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WEAKLY CONTINUOUS OPERATORS. APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Summary. The paper is a supplement to a survey by *J. Franců*: Monotone operators, A survey directed to differential equations, Aplikace Matematiky, 35(1990), 257–301. An abstract existence theorem for the equation $Au = b$ with a coercive weakly continuous operator is proved. The application to boundary value problems for differential equations is illustrated on two examples. Although this generalization of monotone operator theory is not as general as the M-condition, it is sufficient for many technical applications.

Keywords: monotone operators, weakly continuous operators, existence theorems, boundary value problems for differential equations, heat conduction equation, Navier-Stokes equations.

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INTRODUCTION

This paper is a supplement to the survey paper [1] dealing with applications of abstract monotone operator theory to existence theorems in boundary value problems for differential equations. Dealing with existence of a solution for a Navier-Stokes problem, I found a simpler abstract existence theorem. Instead of monotony, pseudomonotony, M-condition or strong continuity it is sufficient to assume weak continuity of the operator. Although the theorem is not as general as the M-condition operator case, it covers many technically important cases as we will show by two examples.

The weakly continuous operators have some "users friendly" properties, e.g. their sum and multiple is again weakly continuous; linear continuous operators and most quasilinear differential operators are weakly continuous.

It was surprising for me that studying the extensive literature on monotone operators I have not met this simple and useful existence theorem.

The first section proves the abstract existence theorem, the second contains some "useful" properties of weakly continuous operators and the third illustrates the application to two examples. For details dealing with the weak formulation and further references we refer to [1]. We follow the notation introduced in [1].

1. Existence theorem

1.1. Notation. Let V be a reflexive Banach space and A an operator acting from the space V to its dual space V'. As in [1] we shall denote the strong convergence in V ($\|u_n - u\|_V \to 0$) by $u_n \to u$ and the weak convergence in V ($\langle b, u_n - u\rangle \to 0$ $\forall b \in V'$) by $u_n \rightharpoonup u$. The convergences in V' are denoted in the same way. Due to reflexivity, the weak convergence $b_n \rightharpoonup b$ in V' is equivalent to $\langle b_n - b, u\rangle \to 0$ $\forall u \in V$.

We shall deal with the surjectivity of the operator A, i.e. the existence of a solution to the equation with a right-hand side $b \in V'$

$$Au = b. \tag{1.1}$$

Let us recall that the equation (1.1) is an operator equation. The problem reads as follows:

$$\text{Find } u \in V \text{ such that} \qquad \langle Au, v\rangle = \langle b, v\rangle \quad \forall v \in V. \tag{1.1'}$$

1.2. Theorem. *Let V be a reflexive separable Banach space and $A\colon V \to V'$ an operator which is*

– weakly continuous, i.e.

$$u_n \rightharpoonup u \quad \Longrightarrow \quad Au_n \rightharpoonup Au, \tag{1.2}$$

– coercive, i.e.

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty. \tag{1.3}$$

Then A is surjective, i.e. equation (1.1) has a solution for each $b \in V'$.

P r o o f. We prove the theorem in four steps.

1. *We construct a sequence of finite-dimensional subspaces V_n and a sequence of approximate problems.*

Since the space V is separable, it contains a countable dense subset. Excluding the linearly dependent terms, we obtain a linearly independent sequence $\{w_1, w_2, w_3, \ldots\}$.

The first n terms $w_1, w_2, \ldots, w_n$ generate the space V_n. Thus we obtain a sequence of finite-dimensional subspaces $V_n \subset V$.

The sequence $\{V_n\}$ has the following approximation property:

$$\forall v \in V \quad \exists \{v_n\} \quad \text{such that} \quad v_n \in V_n \quad \text{and} \quad v_n \to v. \tag{1.4}$$

Each finite-dimensional subspace $V_n \subset V$ defines an approximate problem—the so-called Galerkin approximation of the original one:

$$\text{Find } u_n \in V_n \text{ such that} \qquad \langle Au_n, v\rangle = \langle b, v\rangle \quad \forall v \in V_n. \tag{1.5}$$

2. *We prove that the finite-dimensional problem* (1.5) *has a solution* u_n.

Let V_n be a finite-dimensional subspace generated by $\{w_1, w_2, \ldots, w_n\}$. Taking the coordinates of an element $u_n \in V_n$ with respect to the base $\{w_1, \ldots, w_n\}$, the space V_n can be identified with $\mathbf{R}^n$ by the mapping

$$u_n = x_1 w_1 + x_2 w_2 + \ldots + x_n w_n \in V_n \quad \longleftrightarrow \quad x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n.$$

The equality in (1.5) with $v = w_1, \ldots, w_n$ yields a vector equation $f(x) = y$ in $\mathbf{R}^n$. Indeed, the mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is given by $f(x) = \{\langle A(\sum_j x_j w_j), w_i\rangle\}_i$ and $y = \{\langle b, w_i\rangle\}_i \in \mathbf{R}^n$.

Thus the problem (1.5) is equivalent to a vector equation $f(x) = y$ on $\mathbf{R}^n$.

Since the weak and the strong convergence on a finite-dimensional subspace coincide, the mapping f is continuous. The coercivity of a mapping restricted to a subspace is preserved. Therefore the existence of u_n follows e.g. from Theorem 2.5 in [1]:

Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a continuous coercive mapping and $y \in \mathbf{R}^n$.

Then the equation $f(x) = y$ has a solution.

The theorem is a consequence of the well known Brouwer theorem.

3. *We extract a subsequence $\{u_{n'}\}$ weakly converging to an element $u \in V$.*

Let $K = \|b\|$. According to the definition of the limit (1.3) there exists a positive constant L such that

$$\|u\| > L \quad \Longrightarrow \quad \|Au\| \geqslant \frac{\langle Au, u\rangle}{\|u\|} > K.$$

Transposition of this implication yields

$$\|Au_n\| = \|b_n\| \leqslant K \quad \Longrightarrow \quad \|u_n\| \leqslant L. \tag{1.6}$$

We have obtained an estimate for u_n independent of n. Since the space V is reflexive, the sequence $\{u_n\}$ contains a subsequence $\{u_{n'}\}$ weakly converging to an element $u \in V$:

$$u_{n'} \rightharpoonup u. \tag{1.7}$$

4. *We prove that the limit u is a solution of the problem* (1.1).

Since the operator is weakly continuous, (1.7) implies

$$Au_{n'} \rightharpoonup Au. \tag{1.8}$$

A weakly converging sequence is bounded, thus

$$\|Au_{n'}\| \leqslant C. \tag{1.9}$$

We prove that u is a solution to (1.1'). Let $v \in V$ be arbitrary. Due to the approximation property (1.4) there exists a sequence $\{v_n\}$, $v_n \in V_n$, $v_n \to v$. We take (1.5) with $v = v_n$ and pass to the limit using (1.8):

$$\langle Au_{n'}, v_{n'} \rangle = \langle Au_{n'}, v_{n'} - v \rangle + \langle Au_{n'}, v \rangle \quad \longrightarrow \quad \langle Au, v \rangle,$$

since the first term tends to zero due to (1.9) and $\|v_n - v\| \to 0$.

On the other hand, due to (1.5) we have

$$\langle Au_{n'}, v_{n'} \rangle = \langle b, v_{n'} \rangle \quad \longrightarrow \quad \langle b, v \rangle.$$

Thus the limit u is a solution and the proof is completed. □

1.3. Remarks. Since weak continuity implies continuity on finite-dimensional subspaces required by the proof, the usual assumption of continuity can be omitted. Actually this is no generalization since the theorem on Nemyckij operators yields continuity.

The assumption of coercivity can be omitted if we ensure in another way the existence of approximate solutions u_n to finite-dimensional problems and find a bounded convex set containing the approximate solutions u_n. Naturally, then the existence need not hold for all right-hand sides $b \in V'$.

2. Weakly continuous operators

We introduce some propositions which facilitate the verification of the assumption.

2.1. Proposition. *The set of weakly continuous operators $A: V \to V'$ forms a linear space, i.e. if A_1, A_2 are weakly continuous and $c \in \mathbf{R}$ then $A_1 + A_2$ and $c A_1$ is also weakly continuous.*

The assertion is a simple consequence of the definitions. It allows to split the operator into a sum of operators and verify their weak continuity separately.

2.2. Proposition. *A linear continuous operator on a reflexive Banach space is weakly continuous.*

Proof. Let $A: V \to V'$ be a linear continuous operator and let $u_n \rightharpoonup u$. Let us introduce the adjoint operator to A—the operator $A^*: V \to V'$—defined by

$$\langle A^* v, u\rangle = \langle Au, v\rangle \qquad \forall\, u, v \in V.$$

Obviously A^* is also continuous. Then for any $v \in V'$ we have $A^* v \in V'$ and $u_n \rightharpoonup u$ implies $Au_n \rightharpoonup Au$ since

$$\langle Au_n - Au, v\rangle = \langle A^* v, u_n - u\rangle \to 0.$$

□

2.3. In general for nonlinear operators we can only assert that a *strongly continuous operator is weakly continuous*, see [1], Lemma 6.2. In case of differential operators for boundary value problems we can say more. We introduce two propositions. They assert that the operator is both well defined and weakly continuous.

2.4. Differential operators. Let Ω be a bounded domain in $\mathbf{R}^N$ with a Lipschitz boundary and let V be a closed subspace of Sobolev space $W^{k,p}(\Omega)$, ($k = 1, 2, 3, \ldots, p \in (1, \infty)$).

An operator $A: V \to V'$ is defined by its values on V, i.e. by defining $\langle Au, v\rangle$.

2.5. Proposition. *Let $A: V \to V'$ be an operator on V given by*

$$\langle Au, v\rangle = \int_\Omega h(\cdot, \partial^{\alpha_1} u, \partial^{\alpha_2} u, \ldots, \partial^{\alpha_m} u)\, \partial^\beta v \,\mathrm{d}x, \tag{2.2}$$

where $\partial^{\alpha_i}, \partial^\beta$ denote partial derivatives (of orders $0, 1, 2, \ldots, k$).

Let the following assumptions be satisfied with constants $p_1, \dots, p_m, q, r \in [1, \infty]$ satisfying $1/r + 1/q = 1$ (e.g. for $p = \infty$ we put $1/p = 0$).

(1) *The function $h\colon \Omega \times \mathbf{R}^m \to \mathbf{R}$ satisfies the assumptions of the theorem on Nemyckij operators with constants p_i, r (see [1], 8.9), i.e. it satisfies*

(a) *Carathéodory conditions: $h(x,\xi)$ is measurable in x for all $\xi \in \mathbf{R}^m$ and continuous in ξ for almost all $x \in \Omega$,*

(b) *growth condition:*

– *if $p_1, p_2, \dots, p_m, r \in [1, \infty)$ then*

$$|h(x, \xi_1, \dots, \xi_m)| \leqslant g(x) + c \sum_{i=1}^{m} |\xi_i|^{p_i/r}, \tag{2.3}$$

where $g \in L_r(\Omega)$ and c is a positive constant,

– *if $p_1, p_2, \dots, p_s = \infty$, $p_{s+1}, \dots, p_m, r < \infty$ then*

$$|h(x, \xi_1, \dots, \xi_m)| \leqslant c\Big(\sum_{i=1}^{s} |\xi_i|\Big)\Big[g(x) + \sum_{i=s+1}^{m} |\xi_i|^{p_i/r}\Big], \tag{2.3'}$$

where $g \in L_r(\Omega)$ and $c(t)$ is a continuous function,

– *if $r = \infty$ then*

$$|h(x, \xi_1, \xi_2, \dots, \xi_m)| \leqslant const. < \infty. \tag{2.3''}$$

(2) *Linear mappings*

$$\mathcal{L}_i\colon V \to L_{p_i}(\Omega), \qquad \mathcal{L}_i u = \partial^{\alpha_i} u \tag{2.4}$$

are strongly continuous for $i = 1, 2, \dots, m$, i.e. they map weakly converging sequences in V to sequences strongly converging in L_{p_i}; and

$$\mathcal{L}\colon V \to L_q(\Omega), \qquad \mathcal{L}v = \partial^{\beta} v \tag{2.5}$$

is a continuous mapping. Then the operator A is well defined and weakly continuous.

Proof. Let $\{u_n\}$ be a weakly convergent sequence in V. Since the mappings (2.4) are strongly continuous we have strong convergences

$$\partial^{\alpha_i} u_n \to \partial^{\alpha_i} u \quad \text{in } L_{p_i}(\Omega) \quad \text{for } i = 1, 2, \dots, m.$$

According to the theorem on Nemyckij operators, the mapping

$$\partial^{\alpha_1} u, \partial^{\alpha_2} u, \dots, \partial^{\alpha_m} u \;\mapsto\; h(\cdot, \partial^{\alpha_1} u, \partial^{\alpha_2} u, \dots, \partial^{\alpha_m} u)$$

is continuous and thus

$$h(\cdot, \partial^{\alpha_1} u_n, \ldots, \partial^{\alpha_m} u_n) \longrightarrow h(\cdot, \partial^{\alpha_1} u, \ldots, \partial^{\alpha_m} u) \quad \text{in } L_r(\Omega). \tag{2.6}$$

Let $v \in V$. Due to (2.5) $\partial^\beta v \in L_q(\Omega) \doteq [L_r(\Omega)]^*$, thus $\langle Au_n, v\rangle \to \langle Au, v\rangle$ and the proof is completed. □

2.6. Proposition. *Let $A\colon V \to V'$ be an operator on V given by*

$$\langle Au, v\rangle = \int_\Omega \partial^{\alpha_0} u\, h(\cdot, \partial^{\alpha_1} u, \partial^{\alpha_2} u, \ldots, \partial^{\alpha_m} u)\, \partial^\beta v \,\mathrm{d}x, \tag{2.7}$$

where $\partial^{\alpha_0} u$ denotes a partial derivative of order k and the others are the same as in Proposition 2.5.

Let the following assumptions be satisfied with constants $p_1, \ldots, p_m, q, r \in [1, \infty]$:

Let the assumptions (1), (2) of Proposition 2.5 be satisfied with some constants p_1, ..., p_m, r, q (p is the exponent in $W^{k,p}(\Omega)$) satisfying

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{q} = 1. \tag{2.8}$$

Then the operator A is well defined and weakly continuous.

Proof. Let $\{u_n\}$ be a weakly convergent sequence in V. Clearly $\partial^{\alpha_0} u_n \rightharpoonup \partial^{\alpha_0} u$ in $L_p(\Omega)$. Using the same argument as in the previous proof we obtain (2.6). Due to relation (2.8) we can write

$$\begin{aligned}\langle Au_n - Au, v\rangle = &\int_\Omega [\partial^{\alpha_0} u_n - \partial^{\alpha_0} u]\, h(\cdot, \partial^{\alpha_1} u, \ldots, \partial^{\alpha_m} u) \partial^\beta v \,\mathrm{d}x \,+ \\ &+ \int_\Omega \partial^{\alpha_0} u_n \,[h(\cdot, \partial^{\alpha_1} u_n, \ldots, \partial^{\alpha_m} u_n) - h(\cdot, \partial^{\alpha_1} u, \ldots, \partial^{\alpha_m} u)]\, \partial^\beta v \,\mathrm{d}x.\end{aligned}$$

The first term tends to zero due to the weak convergence $\partial^{\alpha_0} u_n \rightharpoonup \partial^{\alpha_0} u$, the second due to the boundedness of $\{\partial^{\alpha_0} u_n\}$ and the strong convergence (2.6) in the corresponding spaces. Hence the weak continuity of operator A follows. □

Roughly speaking, the "well" defined differential operators in divergence form which are linear in the highest derivatives of the unknown (or independent of them) and continuous in the lower order derivatives of the unknown are weakly continuous.

3. Application

We want to illustrate the application of the abstract existence Theorem 1.2. Therefore, we concentrate on verification of weak continuity while for the other analysis (the weak formulation etc.) we refer to [1].

We start with the problem introduced in [1], Section 8, Example III.

Example I. Stationary nonlinear heat-conduction equation

3.1. Formulation of the problem. Let Ω be a bounded domain in $\mathbf{R}^N$ ($N = 2$ or 3) with a Lipschitz boundary $\partial\Omega$ divided into two parts Γ_0, Γ_1. We assume that Γ_0 has a positive surface measure.

We shall consider the equation

$$-\sum_i \frac{\partial}{\partial x_i}\left[a(x,u)\frac{\partial u}{\partial x_i}\right] = f \quad \text{in} \quad \Omega \tag{3.1}$$

(the sums will be from 1 to N) with mixed boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_0, \qquad \sum_i a(x,u)\frac{\partial u}{\partial x_i} n_i = g \quad \text{on} \quad \Gamma_1. \tag{3.2}$$

Let us recall that the problem describes the steady state of heat conduction ($u(x)$ — temperature) in a body occupying the volume Ω with internal heat sources f. Function $a(x,\xi)$ describes heat conduction properties of the material. On the boundary, temperature or heat flow is prescribed. To simplify the problem we consider zero stable boundary conditions only. The nonhomogeneous case $u = U_0$ on Γ_0 causes merely technical difficulties, see [1], 8.15.

3.2. Weak formulation. The space V is defined as the closure of the set $\{u \in C^1(\overline{\Omega}),\ u = 0 \text{ on } \Gamma_0\}$ in the norm of the Sobolev space $W^{1,2}(\Omega)$. The space V is a reflexive separable Banach space.

We define the operator $A\colon V \to V'$ and the functional $b \in V'$ by the relations

$$\langle Au, v\rangle = \int_\Omega \sum_i a(x,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \quad u, v \in V, \tag{3.3}$$

$$\langle b, v\rangle = \int_\Omega f v\,\mathrm{d}x + \int_{\Gamma_1} g\,v\,\mathrm{d}S \quad v \in V. \tag{3.4}$$

The problem can be formulated as follows:

$$\text{Find } u \in V \text{ such that} \qquad \langle Au, v\rangle = \langle b, v\rangle \quad \forall v \in V. \tag{3.5}$$

3.3. Justification and application of the abstract existence theorem. To ensure $b \in V'$ we assume

$$f \in L_2(\Omega), \qquad g \in L_2(\Gamma_1). \tag{3.6}$$

Further, we use Proposition 2.6 to prove that the operator A is "well" defined and weakly continuous.

We assume that the coefficient $a(x, \xi)$ satisfies the Carathéodory conditions and the growth condition (2.3″),

$$|a(x, \xi)| \leqslant c \qquad (c < \infty). \tag{3.7}$$

Each term of the sum (3.3) is of the form (2.7). Indeed, with $m = 1$,

$$\partial^{\alpha_0} u = \frac{\partial u}{\partial x_i}, \qquad \partial^{\alpha_1} u = u, \qquad \partial^{\beta} v = \frac{\partial v}{\partial x_i}$$

the right-hand side of (2.7) converts into one term in (3.3). The assumptions (1) with (2.3″) were supposed with $p = p_1 = q = 2$, $r = \infty$. The identity mapping (2.4) $W^{1,2}(\Omega) \to L^2(\Omega)$ is a compact imbedding while the linear mapping (2.5) $u \mapsto \frac{\partial u}{\partial x_i}$ is continuous from $W^{1,2}(\Omega) \to L^2(\Omega)$.

Thus each term is well defined and weakly continuous and the same holds for their sum due to Proposition 2.1.

It remains to prove the coercivity of the operator A. It is guaranteed by the assumption

$$a(x, \xi) \geqslant \alpha \qquad (\alpha > 0) \tag{3.8}$$

since $[\int \sum (\frac{\partial u}{\partial x_i})^2]^{1/2}$ forms an equivalent norm on V.

We can conclude: *If* (3.6)–(3.8) *is satisfied then the problem* (3.5) *has a solution.*

Example II. Stationary Navier-Stokes equations

3.4. Formulation of the problem. Let Ω be a bounded domain in $\mathbf{R}^N$ ($N = 2$ means the plane case, $N = 3$ the space case) with a Lipschitz boundary Γ. We shall consider the following system of equations called stationary Navier-Stokes equations:

$$-\nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = f_i + \frac{\partial p}{\partial x_i} \qquad i = 1, \dots, N \quad \text{in} \quad \Omega, \tag{3.9}$$

$$\sum_i \frac{\partial u}{\partial x_i} = 0 \quad \text{in} \quad \Omega, \tag{3.10}$$

where the sums are from 1 to N. For the sake of simplicity we consider the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \Gamma. \tag{3.11}$$

The system describes the steady state flow of incompressible viscous liquid occupying the volume Ω subjected to given external volume forces $f \equiv \{f_i(x)\}$. The introduced boundary condition (3.11) means that the liquid is closed within fixed walls.

The flow is described by two unknowns: velocity vector $u \equiv \{u_i(x)\}$ and scalar pressure $p \equiv p(x)$.

The first equation is the equation of motion: the first term on the left-hand side is the viscous term with ν—the constant of viscosity, the second is the convection term. The second equation is the equation of continuity.

3.5. Weak formulation. We shall look for a solution in the space V defined as the closure of the set

$$\left\{u \in \left[C^1(\Omega)\right]^N,\ \sum \frac{\partial u_i}{\partial x_i} = 0 \text{ in } \Omega,\ u = 0 \text{ on } \Gamma\right\} \tag{3.12}$$

in the norm of the vector Sobolev space $[W^{1,2}(\Omega)]^N$. The space V is a separable reflexive Banach space.

We multiply the i-th equation (3.9) by a function v_i, integrate it over Ω and sum them up. Applying the Green theorem to the viscous term and the term with the pressure we obtain

$$\begin{aligned} -\nu \int_\Gamma \sum_{i,j} \frac{\partial u_i}{\partial x_j} v_i n_j \,\mathrm{d}S + \nu \int_\Omega \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x + \int_\Omega \sum_{i,j} u_j \frac{\partial u_i}{\partial x_j} v_i \,\mathrm{d}x \\ = \int_\Omega \sum_i f_i v_i \,\mathrm{d}x + \int_\Gamma p \sum_i v_i n_i \,\mathrm{d}S - \int_\Omega p \sum_i \frac{\partial v_i}{\partial x_i}\,\mathrm{d}x. \end{aligned}$$

Taking $u, v \in V$ the integrals over Γ vanish due to $v = 0$ on Γ. The integral with pressure p vanishes due to $\sum \frac{\partial v_i}{\partial x_i} = 0$. Moreover, $u \in V$ implies that the condition (3.10) is satisfied; thus (3.10) can be omitted.

We define: a bilinear form $a(\cdot,\cdot)$, a trilinear form $b(\cdot,\cdot,\cdot)$, an operator $A\colon V \to V'$ and a functional $b \in V'$ by the relations

$$\begin{aligned} a(u,v) &= \int_\Omega \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x, \qquad & b(u,v,w) &= \int_\Omega \sum_{i,j} u_j \frac{\partial u_i}{\partial x_j} v_i \,\mathrm{d}x, \\ \langle Au, v\rangle &= a(u,v) + b(u,u,v), & \langle f, v\rangle &= \int_\Omega \sum_i f_i v_i \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

In the above introduced notation the problem can be formulated as follows:

(3.14) Find $u \in V$ such that $\langle Au, v\rangle = \langle f, v\rangle \quad \forall v \in V.$

Remark. The weak formulation does not contain the pressure p. One can prove that for any sufficiently smooth weak solution u of (3.14) there exists a function p such that u, p is a solution to (3.9)–(3.11).

3.6. Justification and application of the abstract existence theorem. In order to ensure $f \in V'$ we assume $f \in [L_2(\Omega)]^N$.

The bilinear form $a(\cdot,\cdot)$ is continuous on $V \times V$. Due to Proposition 2.2 the first part of A is weakly continuous.

The trilinear form $b(\cdot,\cdot,\cdot)$ consists of terms $\int u_j \frac{\partial u_i}{\partial x_j} v_i \, dx$ which are of type (2.7) with $m = 1$,

$$h(\cdot,\xi) = \xi, \qquad \partial^{\alpha_0} u = \frac{\partial u_i}{\partial x_j}, \qquad \partial^{\alpha_1} u = u_j, \qquad \partial^{\beta} v = v_i.$$

Using Proposition 2.6 we prove that the term is well defined and weakly continuous. The assumptions are satisfied with $p = 2$, $p_1 = r = q = 4$. Indeed, (2.3), (2.8) hold and the linear mappings (2.4), (2.5) are imbeddings

$$W^{1,p}(\Omega) \subset L_q(\Omega) \qquad \text{for } \frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{N},$$

with $p = 2, q = 4$ which hold for $N \leqslant 4$. In addition, the imbedding is compact in our case $N \leqslant 3$.

Due to Proposition 2.1 the operator A is well defined and weakly continuous. Thus we have justified the weak formulation (3.14) of the problem.

It remains to prove that the operator A is coercive. We make use of the equality

(3.16) $$b(u,u,u) = 0 \qquad u \in V.$$

Applying the Green theorem for $v = 0$ on Γ we obtain

$$b(u,v,w) = \int_\Omega \sum_{i,j} u_j \frac{\partial v_i}{\partial x_j} w_i \, dx = -\int_\Omega \sum_{i,j} \frac{\partial u_j}{\partial x_j} v_i w_i \, dx - \int_\Omega \sum_{i,j} u_j v_i \frac{\partial w_i}{\partial x_j} \, dx.$$

The first integral on the right-hand side vanishes due to $u \in V$ and we can obtain $b(u,v,w) = -b(u,w,v)$. For $u = v = w$ equality (3.16) follows.

Taking (3.16) into account we have $\langle Au, u\rangle = a(u,u) \geqslant \|u\|_V$ and coerciveness follows, since $[a(u,u)]^{\frac{1}{2}}$ forms an equivalent norm on V.

We can conclude: *For $f \in [L_2(\Omega)]^N$ the problem* (3.15) *admits a solution.*

Remark. Nonhomogeneous boundary conditions cause some difficulties in the proof of coerciveness of the operator, a special "cut off" function should be used. For small f also uniqueness can be proved, see e.g. [2].

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For further references see [1].

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