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EXPONENTIAL STABILITY AND EXPONENTIAL INSTABILITY FOR LINEAR SKEW-PRODUCT FLOWS

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Abstract. We give characterizations for uniform exponential stability and uniform exponential instability of linear skew-product flows in terms of Banach sequence spaces and Banach function spaces, respectively. We present a unified approach for uniform exponential stability and uniform exponential instability of linear skew-product flows, extending some stability theorems due to Neerven, Datko, Zabczyk and Rolewicz.

Keywords: linear skew-product flow, uniform exponential stability, uniform exponential instability

MSC 2000: 34D05, 34E05

1. Introduction

In recent years, important progress has been made in the theory of evolution equations with unbounded coefficients in infinite dimensional spaces. Significant questions have been answered using the theory of linear skew-product flows (see [1], [2], [3], [6], [7], [10], [11], [13], [15], [16], [20], [21], [23], [24]). New concepts of exponential dichotomy have been introduced and studied (see [1], [2], [3], [6], [7], [13], [15], [20], [24]). In [24], Sacker and Sell proved that exponential dichotomy of a weakly hyperbolic linear skew-product semiflow can be obtained by imposing the condition of finite dimension for the unstable manifold. Giving up this last condition for the case of skew-product sequences, Chow and Leiva presented in [2] necessary and sufficient conditions for discrete dichotomy. The authors generalized the concept of discrete dichotomy introduced by Henry in [5] and thus, they gave characterizations for exponential dichotomy of linear skew-product semiflows. The case of linear skew-product flows has been also considered by Latushkin, Montgomery-Smith and Randolph in [6] and by Latushkin and Schnaubelt in [7]. In [7], dichotomy has been discretely characterized in terms of the hyperbolicity of a family of weighted shift operators defined on $c_0(\mathbb{Z}, X)$. In the spirit of Henry's theory, in [13] exponential dichotomy is expressed using the uniform admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ for a linear skew-product semiflow on $X \times \Theta$.

The concept of uniform exponential stability for linear skew-product semiflows has been studied in [10] and [16]. In [16], we have obtained stability theorems of Perron type, which connect the uniform exponential stability of a linear skew-product semiflow with $(l^p(\mathbb{N}, X),\ l^q(\mathbb{N}, X))$-stability and $(c_0(\mathbb{N}, X), l^\infty(\mathbb{N},\ X))$-stability. The controllability of systems associated with linear skew-product semiflows has been studied in [11].

One of the most remarkable results in the theory of stability of evolution families has been presented by Rolewicz in [22]:

Theorem 1.1. *Let $\varphi\colon\ \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}$ be a function such that for every $t > 0$, $s \to \varphi(t, s)$ is a continuous non-decreasing function with $\varphi(t, 0) = 0$, $\varphi(t, s) > 0$ for all $s > 0$, and for every $s \geqslant 0$, $t \to \varphi(t, s)$ is non-decreasing. If $\mathcal{U} = \{U(t, s)\}_{t\geqslant s\geqslant 0}$ is a strongly continuous evolution family on a Banach space X such that for every $x \in X$ there is $\alpha(x) > 0$ with*

$$\sup_{s\geqslant 0} \int_s^\infty \varphi(\alpha(x), \|U(t, s)x\|)\, \mathrm{d}t < \infty$$

then $\mathcal{U}$ is uniformly exponentially stable.

Giving up the continuity of the function φ, which was essentially used in the original proof of Theorem 1.1, Neerven gave in [18] a similar characterization for uniform exponential stability of C_0-semigroups, as follows:

Theorem 1.2. *A C_0-semigroup $\mathbb{T} = \{T(t)\}_{t\geqslant 0}$ on a Banach space X is uniformly exponentially stable if and only if there is a non-decreasing function $\varphi\colon\ \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, such that*

$$\int_0^\infty \varphi(\|T(t)x\|)\, \mathrm{d}t < \infty, \quad \forall x \in X.$$

In fact, this theorem is a particular case of another result presented by Neerven in [18], which expresses the uniform exponential stability of C_0-semigroups in terms of Banach function spaces:

Theorem 1.3. *A C_0-semigroup $\mathbb{T} = \{T(t)\}_{t\geqslant 0}$ on the Banach space X is uniformly exponentially stable if and only if there exists a Banach function space B with the property $\lim\limits_{t\to\infty} F_B(t) = \infty$ such that for every $x \in X$ the mapping $t \mapsto \|T(t)x\|$ belongs to B.*

In the above result, F_B denotes the fundamental function of the Banach function space B (for details see its definition in Section 2.2, after Remark 2.1). Theorem 1.3 has been generalized in [10] for the case of linear skew-product semiflows. There, the uniform exponential stability of linear skew-product semiflows has been characterized using Banach sequence spaces and Banach function spaces. Characterizations for exponential stability and exponential instability of C_0-semigroups in terms of Banach function spaces have been presented in [9].

The purpose of this paper is to present a unified approach to uniform exponential stability and uniform exponential instability of linear skew-product flows. We will continue the study begun in [9] and in [10], and thus we will give other generalizations for Neerven's theorem for the general case of stability and instability of linear skew-product flows. We will characterize the uniform exponential stability and the uniform exponential instability, using Banach sequence spaces and Banach function spaces, respectively, which are different compared to the spaces considered in [10]. Thus, we will obtain some versions of the theorems due to Rolewicz (see [22]) and Zabczyk (see [25]) for the case of linear skew-product flows. All the stability theorems presented here will be extended to exponential instability. In this manner, we will give new versions of the theorems due to Neerven, Rolewicz and Zabczyk for the case of uniform exponential instability of linear skew-product flows.

2. Preliminaries

2.1. Linear skew-product flows. Let X be a Banach space, let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$.

Definition 2.1. A continuous mapping $\sigma\colon \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ, if it has the following properties:

(i) $\sigma(\theta, 0) = \theta$ for all $\theta \in \Theta$;

(ii) $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t)$ for all $(\theta, s, t) \in \Theta \times \mathbb{R}^2$.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product flow* on $\mathcal{E} = X \times \Theta$ if σ is a flow on Θ and $\Phi\colon \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$ for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$ (*the cocycle identity*);

(iii) there are $M \geqslant 1$ and $\omega > 0$ such that $\|\Phi(\theta, t)\| \leqslant M\mathrm{e}^{\omega t}$ for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

If, in addition, for every $(x,\theta)\in\mathcal{E}$, the mapping $t\to\Phi(\theta,t)x$ is continuous, then π is called a *strongly continuous linear skew-product flow.*

If $\pi=(\Phi,\sigma)$ is a linear skew-product flow, then the mapping Φ is called the *cocycle* associated with the linear skew-product flow π.

Definition 2.3. Let $\pi=(\Phi,\sigma)$ be a linear skew-product flow on $\mathcal{E}=X\times\Theta$. The cocycle Φ is said to be *injective* if for every $(\theta,t)\in\Theta\times\mathbb{R}_+$, the operator $\Phi(\theta,t)$ is injective.

Example 2.1. Let Θ be a metric space, let σ be a flow on Θ and let $\mathbb{T}=\{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on X. Then the pair $\pi_T=(\Phi_T,\sigma)$, where $\Phi_T(\theta,t)=T(t)$ for all $(\theta,t)\in\Theta\times\mathbb{R}_+$, is a strongly continuous linear skew-product flow on $\mathcal{E}=X\times\Theta$, which is called *the linear skew-product flow generated by the C_0-semigroup $\mathbb{T}$ and the flow σ.*

Example 2.2. Let $\Theta=\mathbb{R}$, $\sigma(\theta,t)=\theta+t$ and let $\mathcal{U}=\{U(t,s)\}_{t\geqslant s}$ be an evolution family on a Banach space X. We define $\Phi(\theta,t)=U(t+\theta,\theta)$ for all $(\theta,t)\in\Theta\times\mathbb{R}_+$. Then $\pi=(\Phi,\sigma)$ is a linear skew-product flow on $\mathcal{E}=X\times\Theta$ called *the linear skew-product flow generated by the evolution family $\mathcal{U}$.*

Classical examples of cocycles appear as operator solutions for variational equations. Let σ be a flow on a locally compact metric space Θ and let $\{A(\theta)\}_{\theta\in\Theta}$ be a family of densely defined closed operators on a Banach space X. A strongly continuous cocycle Φ is said to solve the variational equation

$$\text{(A)}\qquad \dot{x}=A(\sigma(\theta,t))x,\quad \theta\in\Theta,\ t\in\mathbb{R}_+$$

if for every $\theta\in\Theta$ there exists a dense subset $D_\theta\subset D(A(\theta))$ such that for every $x_\theta\in D_\theta$ the function $t\mapsto x(t):=\Phi(\theta,t)x_\theta$ is differentiable for $t\geqslant 0$, $x(t)\in D(A(\sigma(\theta,t)))$ for every $t\geqslant 0$ and $t\mapsto x(t)$ satisfies the differential equation (A).

Example 2.3. Let $C(\mathbb{R},\mathbb{R})$ be the space of all continuous functions $f\colon\ \mathbb{R}\to\mathbb{R}$, which is metrizable by the metric

$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{d_n(f,g)}{1+d_n(f,g)}$$

where $d_n(f,g)=\sup\limits_{t\in[-n,n]}|f(t)-g(t)|$. Let $a\colon\ \mathbb{R}\to\mathbb{R}_+$ be a continuous increasing function with $\lim\limits_{t\to\infty}a(t)<\infty$. If we denote $a_s(t)=a(t+s)$ and Θ is the closure of $\{a_s\colon\ s\in\mathbb{R}\}$ in $C(\mathbb{R},\mathbb{R})$, then $\sigma\colon\ \Theta\times\mathbb{R}\to\Theta$, $\sigma(\theta,t)(s):=\theta(t+s)$, is a flow on Θ.

For every $\theta \in \Theta$ we consider the time dependent parabolic system with Neumann boundary conditions:

$$(\mathrm{S}_1) \qquad \begin{cases} \dfrac{\partial y}{\partial t}(t,\xi) = \theta(t)\dfrac{\partial^2 y}{\partial \xi^2}(t,\xi), \quad t>0,\ \xi \in (0,1) \\ y(0,\xi) = y_0(\xi), \quad \xi \in (0,1) \\ \dfrac{\partial y}{\partial \xi}(t,0) = \dfrac{\partial y}{\partial \xi}(t,1) = 0, \quad t>0. \end{cases}$$

Let $X = L^2(0,1)$, $D(A) = \{x \in H^2(0,1)\colon \dot{x}(0) = \dot{x}(1) = 0\}$ and $Ax = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}x$.

If for every $\theta \in \Theta$ we denote $A(\theta) := \theta(0)A$ the system (S_1) can be rewritten in X as

$$(\mathrm{S}_2) \qquad \begin{cases} \dot{x}(t) = A(\sigma(\theta,t))x(t),\ t \geqslant 0 \\ x(0) = x_0. \end{cases}$$

We have that X is a separable Hilbert space with respect to the inner product

$$\langle x, y\rangle = \int_0^1 x(\xi)y(\xi)\,\mathrm{d}\xi.$$

If $\varphi_0 = 1$ and

$$\varphi_n(\xi) = \sqrt{2}\cos n\pi\xi, \quad \forall \xi \in [0,1], \forall n \in \mathbb{N}^*,$$

then $\{\varphi_n\}_{n\geqslant 0}$ is an orthonormal basis in X. The operator A generates a C_0-semigroup $\mathbb{T} = \{T(t)\}_{t\geqslant 0}$, given by

$$T(t)x = \sum_{n=0}^{\infty} \mathrm{e}^{-n^2\pi^2 t}\,\langle x, \varphi_n\rangle\,\varphi_n, \quad \forall x \in X.$$

Then

$$\Phi\colon\ \Theta \times \mathbb{R}_+ \to \mathcal{B}(X), \quad \Phi(\theta,t)x = T\biggl(\int_0^t \theta(s)\,\mathrm{d}s\biggr)x$$

is a cocycle and $\pi = (\Phi,\sigma)$ is a strongly continuous linear skew-product flow on $\mathcal{E} = X \times \Theta$. Moreover, for every $x_0 \in D(A)$

$$x(t) = \Phi(\theta,t)x_0, \quad t \geqslant 0$$

is a strong solution of the system (S_2).

E x a m p l e 2.4. Let Ω be a suitable region in $\mathbb{R}^p$ (with $p=2$ or $p=3$). Let D be the set of all mappings $u \in C_0^\infty(\Omega,\mathbb{R}^p)$ with $\nabla u = 0$ and let Y be the closure of D in $L^2(\Omega,\mathbb{R}^p)$. Let $D(A) = H^2(\Omega,\mathbb{R}^p) \cap Y$ and let $-A$ be the generator of an analytic semigroup.

Consider the Navier-Stokes equation as a nonlinear abstract evolution equation on Y:

$$\text{(NS)} \qquad \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f \\ u(0) = u_0. \end{cases}$$

If f is a time forcing function, denote by $\Omega(f)$ the ω-limit set of f. Then (NS) has a compact attractor $\Theta \subset D(A) \times \Omega(f)$. If $\theta = (u,v) \in \Theta$, we define the flow $\sigma(\theta,t) = (u_t, v_t)$, where $u_t(s,\cdot) = u(t+s,\cdot)$ and $v_t(s,\cdot) = v(t+s,\cdot)$. The linearized Navier-Stokes equation has the form

$$\text{(LNS)} \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} + \nu Ax + B(u(t),x) + B(x,u(t)) = 0 \\ x(0) = x_0. \end{cases}$$

Then there is a linear skew-product flow $\pi = (\Phi,\sigma)$ on $\mathcal{E} = D(A^{1/2}) \times \Theta$ such that for every $x_0 \in D(A^{1/2})$

$$x(t) = \Phi(\theta,t)x_0, \quad t \geqslant 0$$

is a strong solution of the above equation. For notation and details concerning the results presented in this example we refer to [24], Section 4.1 and the references therein.

Definition 2.4. A linear skew-product flow $\pi = (\Phi,\sigma)$ on $\mathcal{E} = X \times \Theta$ is said to be

(i) *uniformly exponentially stable* if there are $N, \nu > 0$ such that

$$\|\Phi(\theta,t)\| \leqslant N\mathrm{e}^{-\nu t}, \quad \forall (\theta,t) \in \Theta \times \mathbb{R}_+;$$

(ii) *uniformly exponentially unstable* if there are $N, \nu > 0$ such that

$$\|\Phi(\theta,t)x\| \geqslant N\mathrm{e}^{\nu t}\|x\|, \quad \forall (x,\theta,t) \in \mathcal{E} \times \mathbb{R}_+.$$

Example 2.5. Let X be a Banach space. Consider $C(\mathbb{R},\mathbb{R})$-the space defined in Example 2.3. Let $a,b\colon \mathbb{R} \to \mathbb{R}_+$, where a is an increasing continuous function with $\alpha := \lim\limits_{t\to\infty} a(t) < \infty$ and b is a decreasing continuous function such that there exists $\beta := \lim\limits_{t\to\infty} b(t) > 0$.

If $a_s(t) = a(t+s)$ and $\Theta = \{a_s\colon\ s \in \mathbb{R}\}$, then $\sigma\colon\ \Theta\times\mathbb{R} \to \Theta$, $\sigma(\theta,t)(s) := \theta(t+s)$, is a flow on Θ. If $\delta > \alpha$ and

$$\Phi\colon\ \Theta \times \mathbb{R}_+ \to \mathcal{B}(X), \quad \Phi(\theta,t)x = \mathrm{e}^{-\delta t + \int_0^t \theta(\tau)\,\mathrm{d}\tau}x$$

we have that $\pi = (\Phi, \sigma)$ is a strongly continuous linear skew-product flow on $X \times \Theta$. Moreover, π is uniformly exponentially stable.

In the same manner, if $b_s(t) = b(t+s)$ and $\widetilde{\Theta} = \{b_s\colon s \in \mathbb{R}\}$, then $\tilde{\sigma}\colon \widetilde{\Theta} \times \mathbb{R} \to \widetilde{\Theta}$, $\tilde{\sigma}(\theta, t)(s) := \theta(t+s)$, is a flow on $\widetilde{\Theta}$. Moreover, for

$$\widetilde{\Phi}\colon \widetilde{\Theta} \times \mathbb{R}_+ \to \mathcal{B}(X), \quad \widetilde{\Phi}(\theta, t)x = \mathrm{e}^{\int_0^t \theta(\tau)\,\mathrm{d}\tau} x$$

we have that $\tilde{\pi} = (\widetilde{\Phi}, \tilde{\sigma})$ is a strongly continuous linear skew-product flow on $X \times \widetilde{\Theta}$. It is easy to see that

$$\|\widetilde{\Phi}(\theta, t)x\| \geqslant \mathrm{e}^{\beta t}\|x\|, \quad \forall (x, \theta, t) \in X \times \widetilde{\Theta} \times \mathbb{R}_+,$$

so $\tilde{\pi}$ is uniformly exponentially unstable.

Other examples of linear skew-product flows can be found in [1]–[3], [6], [7], [11], [13], [15], [20], [21], [23], [24].

Proposition 2.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. If there are $t_0 > 0$ and $c \in (0, 1)$ such that $\|\Phi(\theta, t_0)\| \leqslant c$ for all $\theta \in \Theta$, then π is uniformly exponentially stable.*

P r o o f. Let $M \geqslant 1$ and $\omega > 0$ be given by Definition 2.2. Let ν be a positive number such that $c = \mathrm{e}^{-\nu t_0}$.

Let $\theta \in \Theta$ be fixed. For $t \geqslant 0$ there are $n \in \mathbb{N}$ and $r \in [0, t_0)$ such that $t = nt_0 + r$. Then we obtain

$$\|\Phi(\theta, t)\| \leqslant \|\Phi(\sigma(\theta, nt_0), r)\|\,\|\Phi(\theta, nt_0)\| \leqslant M\mathrm{e}^{\omega t_0}\mathrm{e}^{-n\nu t_0} \leqslant N\mathrm{e}^{-\nu t},$$

where $N = M\mathrm{e}^{(\omega+\nu)t_0}$. So, π is uniformly exponentially stable. $\square$

Proposition 2.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. If there are $t_0 > 0$ and $\delta > 1$ such that*

$$\|\Phi(\theta, t_0)x\| \geqslant \delta\|x\|, \qquad \forall (x, \theta) \in \mathcal{E},$$

then π is uniformly exponentially unstable.

P r o o f. Let $M \geqslant 1$, $\omega > 0$ be given by Definition 2.2 and let $\nu > 0$ be such that $\delta = \mathrm{e}^{\nu t_0}$. Let $(x, \theta) \in \mathcal{E}$. For $t \geqslant 0$ there are $k \in \mathbb{N}$ and $r \in [0, t_0)$ such that $t = kt_0 + r$. Using the cocycle identity and the hypothesis, it follows that

$$\delta^{k+1}\|x\| \leqslant \|\Phi(\theta, (k+1)t_0)x\| \leqslant M\mathrm{e}^{\omega t_0}\|\Phi(\theta, t)x\|.$$

Denoting $N = 1/M\mathrm{e}^{\omega t_0}$, we deduce from the above that

$$\|\Phi(\theta, t)x\| \geqslant N\mathrm{e}^{\nu t}\|x\|, \quad \forall (x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+,$$

so π is uniformly exponentially unstable. $\square$

2.2. Banach function spaces. Let (Ω, Σ, μ) be a positive σ-finite measure space. We denote by $M(\mu)$ the linear space of μ-measurable functions $f\colon \Omega \to \mathbb{C}$, identifying the functions which are equal μ-a.e.

Definition 2.5. A *Banach function norm* is a function $N\colon M(\mu) \to [0,\infty]$ with the following properties:

(i) $N(f) = 0$ if and only if $f = 0\mu$-a.e.;
(ii) if $|f| \leqslant |g|\mu$-a.e. then $N(f) \leqslant N(g)$;
(iii) $N(af) = |a|N(f)$ for all $a \in \mathbb{C}$ and all $f \in M(\mu)$ with $N(f) < \infty$;
(iv) $N(f+g) \leqslant N(f) + N(g)$ for all $f, g \in M(\mu)$.

Let $B = B_N$ be the set defined by $B := \{f \in M(\mu)\colon |f|_B := N(f) < \infty\}$. It is easy to see that $(B, |\cdot|_B)$ is a linear space. If B is complete then B is called the *Banach function space over* Ω.

R e m a r k 2.1. B is an ideal in $M(\mu)$, i.e., if $|f| \leqslant |g|\mu$-a.e. and $g \in B$ then also $f \in B$ and $|f|_B \leqslant |g|_B$.

Let $(\Omega, \Sigma, \mu) = (\mathbb{R}_+, \mathcal{M}, m)$, where $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable sets $A \subset \mathbb{R}_+$ and m is the Lebesgue measure. For a Banach function space over $\mathbb{R}_+$ we define

$$F_B\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+, \quad F_B(t) := \begin{cases} |\chi_{[0,t)}|_B, & \text{if } \chi_{[0,t)} \in B \\ \infty, & \text{if } \chi_{[0,t)} \notin B \end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of $[0,t)$. The function F_B is called *the fundamental function* of the Banach function space B.

In what follows, we will denote by $\mathcal{L}(\mathbb{R}_+)$ the set of all Banach function spaces B with the property that for every $\varepsilon > 0$ there exists $t_0 \in \mathbb{R}_+$ such that

$$|\chi_{[t-t_0,t)}|_B \geqslant \varepsilon, \quad \forall t \geqslant t_0.$$

R e m a r k 2.2. If B is a Banach function space with the property that $B \in \mathcal{L}(\mathbb{R}_+)$ then $\lim\limits_{t\to\infty} F_B(t) = \infty$.

Similarly, let $(\Omega, \Sigma, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$, where μ_c is the countable measure and let B be a Banach function space over $\mathbb{N}$ (in this case B is called a *Banach sequence space*). We define

$$F_B\colon \mathbb{N}^* \to \overline{\mathbb{R}}_+, \quad F_B(n) := \begin{cases} |\chi_{\{0,\dots,n-1\}}|_B, & \text{if } \chi_{\{0,\dots,n-1\}} \in B \\ \infty, & \text{if } \chi_{\{0,\dots,n-1\}} \notin B \end{cases}$$

and call it *the fundamental function* of the Banach sequence space B.

We denote by $\mathcal{L}(\mathbb{N})$ the set of all Banach sequence spaces B with the property that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|\chi_{\{k-n_0,\dots,k\}}|_B \geqslant \varepsilon, \quad \forall k \in \mathbb{N},\ k \geqslant n_0.$$

R e m a r k 2.3. If B is a Banach sequence space with the property that $B \in \mathcal{L}(\mathbb{N})$ then $\lim\limits_{n\to\infty} F_B(n) = \infty$.

R e m a r k 2.4. If B is a Banach function space over $\mathbb{R}_+$ which belongs to $\mathcal{L}(\mathbb{R}_+)$, then

$$S_B := \Big\{(\alpha_n)_n \colon \sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)} \in B\Big\},$$

with the norm

$$|(\alpha_n)_n|_{S_B} := \Big|\sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)}\Big|_B,$$

is a Banach sequence space which belongs to $\mathcal{L}(\mathbb{N})$. Indeed, this assertion easily follows by observing that

$$|\chi_{\{k-n_0,\dots,k\}}|_{S_B} = |\chi_{[k-n_0,k+1)}|_B, \quad \forall k, n_0 \in \mathbb{N},\ k \geqslant n_0.$$

E x a m p l e 2.6. (*Orlicz spaces*) Let $g\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be a non-decreasing, left continuous function, which is not identically 0 or ∞ on $(0,\infty)$. We define the function

$$Y_g(t) = \int_0^t g(s)\,\mathrm{d}s,$$

which is called the *Young function* associated with g.

Let $(\Omega, \Sigma, \mu) \in \{(\mathbb{R}_+, \mathcal{M}, m), (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)\}$. For every $h\colon \Omega \to \mathbb{C}$ we consider

$$M_g(h) = \int_\Omega Y_g(|h(\omega)|)\,\mathrm{d}\mu.$$

The set of all functions $h\colon \Omega \to \mathbb{C}$ with the property that there exists $k > 0$ such that $M_g(kf) < \infty$, is easily checked to be a linear space. With respect to the norm $|h|_g := \inf\{k > 0\colon M_g(h/k) \leqslant 1\}$, it is a Banach function space over Ω called the *Orlicz function space*. For $\Omega = \mathbb{R}_+$ we will denote it by E_g and for $\Omega = \mathbb{N}$ by O_g (in this case it is called the *Orlicz sequence space*).

R e m a r k 2.5. Let $p \in [1,\infty]$. The Orlicz function spaces and the Orlicz sequence spaces associated with

$$g_p(t) = pt^{p-1} \ \text{ for } 1 \leqslant p < \infty \ \text{ and } g_\infty(t) = \begin{cases} 0, & t \in [0,1] \\ \infty, & t > 1 \end{cases} \quad \text{for } \ p = \infty$$

are $L^p(\mathbb{R}_+, \mathbb{C})$ and $l^p(\mathbb{N}, \mathbb{C})$, respectively.

R e m a r k 2.6. If $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing left continuous function with $g(t) > 0$ for all $t > 0$ and $g(0) = 0$, then the Orlicz function space E_g belongs to $\mathcal{L}(\mathbb{R}_+)$ and the Orlicz sequence space O_g belongs to $\mathcal{L}(\mathbb{N})$.

3. Uniform exponential stability in terms of Banach function spaces

In this section we will give necessary and sufficient conditions for uniform exponential stability of linear skew-product flows using Banach sequence spaces and Banach function spaces.

Throughout this section, we denote by $\mathcal{F}$ the set of all non-decreasing functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.

Let X be a Banach space, let Θ be a metric space and let $\mathcal{E} = X \times \Theta$. We denote $D = \{x \in X\colon \|x\| \leqslant 1\}$.

We start with a characterization for uniform exponential stability of linear skew-product flows in terms of Banach sequence spaces.

Theorem 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a Banach sequence space $B \in \mathcal{L}(\mathbb{N})$, a function $\varphi \in \mathcal{F}$ and an increasing sequence $(t_n) \subset \mathbb{R}_+$ with $\delta = \sup\limits_n (t_{n+1} - t_n) < \infty$ such that*

(i) *for every $(x, \theta) \in D \times \Theta$, the sequence*

$$\gamma_{x,\theta}\colon \mathbb{N} \to \mathbb{R}_+, \quad \gamma_{x,\theta}(n) = \varphi(\|\Phi(\theta, t_n)x\|)$$

belongs to B;

(ii) *there is a constant $K > 0$ such that $|\gamma_{x,\theta}|_B \leqslant K$ for all $(x, \theta) \in D \times \Theta$.*

P r o o f. N e c e s s i t y results by considering $B = l^1(\mathbb{N}, \mathbb{C})$, $\varphi(t) = t$ for all $t \geqslant 0$ and $t_n = n$ for all $n \in \mathbb{N}$.

S u f f i c i e n c y. We distinguish two possible cases:

C a s e 1. $T = \sup\limits_n t_n < \infty$.

Let M, ω be given by Definition 2.2. For every $x \in D$ we set $\tilde{x} = x/M\mathrm{e}^{\omega T}$. We observe that

$$\|\Phi(\theta, T)\tilde{x}\| \leqslant M\mathrm{e}^{\omega T}\|\Phi(\theta, t_n)\tilde{x}\| = \|\Phi(\theta, t_n)x\|, \quad \forall (\theta, n) \in \Theta \times \mathbb{N}.$$

This yields

$$\chi_{\{0,\ldots,n\}}\varphi(\|\Phi(\theta, T)\tilde{x}\|) \leqslant \gamma_{x,\theta}, \quad \forall (\theta, n) \in \Theta \times \mathbb{N},$$

and hence we obtain that

$$F_B(n)\varphi(\|\Phi(\theta,T)\tilde{x}\|) \leqslant K, \quad \forall(\theta,n)\in\Theta\times\mathbb{N}.$$

Because $B\in\mathcal{L}(\mathbb{N})$ using Remark 2.3, we obtain from the last inequality that $\varphi(\|\Phi(\theta,T)\tilde{x}\|)=0$. Taking into account that $\varphi\in\mathcal{F}$ it follows that

$$\Phi(\theta,T)x=0, \quad \forall(x,\theta)\in D\times\Theta,$$

so π is uniformly exponentially stable.

Case 2. $\sup\limits_n t_n=\infty$. Let $n_0\in\mathbb{N}^*$ be such that

$$|\chi_{\{k-n_0,\ldots,k\}}|_B > \frac{K}{\varphi(1)}, \quad \forall k\geqslant n_0.$$

Let $(x,\theta)\in D\times\Theta$ and $\tilde{x}=x/M\mathrm{e}^{\omega n_0\delta}$. For every $k\geqslant n_0$ we have

$$\|\Phi(\theta,t_k)\tilde{x}\| \leqslant \|\Phi(\theta,t_n)x\|, \quad \forall n\in\{k-n_0,\ldots,k\}.$$

It follows that

$$\chi_{\{k-n_0,\ldots,k\}}\varphi(\|\Phi(\theta,t_k)\tilde{x}\|) \leqslant \gamma_{x,\theta},$$

so

$$\frac{K}{\varphi(1)}\varphi(\|\Phi(\theta,t_k)\tilde{x}\|) < |\gamma_{x,\theta}|_B \leqslant K, \quad \forall k\geqslant n_0.$$

From $\varphi\in\mathcal{F}$ we deduce that

$$\|\Phi(\theta,t_k)\| \leqslant M\mathrm{e}^{\omega n_0\delta}, \quad \forall k\geqslant n_0.$$

For $K_1=M\mathrm{e}^{\omega(n_0\delta+t_{n_0})}$ we obtain

$$\|\Phi(\theta,t_k)\| \leqslant K_1, \quad \forall(\theta,k)\in\Theta\times\mathbb{N}.$$

Let $t\geqslant t_0$. Then there exists $k\in\mathbb{N}$ such that $t_k\leqslant t\leqslant t_{k+1}$. Thus we deduce that

$$\|\Phi(\theta,t)\| \leqslant M\mathrm{e}^{\omega\delta}\|\Phi(\theta,t_k)\| \leqslant M\mathrm{e}^{\omega\delta}K_1.$$

Denoting $K_2=M(\mathrm{e}^{\omega\delta}K_1+\mathrm{e}^{\omega t_0})$, we have

$$\|\Phi(\theta,t)\| \leqslant K_2, \quad \forall(\theta,t)\in\Theta\times\mathbb{R}_+.$$

Let $(x, \theta) \in D \times \Theta$. For every $n \in \mathbb{N}$ we observe that

$$\|\Phi(\theta, t_n)x\| \leqslant K_2\|\Phi(\theta, t_k)x\|, \quad \forall k \in \{0, \dots, n\}.$$

Setting $\tilde{x} = x/K_2$, from the last inequality we obtain

$$\varphi(\|\Phi(\theta, t_n)\tilde{x}\|) \leqslant \gamma_{x,\theta}(k), \quad \forall n, k \in \mathbb{N},\ n \geqslant k$$

and so

$$F_B(n)\varphi(\|\Phi(\theta, t_n)\tilde{x}\|) \leqslant |\gamma_{x,\theta}|_B \leqslant K, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

According to Remark 2.3 there is $m_0 \in \mathbb{N}^*$ such that

$$F_B(m_0) > \frac{K}{\varphi(\frac{1}{2K_2})}.$$

Since $\varphi \in \mathcal{F}$, it results from (3.1) that

$$\|\Phi(\theta, t_{m_0})x\| < 1/2.$$

Taking into account that m_0 does not depend on θ and x, we obtain that

$$\|\Phi(\theta, t_{m_0})x\| < 1/2, \quad \forall (x, \theta) \in D \times \Theta.$$

From Proposition 2.1 we conclude that π is uniformly exponentially stable. $\square$

As a consequence we obtain a new version of the theorem of Zabczyk (see [25]), for linear skew-product flows:

Corollary 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a sequence $(t_n) \subset \mathbb{R}_+$ with $\sup_n |t_{n+1} - t_n| < \infty$, a function $\varphi \in \mathcal{F}$ and a constant $K > 0$ such that*

$$\sum_{n=0}^{\infty} \varphi(\|\Phi(\theta, t_n)x\|) \leqslant K, \quad \forall (x, \theta) \in D \times \Theta.$$

P r o o f. N e c e s s i t y immediately follows for $t_n = n$ for all $n \in \mathbb{N}$ and $\varphi(t) = t$ for all $t \geqslant 0$.

S u f f i c i e n c y. If $T = \sup_{n\in\mathbb{N}} t_n < \infty$ then the hypothesis yields that

$$n\varphi\Big(\Big\|\Phi(\theta,T)\frac{x}{M\mathrm{e}^{\omega T}}\Big\|\Big) \leqslant K, \quad \forall (x,\theta,n)\in D\times\Theta\times\mathbb{N}^*,$$

where M, ω are given by Definition 2.2. It follows that $\Phi(\theta,T)=0$ for all $\theta\in\Theta$, so by Proposition 2.1 π is uniformly exponentially stable.

If $\sup_{n\in\mathbb{N}} t_n = \infty$, then without loss of generality we may assume that the sequence (t_n) is increasing (if not, we can consider a subsequence and the proof is the same). Thus, from Theorem 2.1 for $B = l^1(\mathbb{N},\mathbb{C})$ we conclude that π is uniformly exponentially stable. □

Now, we give a characterization of uniform exponential stability in terms of Banach function spaces obtaining a theorem of Neerven type for linear skew-product flows.

Theorem 3.2. *Let $\pi = (\Phi,\sigma)$ be a strongly continuous linear skew-product flow on $\mathcal{E} = X\times\Theta$. Then π is uniformly exponentially stable if and only if there exist a Banach function space $B\in\mathcal{L}(\mathbb{R}_+)$ and a function $\varphi\in\mathcal{F}$ such that*

(i) *for every $(x,\theta)\in D\times\Theta$, the mapping*

$$f_{x,\theta}\colon\ \mathbb{R}_+\to\mathbb{R}_+, \quad f_{x,\theta}(t) = \varphi(\|\Phi(\theta,t)x\|)$$

belongs to B;

(ii) *there is a constant $K>0$ such that $|f_{x,\theta}|_B \leqslant K$ for all $(x,\theta)\in D\times\Theta$.*

P r o o f. N e c e s s i t y easily follows for $\varphi(t)=t$ for all $t\geqslant 0$ and $B = L^1(\mathbb{R}_+,\mathbb{C})$.

S u f f i c i e n c y. We consider $S_B = \big\{(\alpha_n)_n\colon \sum\limits_{n=0}^{\infty}\alpha_n\chi_{[n,n+1)}\in B\big\}$. From Remark 2.4 it follows that $S_B\in\mathcal{L}(\mathbb{N})$. We define

$$\psi\colon\ \mathbb{R}_+\to\mathbb{R}_+, \quad \psi(t) = \varphi\Big(\frac{1}{M\mathrm{e}^{\omega}}t\Big),$$

where M,ω are given by Definition 2.2.

Let $(x,\theta)\in D\times\Theta$. We consider the sequence

$$\gamma_{x,\theta}\colon\ \mathbb{R}_+\to\mathbb{R}_+, \quad \gamma_{x,\theta}(n) = \psi(\|\Phi(\theta,n+1)x\|).$$

We deduce that

$$\psi(\|\Phi(\theta,n+1)x\|)\leqslant\varphi(\|\Phi(\theta,t)x\|), \quad \forall t\in[n,n+1), \forall n\in\mathbb{N}.$$

It results that

$$\gamma_{x,\theta}(n)\chi_{[n,n+1)} \leqslant f_{x,\theta}(t), \quad \forall t \in [n, n+1), \forall n \in \mathbb{N}.$$

We obtain that

$$\sum_{n=0}^{\infty} \gamma_{x,\theta}(n)\chi_{[n,n+1)} \leqslant f_{x,\theta}.$$

It follows that $\gamma_{x,\theta} \in S_B$ and

$$|\gamma_{x,\theta}|_B \leqslant K, \quad \forall (x, \theta) \in D \times \Theta.$$

From Theorem 3.1 we conclude that π is uniformly exponentially stable. $\square$

A theorem of Rolewicz type, for the case of linear skew-product flows, can be expressed as follows:

Corollary 3.2. *Let $\pi = (\Phi, \sigma)$ be a strongly continuous linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a function $\varphi \in \mathcal{F}$ and a constant $K > 0$ such that*

$$\int_0^{\infty} \varphi(\|\Phi(\theta, t)x\|)\,\mathrm{d}t \leqslant K, \quad \forall (x, \theta) \in D \times \Theta.$$

P r o o f. N e c e s s i t y is obvious, for $\varphi(t) = t$, for all $t \geqslant 0$.

S u f f i c i e n c y results from Theorem 3.2 for $B = L^1(\mathbb{R}_+, \mathbb{C})$. $\square$

R e m a r k 3.1. Let $p \in [1, \infty)$. If $\varphi(t) = t^p$ for all $t \geqslant 0$ and π is generated by an evolution family, then from Corollary 3.2 we obtain the theorem of Datko (see [4]).

4. Uniform exponential instability and Banach function spaces

In this section we will characterize the concept of uniform exponential instability in terms of Banach sequence spaces and Banach function spaces, our purpose being to give modifications of the theorems from the previous section for the case of uniform exponential instability of linear skew-product flows.

Let X be a Banach space, let Θ be a metric space and let $\mathcal{E} = X \times \Theta$. We denote $C = \{x \in X \colon \|x\| = 1\}$.

We maintain the notation $\mathcal{F}$ for the set of all non-decreasing functions $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.

First, we give a characterization for uniform exponential instability of linear skew-product flows, using Banach sequence spaces.

Theorem 4.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially unstable if and only if Φ is injective and there exist a Banach sequence space $B \in \mathcal{L}(\mathbb{N})$ and a function $\varphi \in \mathcal{F}$ such that*

(i) *for every $(x, \theta) \in C \times \Theta$, the mapping*

$$\gamma_{x,\theta}\colon \mathbb{N} \to \mathbb{R}_+, \quad \gamma_{x,\theta}(n) = \varphi\Big(\frac{1}{\|\Phi(\theta, n)x\|}\Big)$$

belongs to B;

(ii) *there is a constant $K > 0$ such that $|\gamma_{x,\theta}|_B \leqslant K$ for all $(x, \theta) \in C \times \Theta$.*

Proof. Necessity results for $\varphi(t) = t$ for all $t \geqslant 0$ and $B = l^1(\mathbb{N}, \mathbb{C})$.

Sufficiency. Because $B \in \mathcal{L}(\mathbb{N})$ there is $n_0 \in \mathbb{N}^*$ with the property that

$$|\chi_{\{k-n_0,\dots,k\}}|_B > \frac{K}{\varphi(1)}, \quad \forall k \geqslant n_0. \tag{4.1}$$

Let $x \in C$. We set $\tilde{x} = Lx$, where $L = M\mathrm{e}^{\omega n_0}$ and M, ω are given by Definition 2.2. Let $\theta \in \Theta$ and $p \in \mathbb{N}$. We have

$$\|\Phi(\theta, p+k)x\| \leqslant M\mathrm{e}^{\omega k}\|\Phi(\theta, p)x\| \leqslant \|\Phi(\theta, p)\tilde{x}\|, \quad \forall k \in \{0, \dots, n_0\}.$$

It results that

$$\varphi\Big(\frac{1}{\|\Phi(\theta, p)\tilde{x}\|}\Big) \leqslant \varphi\Big(\frac{1}{\|\Phi(\theta, p+k)x\|}\Big), \quad \forall k \in \{0, \dots, n_0\}.$$

Hence, we have

$$\chi_{\{p,\dots,p+n_0\}}\varphi\Big(\frac{1}{\|\Phi(\theta, p)\tilde{x}\|}\Big) \leqslant \gamma_{x,\theta}.$$

Using (4.1) we obtain that

$$\frac{K}{\varphi(1)}\varphi\Big(\frac{1}{\|\Phi(\theta, p)\tilde{x}\|}\Big) < K.$$

From the last inequality we deduce

$$\frac{1}{\|\Phi(\theta, p)x\|} \leqslant L, \quad \forall (x, \theta, p) \in C \times \Theta \times \mathbb{N}. \tag{4.2}$$

Because $B \in \mathcal{L}(\mathbb{N})$ there is $m_0 \in \mathbb{N}^*$ such that

$$|\chi_{\{k-m_0,\dots,k\}}|_B > \frac{K}{\varphi(\frac{1}{2L})}, \quad \forall k \geqslant m_0. \tag{4.3}$$

Let $(x,\theta) \in C \times \Theta$ and let $\tilde{x} = Lx$. From (4.2) we have

$$\|\Phi(\theta,k)x\| \leqslant L\|\Phi(\theta,m_0)x\|, \quad \forall k \in \{0,\dots,m_0\}.$$

It follows that

$$\varphi\Big(\frac{1}{\|\Phi(\theta,m_0)\tilde{x}\|}\Big) \leqslant \varphi\Big(\frac{1}{\|\Phi(\theta,k)x\|}\Big), \quad \forall k \in \{0,\dots,m_0\}.$$

This yields that

$$\chi_{\{0,\dots,m_0\}}\varphi\Big(\frac{1}{\|\Phi(\theta,m_0)\tilde{x}\|}\Big) \leqslant \gamma_{x,\theta},$$

and hence from (4.3) and $\varphi \in \mathcal{F}$ we deduce that

$$\frac{1}{\|\Phi(\theta,m_0)\tilde{x}\|} < \frac{1}{2L}.$$

So, we obtain that

$$\|\Phi(\theta,m_0)x\| \geqslant 2, \quad \forall (x,\theta) \in C \times \Theta.$$

From Proposition 2.2. we conclude that π is uniformly exponentially unstable. $\square$

As a consequence, we obtain a theorem of Zabczyk type for uniform exponential instability, given by

Corollary 4.1. *Let $\pi = (\Phi,\sigma)$ be a linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially unstable if and only if Φ is injective and there exist a function $\varphi \in \mathcal{F}$ and a constant $K > 0$ such that*

$$\sum_{n=0}^{\infty} \varphi\Big(\frac{1}{\|\Phi(\theta,n)x\|}\Big) \leqslant K, \quad \forall (x,\theta) \in C \times \Theta.$$

Proof. Necessity holds for $\varphi(t) = t$ for all $t \geqslant 0$.

Sufficiency is immediate from Theorem 4.1, considering $B = l^1(\mathbb{N},\mathbb{C})$. $\square$

Theorem 4.2. *Let $\pi = (\Phi, \sigma)$ be a strongly continuous linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially unstable if and only if Φ is injective and there exist a Banach function space $B \in \mathcal{L}(\mathbb{R}_+)$ and a function $\varphi \in \mathcal{F}$ such that*

(i) *for every $(x, \theta) \in C \times \Theta$, the mapping*

$$f_{x,\theta}\colon \mathbb{R}_+ \to \mathbb{R}_+, \quad f_{x,\theta}(t) = \varphi\Big(\frac{1}{\|\Phi(\theta, t)x\|}\Big)$$

belongs to B;

(ii) *there is a constant $K > 0$ such that $|f_{x,\theta}|_B \leqslant K$ for all $(x, \theta) \in C \times \Theta$.*

Proof. Necessity is immediate for $\varphi(t) = t$ for all $t \geqslant 0$ and $B = L^1(\mathbb{R}_+, C)$.

Sufficiency. If M, ω are given by Definition 2.2, then we consider

$$\psi\colon \mathbb{R}_+ \to \mathbb{R}_+, \qquad \psi(t) = \varphi\Big(\frac{t}{M\mathrm{e}^{\omega}}\Big).$$

Let $(x, \theta) \in C \times \Theta$. We define

$$\gamma_{x,\theta}\colon \mathbb{N} \to \mathbb{R}_+, \quad \gamma_{x,\theta}(n) = \psi\Big(\frac{1}{\|\Phi(\theta, n)x\|}\Big).$$

We have

$$\|\Phi(\theta, t)x\| \leqslant M\mathrm{e}^{\omega}\|\Phi(\theta, n)x\|, \quad \forall t \in [n, n+1), \forall n \in \mathbb{N},$$

and hence

$$\psi\Big(\frac{1}{\|\Phi(\theta, n)x\|}\Big) \leqslant \varphi\Big(\frac{1}{\|\Phi(\theta, t)x\|}\Big), \quad \forall t \in [n, n+1), \forall n \in \mathbb{N}.$$

This leads to

$$\sum_{n=0}^{\infty} \gamma_{x,\theta}(n)\chi_{[n,n+1)} \leqslant f_{x,\theta}. \tag{4.4}$$

Let S_B be the Banach sequence space associated with B according to Remark 2.4. Then $S_B \in \mathcal{L}(\mathbb{R}_+)$ and from (4.4) we deduce that $\gamma_{x,\theta} \in S_B$. Moreover, $|\gamma_{x,\theta}|_{S_B} \leqslant |f_{x,\theta}|_B \leqslant K$ for all $(x, \theta) \in C \times \Theta$. Applying Theorem 4.1 for S_B and ψ we deduce that π is uniformly exponentially unstable. $\square$

The last result of this section is a condition of Rolewicz type for uniform exponential instability. It is given by

Corollary 4.2. *Let $\pi = (\Phi, \sigma)$ be a strongly continuous linear skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially unstable if and only if Φ is injective and there exist a function $\varphi \in \mathcal{F}$ and a constant $K > 0$ such that*

$$\int_0^\infty \varphi\Big(\frac{1}{\|\Phi(\theta,t)x\|}\Big)\,\mathrm{d}t \leqslant K, \quad \forall (x,\theta) \in C \times \Theta.$$

Proof. Necessity is obvious for $\varphi(t) = t$ for all $t \geqslant 0$.

Sufficiency easily results from Theorem 4.2 for $B = L^1(\mathbb{R}_+, \mathbb{C})$. □

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