,u) - L(x).$$

This functional is defined in V which is given in Chapter 2, as well as the forms $A(u,u)$, $a(u,u)$ and $L(u)$. So far we have not considered the fact that the bolt can be prestressed by screwing the nut. This can be mathematically formulated as follows: There are connections between the points which correspond to the points on the wall of the underground opening, and the points which correspond to the points on the bearing plate after the bolt was prestressed.

Due to the fact that we deal with the theory of small deformations, the stretching of the ungrouted part of the bolt is given by the term

$$\big\langle u\big(\eta(x)\big) - u(x),\, w(x)\big\rangle, \quad x \in S_1,$$

where $u(x)$ is the displacement vector in the rock mass. In the case the bolt is additionally prestressed the term

$$\big\langle u\big(\eta(x)\big) - u(x),\, w(x)\big\rangle + l(x), \quad x \in S_1,$$

represents the stretching of the same part of the bolt where the function $l(x)$ corresponds to the length resulting by screwing the nut.

Introduce another bilinear form by

$$\begin{aligned} b^p(u,v) = \int\limits_{S_1} & d(x)\big(\big\langle u(\eta(x) - u(x),\, w(x)\big\rangle + l(x)\big) \\ & \times \big(\big\langle v(\eta(x)) - v(x),\, w(x)\big\rangle + l(x)\big)\,\mathrm{d}\Gamma, \end{aligned}$$

which corresponds to the deformation energy of the above mentioned part of the bolt after this bolt has been prestressed. Then we have the functional of the whole potential energy

$$\mathscr{L}^p(u) = \frac{1}{2}A(u,u) + \frac{1}{2}\,a(u,u) + \frac{1}{2}\,b^p(u,u) - L(u),$$

where $u \in V$.

Definition 5.1. An element $u \in V$ will be called a solution to the prestressed bolt problem if

$$\mathscr{L}^p(u) \leqslant \mathscr{L}^p(v) \quad \text{for all } v \in V.$$

It is useful to replace the functional $\mathscr{L}^p(u)$ by another one denoted by the same symbol

$$\mathscr{L}^p(u) = \frac{1}{2}A(u,u) + \frac{1}{2}\,a(u,u) + \frac{1}{2}\,b(u,u) - L(u) + L^p(u),$$

where $L^p(u)$ is the linear form

$$L^p(u) = \int\limits_{S_1} h(x)\big\langle u\big(\eta(x)\big) - u(x),\, w(x)\big\rangle\,\mathrm{d}\Gamma,$$

where $h(x) = d(x)l(x)$. It is easy to see that the solution to our initial problem is equivalent to the minimum of the functional $\mathscr{L}^p$ in V.

Theorem 5.1. *Let the assumptions of Theorem* 2.1 *be fulfilled together with the assumptions mentioned above. Moreover, let* $d \in L_\infty(S_1)$, $d(x) \geqslant 0$, *and* $h(x) \in L_2(S_1)$. *Then there exists a unique solution* $u \in V$ *such that the inequality*

$$\|u\|_{[H^1(\Omega))]^3} \leqslant K\big(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3} + \|h\|_{L_2(S_1)}\big)$$

is fulfilled, where K *is a positive constant independent of* u.

Proof. This theorem can be proved in the same way as Theorem 2.1. We leave it to the reader. □

Let us consider an asymptotic process similar to that studied in Chapter 3. We have the sequences $\eta_n\colon S_1 \mapsto S_2$, $\xi_n\colon S_2 \mapsto S_3$, $d_n\colon S_1 \mapsto \mathbb{R}$, $c_n\colon \Omega' \mapsto \mathbb{R}$, such that η_n, d_n b-converge to $\eta\colon S_1 \mapsto S_2$, $d\colon S_1 \mapsto \mathbb{R}$ and η_n, c_n d-converge to $\eta\colon S_2 \mapsto S_3$, $c\colon \Omega' \mapsto \mathbb{R}$. The first step of this process is depicted in Fig. 7. The situation is similar to that in Chapter 3. In this case the surfaces S_1, S_2, S_3 are not fully occupied by bolts as they were above in this chapter but they are wider and remain the same during the whole process. The position and material properties in the subarea occupied by bolts are described by the functions $d_n\colon S_1 \mapsto \mathbb{R}$ and $c_n\colon \Omega' \mapsto \mathbb{R}$,

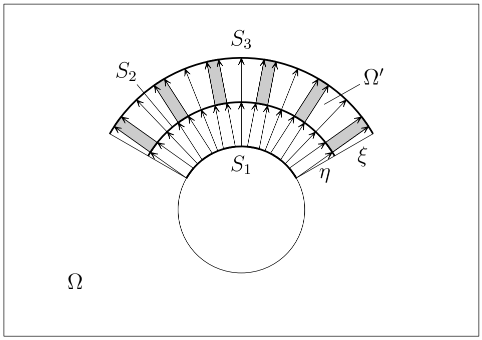

Figure 7. The cross section of a tunnel reinforced by hybrid bolts.

respectively. The b-convergence was defined in [1] and we recall this definition for the reader's convenience.

Definition 5.2. We say that $\eta_n\colon S_1 \mapsto S_2$, $d_n\colon S_1 \mapsto \mathbb{R}$ b-converge to $\eta\colon S_1 \mapsto S_2$, $d\colon S_1 \mapsto \mathbb{R}$ if the following conditions are fulfilled:

1. $\exists K_1, K_2 > 0$, $\forall n$, $\forall x, y \in S_1$:
$$K_1\|x-y\| \leqslant \|\eta_n(x)-\eta_n(y)\| \leqslant K_2\|x-y\|;$$
2. η_n uniformly converges to η on S_1;
3. $\exists K_3 > 0$, $\forall n$: $\|d_n\|_{L_\infty(S_1)} < K_3$, $\|d\|_{L_\infty(S_1)} < K_3$. $L_\infty(S_1)$ is the space of measurable functions on S_1 with the essential norm;
4. $\forall f \in C(S_1)$ (the space of continuous functions on S_1),
$$\int_{S_1} d_n f \,\mathrm{d}\Gamma \to \int_{S_1} d\, f \,\mathrm{d}\Gamma.$$

Introduce a sequence of bilinear forms
$$b_n(u,v) = \int_{S_1} d_n(x)\bigl\langle u\bigl(\eta_n(x)\bigr) - u(x),\, w_n(x)\bigr\rangle\bigl\langle v\bigl(\eta_n(x)\bigr) - v(x),\, w_n(x)\bigr\rangle \,\mathrm{d}\Gamma,$$
where $w_n(x) = \dfrac{\eta_n(x)-x}{\|\eta_n(x)-x\|}$.

Lemma 5.1. *Let η_n, d_n b-converge to η, d and moreover, let $u_n \in \bigl[H^1(\Omega)\bigr]^3$ be a sequence which weakly converges to $u \in \bigl[H^1(\Omega)\bigr]^3$. Then for each $v \in \bigl[H^1(\Omega)\bigr]^3$, we have*
$$b_n(u_n,v) \to b(u,v).$$

P r o o f. This lemma is a simple consequence of Lemma 3.1 in [1], which is a bit stronger assertion than this one. $\square$

Let us consider another sequence of functions $h_n\colon S_1 \mapsto \mathbb{R}$, which corresponds to the prestressed bolts in the asymptotic process. Assume that $h_n \in L_2(S_1)$ and introduce a sequence of forms by

$$L_n^p(u) = \int\limits_{S_1} h_n(x)\big\langle u\big(\eta_n(x)\big) - u(x),\, w_n(x)\big\rangle\,\mathrm{d}\Gamma,$$

where $u \in \big[H^1(\Omega)\big]^3$.

Definition 5.3. We say that $\eta_n\colon S_1 \mapsto S_2$ and $h_n\colon S_1 \mapsto \mathbb{R}$ *a*-converge to $\eta\colon S_1 \mapsto S_2$ and $h\colon S_1 \mapsto \mathbb{R}$, respectively, if the following conditions are fulfilled:

1. $\exists K_1, K_2 > 0$, $\forall n$, $\forall x, y \in S_1$:

$$K_1\|x-y\| \leqslant \|\eta_n(x) - \eta_n(y)\| \leqslant K_2\|x-y\|;$$

2. the sequence η_n uniformly converges to η on S_1;
3. the sequence h_n weakly converges to h in $L_2(S_1)$.

Lemma 5.2. *Let η_n, h_n a-converge to η, h and moreover, let v_n be a sequence in $\big[H^1(\Omega)\big]^3$ which weakly converges to $v \in \big[H^1(\Omega)\big]^3$. Then*

$$L_n^p(v_n) \to L^p(v).$$

Proof. Due to the first condition of Definition 5.3 there exist two positive constants K, K' independent of n such that $\forall u \in L_2(S_1)$ we have

$$K\big\|u(x)\big\|_{L_2(S_2)} \leqslant \big\|u\big(\eta_n(x)\big)\big\|_{L_2(S_1)} \leqslant K'\big\|u(x)\big\|_{L_2(S_2)}. \tag{5.1}$$

Taking into account that the trace operator from $\big[H^1(\Omega)\big]^3$ to $\big[L_2(S_1 \cup S_2)\big]^3$ is compact [5], we find that v_n strongly converges to v in $\big[L_2(S_1 \cup S_2)\big]^3$. This fact together with (5.1) implies

$$\big\|v_n\big(\eta_n(x)\big) - v\big(\eta_n(x)\big) - v_n(x) + v(x)\big\|_{[L_2(S_1)]^3} \to 0. \tag{5.2}$$

Moreover, the space $[C(S_1 \cup S_2)]^3$ is dense in $\big[L_2(S_1 \cup S_2)\big]^3$, which together with (5.1) and (5.2) results in the inequalities $\forall \varepsilon > 0 \quad \exists u \in [C(S_1 \cup S_2)]^3$, $\exists n_0$, $\forall n > n_0$:

$$\begin{aligned} &\big\|v_n\big(\eta_n(x)\big) - u\big(\eta_n(x)\big) - v_n(x) + u(x)\big\|_{[L_2(S_1)]^3} < \varepsilon, \\ &\big\|v\big(\eta(x)\big) - u\big(\eta(x)\big) - v(x) + u(x)\big\|_{[L_2(S_1)]^3} < \varepsilon. \end{aligned} \tag{5.3}$$

Then

$$
\begin{aligned}
(5.4)\qquad & L_n^p(v_n) - L^p(v) \\
&= \int_{S_1} h_n(x)\big\langle v_n\big(\eta_n(x)\big) - u\big(\eta_n(x)\big) - v_n(x) + u(x),\, w_n(x)\big\rangle \,\mathrm{d}\Gamma \\
&\quad - \int_{S_1} h(x)\,\big\langle v\big(\eta(x)\big) - u\big(\eta(x)\big) - v(x) + u(x),\, w(x)\big\rangle \,\mathrm{d}\Gamma \\
&\quad + \int_{S_1} h_n(x)\big\langle u\big(\eta_n(x)\big) - u\big(\eta(x)\big) - u(x) + u(x),\, w_n(x)\big\rangle \,\mathrm{d}\Gamma \\
&\quad + \int_{S_1} h_n(x)\big\langle u\big(\eta(x)\big) - u(x),\, w_n(x) - w(x)\big\rangle \,\mathrm{d}\Gamma \\
&\quad + \int_{S_1} \big(h_n(x) - h(x)\big)\big\langle u\big(\eta(x)\big) - u(x),\, w(x)\big\rangle \,\mathrm{d}\Gamma.
\end{aligned}
$$

Due to the Cauchy-Schwarz inequality, the two estimates (5.3) and the fact that w_n, h_n are bounded in $[C(S_1)]^3$, $L_2(S_1)$, both the first and the second integrals are sufficiently small for all sufficiently large n. Due to the Cauchy-Schwarz inequality and the fact h_n is bounded in $L_2(S_1)$ and η_n, w_n uniformly converge to η, w, both the third and the forth integrals converge to zero. The convergence to zero of the last integral is a consequence of the weak convergence of h_n to h in $L_2(S_1)$. □

Theorem 5.2. *Let the assumptions of Theorem* 2.1 *be fulfilled, let η_n, d_n b-converge to η, d, let ξ_n, c_n d-converge to ξ, c and let η_n, h_n a-converge to η, h. Then the sequence u_n of solutions to the prestressed bolt problems which correspond to η_n, ξ_n, d_n, c_n, h_n converges to the solution of the prestressed bolt problem which corresponds to η, ξ, d, c, h in the space $\big[H^1(\Omega)\big]^3$.*

Proof. First let us prove that the sequence u_n of the solutions is bounded in the norm of $\big[H^1(\Omega)\big]^3$. We have the sequence of variational equalities

$$
A(u_n, v) + a_n(u_n, v) + b_n(u_n, v) = L(v) - L_n^p(v) \tag{5.5}
$$

which hold for all $v \in V$. After replacing v with u_n and applying Korn's inequality we have

$$
\begin{aligned}
K\|u_n\|^2_{[H^1(\Omega)]^3} \leqslant A(u_n, u_n) &\leqslant A(u_n, u_n) + a_n(u_n, u_n) + b_n(u_n, u_n) \\
&= L(u_n) - L_n^p(u_n),
\end{aligned} \tag{5.6}
$$

where K is a positive constant independent of u_n. Because of the definitions of $L(\cdot)$ and $L_n^p(\cdot)$ and the second condition of Definition 5.3 it is evident that there exists a positive constant K_1 independent of u_n such that

$$L(u_n) - L_n^p(u_n) \leqslant K_1 \|u_n\|_{[H^1(\Omega)]^3}.$$

The inequality (5.6) together with this inequality yield the boundedness of u_n. So there is a subsequence u_{n_k} which weakly converges to u^*. Denote this subsequence by u_n again. If we consider the weak convergence of u_n to u^*, Lemma 3.1, Lemma 5.1, Lemma 5.2 and the sequence of equalities (5.5), then the equality

$$A(u^*, v) + a(u^*, v) + b(u^*, v) = L(v) - L^p(v)$$

is fulfilled and u^* is the solution to our problem which corresponds to η, ξ, d, c, h. Because the problem in question has a unique solution the initial sequence u_n weakly converges to u^* which is the solution to the problem and we can denote it by u. It is sufficient to prove that the sequence u_n strongly converges to u.

Because u_n, u are solutions to the above problems we have

$$\begin{aligned} A(u_n, u_n - u) + a_n(u_n, u_n - u) + b_n(u_n, u_n - u) &= L(u_n - u) - L_n^p(u_n - u), \\ A(u,\, u_n - u) + a\,(\,u,\, u_n - u) + b(\,u,\, u_n - u) &= L(u_n - u) + L^p(u_n - u). \end{aligned}$$

Subtracting these equations and adding the term

$$-\big(a_n(u,\, u_n - u) + b_n(u,\, u_n - u)\big)$$

to both sides of the new equation we obtain

$$\begin{aligned} A(u_n - u,\, u_n - u) &+ a_n(u_n - u,\, u_n - u) + b_n(u_n - u,\, u_n - u) \\ = a(u,\, u_n - u) &- a_n(u,\, u_n - u) + b(u,\, u_n - u) - b_n(u,\, u_n - u) \\ &\qquad - L_n^p(u_n - u) + L^p(u_n - u). \end{aligned} \tag{5.7}$$

Applying Korn's inequality we have

$$\begin{aligned} K\,\|u_n - u\|^2_{[H^1(\Omega)]^3} \leqslant A(u_n - u,\, u_n - u) &\leqslant A(u_n - u,\, u_n - u) \\ + a_n(u_n - u,\, u_n - u) &+ b_n(u_n - u,\, u_n - u), \end{aligned}$$

where K is a positive constant independent of a_n. Combining this inequality with (5.7) we conclude

$$\begin{aligned} K\,\|u_n - u\|^2 \leqslant a(u,\, u_n - u) &- a_n(u,\, u_n - u) + b(u,\, u_n - u) \\ - b_n(u,\, u_n - u) &- L_n^p(u_n - u) + L^p(u_n - u), \end{aligned}$$

which together with Lemma 3.1, Lemma 5.1, Lemma 5.2 yields that u_n strongly converges to u. □

R e m a r k 5.1. Let us consider the process of the convergence in Definition 5.3. If we assume that we screwed all the bolts by the same value l we can understand the resulting function h as follows: $h(x) = Epl\varrho(x)/\|\eta(x) - x\|$, where E is Young's modulus, p is the area of the cross section of a bolt, $\varrho(x)$ is the "bolt density" on S_1, and $\|\eta(x) - x\|$ is the length of the bolt at the point x.

6. Another variational problem

In this chapter we will deal with bolts which are partially grouted and partially inserted in the surrounding rock mass without any contacts with it as we did in the previous chapter. There is a stiff plate fixed to the rock mass by the set of prestressed bolts. Moreover, there are other bolts, which are not connected to the stiff plate and which are furnished with bearing plates and nuts and which are prestressed, too. The whole situation is depicted in Figs. 8a–b.

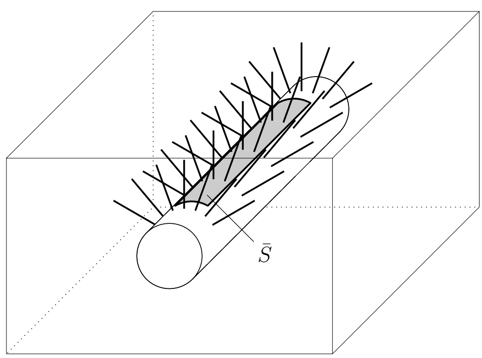

Figure 8a. The model of a tunnel reinforced by a stiff plate and bolts.

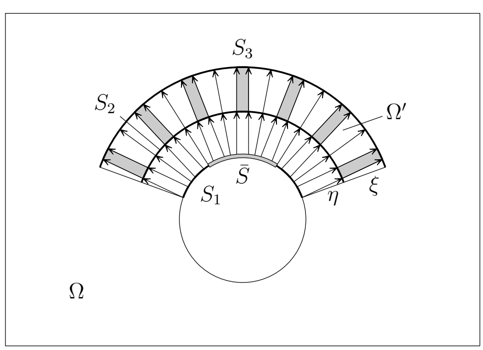

Figure 8b. The cross section of a tunnel reinforced by a stiff plate and bolts.

The main goal of this chapter is to give a variational formulation of the bolt problem including the presence of the stiff plate and the prestressed bolts. Let the conditions formulated at the beginning of Chapter 5 remain fulfilled.

Introduce a subspace $V_0 \subset V$ by

$$V_0 = \{u \in V \mid u(x) = v(x) \quad \text{on } \overline{S} \ \wedge \ v \in R\},$$

where $\overline{S}$ is the subsurface of the surface S_1 which corresponds to the stiff plate. The subspace R is the space introduced in Chapter 2 and corresponds to the rigid-body translations and the rigid-body rotations, which harmonizes with the fact that this part of S_1 is fully connected with the stiff plate. Let us preserve all the notation from Chapter 5 and let us define the functional of the total potential energy by

$$\mathscr{L}^p(u) = \frac{1}{2}A(u,u) + \frac{1}{2}\,a(u,u) + \frac{1}{2}\,b(u,u) - L(u) + L^p(u),$$

which is the same functional as the one in Chapter 5 with the only exception that this functional is defined in the subspace V_0.

Definition 6.1. An element $u \in V_0$ will be called the solution of the modified prestressed bolt problem if

$$\mathscr{L}^p(u) \leqslant \mathscr{L}^p(v) \quad \text{for all } v \in V_0.$$

Let us formulate the following two theorems.

Theorem 6.1. *Let the assumptions of Theorem* 5.1 *be fulfilled. Then there exists a unique solution to the modified prestressed bolt problem and the inequality*

$$\|u\|_{[H^1(\Omega)]^3} \leqslant K\big(\|F\|_{[L_2(\Omega)]^3} + \|P\|_{[L_2(\Gamma_\tau)]^3} + \|h\|_{L_2(S_1)}\big)$$

holds, where K is a positive constant independent of u.

Theorem 6.2. *Let the assumptions of Theorem* 5.2 *be fulfilled and moreover, let η_n, ξ_n, d_n, c_n, h_n, η, ξ, d, c, h be defined in the same way as in Chapter* 5. *Then the sequence u_n of solutions to the modified prestressed bolt problems corresponding to η_n, ξ_n, d_n, c_n, h_n converges to the solution u to the modified prestressed bolt problem corresponding to η, ξ, d, c, h in the space $\big[H^1(\Omega)\big]^3$.*

The proofs of these two theorems are completely parallel to the proofs of Theorems 5.1 and 5.2. We only have to restrict ourselves to the subspace V_0 when giving the proofs. Let us formulate the problem in a bit different way which is more suitable for numerical approximations. Let us define

$$\overline{V} = \{v \in V_0 \mid v(x) = 0 \text{ on } \overline{S}\},$$
$$K_i = \{v \in V_0 \mid v(x) = h_i(x) \text{ on } \overline{S}\},$$

where $h_i\colon \Omega \mapsto \mathbb{R}^3$, $i = 1, \ldots, 6$ form a basis of the 6-dimensional space R which corresponds to the rigid-body translations and the rigid-body rotations. Moreover, let $u_1, \ldots, u_6$ be solutions to the problems

$$A(u_i, v) + a(u_i, v) + b(u_i, v) = 0, \quad \forall v \in \overline{V},\ u_i \in K_i,\ i = 1, \ldots, 6. \tag{6.1}$$

It is evident that these problems have unique solutions and using the Trace Theorem, we can show that $u_1, \ldots, u_6$ are linearly independent. Moreover, the space V_0 can be decomposed into $\overline{V} \oplus D$, where $D = \operatorname{span}\{u_1, \ldots, u_6\}$. Let u_0 be the solution to the variational problem

$$A(u_0, v) + a(u_0, v) + b(u_0, v) - L(v) + L^p(v) = 0, \quad \forall v \in \overline{V},\ u_0 \in \overline{V}. \tag{6.2}$$

Then the solution to the initial problem can be given as $u = u_0 + \sum_{i=1}^{6} c_i u_i$, where $c = (c_1, \ldots, c_6)^T$ is the solution to the system of linear equations

$$Hx = b, \quad H = (H_{ij}),\ b = (b_i),\ i, j = 1, \ldots, 6, \tag{6.3}$$

where

$$\begin{aligned} H_{ij} = {}& A(u_j, u_i) + a(u_j, u_i) + b(u_j, u_i), \\ b_i = {}& -A(u_0, u_i) - a(u_0, u_i) - b(u_0, u_i) + L(u_i) - L^p(u_i). \end{aligned}$$

Due to Korn's inequality it is evident that the matrix H is positive definite, so there is a unique solution to the problem (6.3). If we insert such u in the variational equality

$$A(u, v) + a(u, v) + b(u, v) = L(v) - L^p(v),$$

then due to (6.2), (6.1) it is evident that this equation holds for all $v \in \overline{V}$ and due to (6.3) it holds for every $v \in \operatorname{span}\{u_1, \ldots, u_6\}$. Consequently, u is the solution to our initial problem.

Remark 6.1. We have formulated the modified prestressed bolt problem in the case that there is only one stiff plate. We can formulate such a problem for more stiff plates. In this case the dimension of the matrix H (6.3) will be $3l$, where l corresponds to the number of separate stiff plates.

7. Conclusions

We have formulated some variational problems arising in geomechanics. Moreover, we have developed the asymptotic technique which makes it possible to approximate numerically some problems described above.

The numerical codes based on the models of rock bolt systems described in this paper have been developed and inserted in GEM22, which is the numerical code developed in the Institute of Geonics and applied to solving geomechanical problems.

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