Rudolf Hlavička

# Finite element solution of a hyperbolic-parabolic problem

*Applications of Mathematics*, Vol. 39 (1994), No. 3, 215–239

Persistent URL: http://dml.cz/dmlcz/134254

39 (1994) APPLICATIONS OF MATHEMATICS No. 3, 215–239

# FINITE ELEMENT SOLUTION OF A HYPERBOLIC-PARABOLIC PROBLEM

RUDOLF HLAVIČKA, Brno

(Received January 28, 1993)

*Summary.* Existence and finite element approximation of a hyperbolic-parabolic problem is studied. The original two-dimensional domain is approximated by a polygonal one (external approximations). The time discretization is obtained using Euler's backward formula (Rothe's method).

Under certain smoothing assumptions on the data (see (2.6), (2.7)) the existence and uniqueness of the solution and the convergence of Rothe's functions in the space $C(\overline{I}, V)$ is proved.

*Keywords*: Rothe's method, finite elements.

*AMS classification*: 65N30, 65M60.

## 1. FORMULATION OF THE PROBLEM

By a two-dimensional hyperbolic-parabolic initial boundary value problem we understand a problem of the following type: Let $\Omega$, $\Omega_H$, $\Omega_P$ be two-dimensional bounded domains with Lipschitz continuous boundaries such that $\overline{\Omega} = \overline{\Omega}_H \cup \overline{\Omega}_P$, $\Omega_H \cap \Omega_P = \emptyset$, mes $\Omega_H > 0$. If mes $\Omega_P = 0$, we get only equation (1.1)—a hyperbolic problem. Find a function $u\colon \overline{\Omega} \to \mathbb{R}$ such that $u_M := u|_{\Omega_M} (M = H, P)$ satisfy the equations

$$(1.1) \qquad \frac{\partial^2 u_H}{\partial t^2} = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(k_{ij}^H \frac{\partial u_H}{\partial x_j}\Big) + f^H \quad \text{in} \quad \Omega_H \times (0,T),$$

$$(1.2) \qquad \frac{\partial u_P}{\partial t} = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(k_{ij}^P \frac{\partial u_P}{\partial x_j}\Big) + f^P \quad \text{in} \quad \Omega_P \times (0,T),$$

where $0 < T < \infty$ and $k_{ij}^M(x)$, $f^M(x,t)$ are given functions satisfying (2.3)–(2.5). Equation (1.1) is of hyperbolic type and Eq. (1.2) is of parabolic type. The function $u$ has to satisfy some boundary condition, for instance

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0,T),$$
$$\sum_{i,j=1}^{2} k_{ij} \frac{\partial u}{\partial x_i} \nu_j = 0 \quad \text{on} \quad \Gamma_N \times (0,T),$$

where $\Gamma_D$ is a nonempty relatively open subset of $\Gamma := \partial\Omega$, $\Gamma_N = \Gamma - \overline{\Gamma}_D$ and $\nu_j$ denote the components of the unit outer normal to $\Gamma$. The initial conditions are

$$u(x,0) = u^0 \qquad \forall x \in \Omega,$$
$$\frac{\partial u_H(x,0)}{\partial t} = z_H^0 \qquad \forall x \in \Omega_H,$$

where $u^0$, $z_H^0$ are given functions. Finally, on the interface $\Lambda := \partial\Omega_H \cap \partial\Omega_P$ the function $u$ has to satisfy for all $t \in (0,T)$ so called transition conditions: At every $x \in \Lambda$ the limit value of $u_H$ is equal to the limit value of $u_P$ and the limit value of $\sum_{i,j=1}^{2} k_{ij}^H \frac{\partial u_H}{\partial x_i} \nu_j$ is equal to the limit value of $\sum_{i,j=1}^{2} k_{ij}^P \frac{\partial u_P}{\partial x_i} \nu_j$, where $\nu_j$ denote the components of the unit normal to $\Lambda$ oriented in a unique way. The transition conditions are briefly written in the form

$$[u]_P^H = \left[ \sum_{i,j=1}^{2} k_{ij} \frac{\partial u}{\partial x_i} \nu_j \right]_P^H = 0.$$

A motivation for studying this type of a problem is the computation of two-dimensional electromagnetic fields in the case when in $\Omega_H$ the electrical conductivity is $\sigma = 0$ and in $\Omega_P$ it is $\sigma \gg 0$ (for similar situations see e.g. [1, 3, 5, 6, 8, 9]).

Note. The symbol $C$ is used as a generic constant, which means that this constant may represent various values on different places in the paper.

## 2. Variational formulation

Let $\Gamma_D$ and $\Gamma_N$ is relatively open subsets of $\Gamma$ such that

$$\Gamma_D \cap \Gamma_N = \emptyset, \ \operatorname{mes}_1 \Gamma_D + \operatorname{mes}_1 \Gamma_N = \operatorname{mes}_1 \Gamma, \ \operatorname{mes}_1 \Gamma_D > 0,$$

where $\Gamma_D$ consists of a finite number of disjoint arc-components, each component being of a positive one-dimensional measure. We set

$$V := \{v \in H^1(\Omega) \colon v = 0 \text{ on } \Gamma_D\}.$$

Let $\Omega_H$ and $\Omega_P$ be subdomains of $\Omega$ with the properties

$$\overline{\Omega} = \overline{\Omega}_H \cup \overline{\Omega}_P, \ \Omega_H \cap \Omega_P = \emptyset.$$

We shall assume that $\Gamma_D$ satisfies one of the following three possibilities:

$$\Gamma \cap \partial\Omega_H \cap \partial\Omega_P \subset \Gamma_D,$$
$$\partial\Omega_H \cap \partial\Omega_P \cap \overline{\Gamma}_D = \emptyset,$$
$$\partial\Omega_H \cap \partial\Omega_P \cap \Gamma = \{Q_1, Q_2 \colon Q_1 \in \Gamma_D, \ Q_2 \notin \overline{\Gamma}_D\}$$

and that the boundaries $\Gamma = \partial\Omega$, $\partial\Omega_H$, $\partial\Omega_P$ are piecewise of class $C^3$.

We define

$$V_M = \{v_M := v\big|_{\Omega_M} : v \in V\} \quad (M = H, P).$$

Let $G \subset \mathbb{R}^2$ be an arbitrary domain, the norm in the Sobolev space $H^k(G)$ will be denoted by $\|.\|_{k,G}$; $\|\cdot\|_k := \|\cdot\|_{k,\Omega}$. The scalar products in the spaces $L_2(\Omega)$ and $L_2(G)$ will be denoted by $(.,.)$ and $(.,.)_G$; $(.,.)_M := (.,.)_{L_2(\Omega_M)}$. The norms in $V$ and in $V_M$ are induced by the norms $\|\cdot\|_1$, $\|\cdot\|_{1,\Omega_M}$, respectively. We shall work with the spaces $C(\overline{I}, B)$, $L_2(I, B)$, $L_\infty(I, B)$, where B is a Banach space and $I = (0, T)$. The symbol $V_H^*$ denotes the normed dual of $V_H$ and for $f \in V_H^*$, $u \in V_H$ we write $f(u) = \langle f, u \rangle_H$. We set

$$v_M(t) := v(t)|_{\Omega_M} \qquad \forall t \in I.$$

We shall use a form

$$a \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R},$$

where

$$a_M(v, w) = \sum_{i,j=1}^{2} \int_{\Omega_M} k_{ij}^M \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, \mathrm{d}x \qquad \forall v, w \in H^1(\Omega), \tag{2.1}$$

$$a(v, w) = a_H(v, w) + a_P(v, w) \qquad \forall v, w \in H^1(\Omega). \tag{2.2}$$

We assume

(2.3) $$k_{ij}^M(x) = k_{ji}^M(x) \qquad (i,j = 1,2; M = H,P),$$

then $a(v,w) = a(w,v)$ and further

(2.4) $$k_{ij}^M \in W_\infty^1(\tilde{\Omega}_M) \qquad (i,j = 1,2; M = H,P),$$

(2.5) $$\sum_{i,j=1}^{2} k_{ij}^M(x)\xi_i\xi_j \geqslant \gamma_M(\xi_1^2 + \xi_2^2) \qquad \forall x \in \tilde{\Omega}_M, \forall \xi_1, \xi_2 \in \mathbf{R},$$

where $\gamma_M > 0$ and the domains $\tilde{\Omega}_M \supset \Omega_M (M = H,P)$ will be specified later in Section 3. The symbols $\dot{u}, \ddot{u}$ will denote the strong derivatives with respect to $t$ of an abstract function $u$ (for the definition see [2, Chap. IV]).

**Problem 2.1.** *Let the form $a(v,w)$ be defined by (2.1), (2.2), where the functions $k_{ij}^M$ satisfy (2.3)–(2.5). Let $u^0$, $z_H^0$, $f$ be given functions such that*

(2.6) $$u^0 \in V, \quad z_H^0 \in L_2(\Omega_H),$$

(2.7) $$f^M \in L_2(I, W_\infty^1(\tilde{\Omega}_M)), \quad \dot{f}^M \in L_2(I, W_\infty^1(\tilde{\Omega}_M)), \quad (M = H,P).$$

*Find a function $u\colon \bar{I} \to V$ with the properties*

$$\begin{gathered} u \in C(\bar{I}, V), \\ \dot{u} \in L_2(I, L_2(\Omega)), \\ \ddot{u}_H \in L_2(I, V_H^*), \\ u(0) = u^0 \in V, \\ \dot{u}_H(0) = z_H^0 \in L_2(\Omega_H), \\ \int_0^T \{\langle \ddot{u}, v\rangle_H + (\dot{u}, v)_P + a(u,v)\}\,\mathrm{d}t = \int_0^T (f,v)\,\mathrm{d}t \quad \forall v \in L_2(I,V). \end{gathered}$$

## 3. Discretization

We shall approximate the domain $\Omega$ by a domain $\Omega_n$ with a polygonal boundary the vertices of which lie on $\Gamma$. Let $\mathcal{T}_n = \{K_1, \dots, K_m\}$ be a triangulation of $\Omega_n$. Let $\sigma_n$ be the set of all nodes in $\mathcal{T}_n$. We shall assume

$$\sigma_n \subset \overline{\Omega}, \qquad \sigma_n \cap \partial\Omega_n \subset \Gamma,$$
$$\overline{\Gamma}_D \cap \overline{\Gamma}_N \subset \sigma_n;$$

the points of $\Gamma$ where the condition of the $C^3$-smoothness is not satisfied, belong to $\sigma_n$.

We shall consider only those triangulations $\mathcal{T}_n$ that at most two vertices of each triangle lie on $\Gamma$; such triangles are called boundary triangles.

Let $K$ be a boundary triangle and let $B_1, B_2, B_3$ be its vertices, $B_1, B_3 \in \Gamma$. Let $\Sigma$ be the part of $\Gamma$ which is approximated by the segment $\overline{B_1B_3}$. The closed curved triangle $K^{\mathrm{id}}$ with two straight sides $\overline{B_1B_2}$, $\overline{B_2B_3}$ and one curved side $\Sigma$ is called the ideal triangle associated with the triangle $K$. If we replace all boundary triangles in $\mathcal{T}_n$ by their associated ideal triangles we obtain the ideal triangulation $\mathcal{T}_n^{\mathrm{id}}$. We shall assume that $K^{\mathrm{id}} \subset K$ or $K \subset K^{\mathrm{id}}$ for every $K \in \mathcal{T}_n$.

Every triangulation $\mathcal{T}_n$ consists of two subtriangulations $\mathcal{T}_{n,H}$, $\mathcal{T}_{n,P}$ such that

$$\mathcal{T}_n = \mathcal{T}_{n,H} \cup \mathcal{T}_{n,P}, \qquad \mathcal{T}_{n,H} \cap \mathcal{T}_{n,P} = \emptyset.$$

The subtriangulations $\mathcal{T}_{n,M}$ has all properties described in the preceeding text.

With every triangulation we associate three parameters

$$h = \max_{K \in \mathcal{T}_n} h_K, \quad \overline{h} = \min_{K \in \mathcal{T}_n} h_K, \quad \theta = \min_{K \in \mathcal{T}_n} \theta_K,$$

where $h_K$ is the length of the greatest side and $\theta_K$ is the magnitude of the smallest angle of the triangle $K \in \mathcal{T}_n$. We shall assume that the following conditions are satisfied:

$$\frac{\overline{h}_n}{h_n} \geqslant C_0 > 0 \qquad (n = 1, 2, 3 \dots),$$
$$\theta_n \geqslant \theta_0 > 0 \qquad (n = 1, 2, 3 \dots),$$
$$\lim_{n\to\infty} h_n = 0.$$

Let $\{\Delta t_n\}_{n=1}^{\infty}$ be a sequence independent of $\{h_n\}_{n=1}^{\infty}$ with the properties

$$\Delta t_n > 0, \quad \lim_{n\to\infty} \Delta t_n = 0, \quad r_n := \frac{T}{t_n} \text{ is integer.}$$

Let the bounded domains $\tilde{\Omega}$, $\tilde{\Omega}_M$ satisfy

$$\bigcup_{n=1}^{\infty} \left(\overline{\Omega}_M \cup M_n\right) \subset \tilde{\Omega}_M \qquad (M = H, P),$$

$$\bigcup_{n=1}^{\infty} (\overline{\Omega} \cup \Omega_n) \subset \tilde{\Omega},$$

where $M_n$ is the domain with a polygonal boundary associated with the triangulation $\mathcal{T}_{n,M}$.

We define finite dimensional spaces

$$X_n = \{v \in C(\overline{\Omega}_n) : v|_K \text{ is linear for all } K \in \mathcal{T}_n\},$$
$$V_n = \{v \in X_n : v(B_i) = 0 \quad \forall B_i \in \sigma_n \cap \overline{\Gamma}_D\}.$$

For $v \in X_n$ the symbol $v_M$ denotes the function from the space $C(\overline{M}_n)$ which is linear in every triangle $K \in \mathcal{T}_{n,M}$ satisfying

$$v_M(B_i) = v(B_i) \qquad \forall B_i \in \sigma_n \cap \overline{M}_n.$$

We define

$$V_{nM} = \{v_M ; v \in V_n\}.$$

Further, we define the forms

$$a_{M_n}(v,w) = \sum_{i,j=1}^{2} \int_{M_n} k_{ij}^M \frac{\partial v_M}{\partial x_i} \frac{\partial w_M}{\partial x_j} \, \mathrm{d}x \qquad \forall v, w \in X_n, \tag{3.1}$$

$$a_n = a_{H_n} + a_{P_n}.$$

We shall approximate the terms $a_n(v,w)$ and

$$(f,v)_n := \sum_{K \in \mathcal{T}_n} (f,v)_K$$

using a quadrature formula of degree of precision $d = 1$ on each triangle. The results will be denoted by $a_n^I(v,w)$ and $(f,v)_n^I$.The following lemma can be found in [9, 10].

**Lemma 3.1.** *Let the assumptions* (2.3)-(2.5), (2.7) *be fulfilled. Then for all* $v, w \in X_n$ *we have*

$$|a_n(v,w) - a_n^I(v,w)| \leqslant C h_n \|v\|_{1,\Omega_n} \|w\|_{1,\Omega_n}, \tag{3.2}$$

$$|(f(t),v)_n - (f(t),v)_n^I| \leqslant C h_n \|f(t)\|_{1,\infty,\tilde{\Omega}} \|v\|_{1,\Omega_n} \quad \forall t \in \overline{I}, \tag{3.3}$$

$$|a_n^I(v,w)| \leqslant M \|v\|_{1,\Omega_n} \|w\|_{1,\Omega_n}, \tag{3.4}$$

$$\beta \|v\|_{1,\Omega_n}^2 \leqslant a_n^I(v,v), \tag{3.5}$$

*where the constants $C$, $M$, $\beta$ are independent of $n$ and of the functions $v$, $w$.*

Let $\tilde{v} \in H^1(\mathbb{R}^2)$ be the Calderon extension of the function $v \in V$. Then by [9, 10] there exists a sequence $\{v_n\}_{n=1}^{\infty}$ of functions $v_n \in V_n$ with the properties

$$\lim_{n\to\infty} \|v_n - \tilde{v}\|_{1,\Omega_n} = 0, \quad \lim_{n\to\infty} \|v_{nM} - \tilde{v}\|_{1,M_n} = 0.$$

Now we can formulate the discrete form of Problem 2.1:

**Problem 3.2.** *Let $n$ be a given integer and let*

$$t_i = i\Delta t_n \qquad (i = 1, \ldots, r_n).$$

*Find $U_n^i \in V_n (i = 1, \ldots, r_n)$ such that*

$$(3.6) \qquad (\delta_n^2 U_n^i, v)_{H_n} + (\delta_n U_n^i, v)_{P_n} + a_n^I(U_n^i, v) = (f(t_i), v)_n^I \quad \forall v \in V_n,$$

*where $\delta_n U_n^i = (U_n^i - U_n^{i-1})/\Delta t_n$, $\delta_n^2 U_n^i = (\delta_n U_n^i - \delta_n U_n^{i-1})/\Delta t_n$ with $U_{nH}^{-1} := U_{nH}^0 - \Delta t_n Z_{nH}^0$; here $\{U_n^0\}_{n=1}^{\infty}$ is a sequence of functions $U_n^0 \in V_n$ with the property*

$$(3.7) \qquad \lim_{n\to\infty} \|U_n^0 - \tilde{u}^0\|_{1,\Omega_n} = 0,$$

*where $\tilde{u}^0$ denotes the Calderon extension of the function $u^0 \in V$, and $\{Z_{nH}^0\}_{n=1}^{\infty}$ is a sequence of functions $Z_{nH}^0 \in V_{nH}$ with the property*

$$(3.8) \qquad \lim_{n\to\infty} \|Z_{nH}^0 - \tilde{z}_H^0\|_{0,H_n} = 0,$$

*where $\tilde{z}_H^0 \in L_2(\mathbb{R}^2)$ is the extension of the function $z_H^0$ by zero.*

Relations (3.4) and (3.5) imply:

**Theorem 3.3.** *The solution $U_n^i$ of Problem 3.2 exists and is unique.*

First we shall define the set

$$V_a = \bigcup_{f\in L_2(\Omega)} \{u \in V\colon a(u,v) = (f,v) \quad \forall v \in V\}.$$

By [4] we have

**Lemma 4.1.** *The set $V_a$ is dense in the space $V$.*

**Lemma 4.2.** *There exist sequences $\{{}^mS_H^0\}$, $\{{}^mZ_H^0\}$, $\{{}^mZ_P^0\}$, $\{{}^mU^0\}$, $\{{}^mf^M\}$ such that ${}^mS_H^0 \in C_0^\infty(\Omega_H)$, ${}^mZ_H^0 \in C_0^\infty(\Omega_H)$, ${}^mZ_P^0 \in C_0^\infty(\Omega_P)$, ${}^mU^0 \in V$ and ${}^mf^M$, ${}^m\dot{f}^M$, ${}^m\ddot{f}^M \in L_2\big(I, W_\infty^1(\tilde{\Omega}_M)\big)(M = H, P)$ with the properties*

$$({}^mS_H^0, v)_H + ({}^mZ_P^0, v)_P + a({}^mU^0, v) = ({}^mf(0), v) \quad \forall v \in V, \tag{4.1}$$

$$\lim_{m\to\infty} \|{}^mZ_H^0 - z_H^0\|_{0,H} = 0, \tag{4.2}$$

$$\lim_{m\to\infty} \|{}^mU^0 - u^0\|_1 = 0, \tag{4.3}$$

$$\lim_{m\to\infty} \|{}^mf^M - f^M\|_{C\left(\bar{I}, W_\infty^1(\tilde{\Omega}_M)\right)} = 0, \tag{4.4}$$

$$\lim_{m\to\infty} \|{}^m\dot{f}^M - \dot{f}^M\|_{L_2\left(I, W_\infty^1(\tilde{\Omega}_M)\right)} = 0, \tag{4.5}$$

*where $u^0$, $z_H^0$, $f^M$ are the data of Problem 2.1.*

Proof. By regularization [4] we can construct sequences $\{{}^mf^M\}(M = H, P)$ with the properties (4.4) and (4.5).

Lemma 4.1 implies the existence of sequences

$${}^ms_H^0 \in L_2(\Omega_H), \quad {}^mz_P^0 \in L_2(\Omega_P)$$

such that the solutions ${}^mu^0 \in V$ of the elliptic problems

$$a({}^mu^0, v) = ({}^mf(0), v) - ({}^ms_H^0, v)_H - ({}^mz_P^0, v)_P \quad \forall v \in V \tag{4.6}$$

fulfil

$$\lim_{m\to\infty} \|{}^mu^0 - u^0\|_1 = 0. \tag{4.7}$$

The density of $C_0^\infty(\Omega_M)$ in $L_2(\Omega_M)(M = H, P)$ implies the existence of sequences

$${}^mS_H^0 \in C_0^\infty(\Omega_H), \quad {}^mZ_H^0 \in C_0^\infty(\Omega_H), \quad {}^mZ_P^0 \in C_0^\infty(\Omega_P) f \tag{4.8}$$

with the following properties:

(4.9) $$\lim_{m\to\infty} \|{}^mS_H^0 - {}^ms_H^0\|_{0,H} = 0,$$

(4.10) $$\lim_{m\to\infty} \|{}^mZ_H^0 - z_H^0\|_{0,H} = 0,$$

(4.11) $$\lim_{m\to\infty} \|{}^mZ_P^0 - {}^mz_P^0\|_{0,P} = 0.$$

Let ${}^mU^0 \in V$ be the solution of the problem (4.1). From (4.1) and (4.6) it follows that

$$\|{}^mU^0 - {}^mu^0\| \leqslant C(\|{}^mS_H^0 - {}^ms_H^0\|_{0,H} + \|{}^mZ_P^0 - {}^mz_P^0\|_{0,P}).$$

Thus, by (4.7), (4.9), (4.11),

(4.12) $$\lim_{m\to\infty} \|{}^mU^0 - u^0\|_1 = 0.$$

The results now follow from (4.8)–(4.12). □

Now, to obtain an a priori estimate in Lemma 4.5, we shall formulate the following auxiliary discrete problem using better input data than in Problem 2.1:

**Problem 4.3.** *Find ${}^mU_n^i \in V_n$ $(i = 1, \dots, r_n)$ such that*

(4.13) $$(\delta_n^2\, {}^mU_n^i, v)_{H_n} + (\delta_n\, {}^mU_n^i, v)_{P_n} + a_n^I({}^mU_n^i, v) = ({}^mf(t_i), v)_n^I \quad \forall v \in V_n,$$

*where ${}^mU_n^0 \in V_n$ is the solution of the discrete problem*

(4.14) $$a({}^mU_n^0, v) = ({}^mf(0), v)_n^I - (I_n^H\, {}^mS_H^0, v)_{H_n} - (I_n^P\, {}^mZ_P^0, v)_{P_n} \quad \forall v \in V_n$$

*and*

(4.15) $${}^mZ_{nH}^0 = I_n^H Z_H^0.$$

*Here $I_n^M w$ denotes the interpolate of a function $w \in C(\overline{\Omega}_{nM})$.*

In the same way as Theorem 3.3 we obtain:

**Lemma 4.4.** *The solution ${}^mU_n^i$ of Problem 4.3 exists and is unique.*

We set

$${}^mZ_n^i = \delta_n {}^mU_n^i, \qquad {}^mS_n^i = \delta_n^2\, {}^mU_n^i,$$

the symbol $\Delta^n$ denotes the backward difference of $n$-th order with respect to $i$.

**Lemma 4.5.** *For the solutions ${}^mU_n^i$ of Problem 4.3 we have*

$$\|{}^mS_{nH}^i\|_{0,H_n} \leqslant C\sum_{i=1}^{r_n} \|{}^mS_{nP}^i\|_{0,P_n}\Delta t_n \leqslant C, \quad \sum_{i=1}^{r_n} \|{}^mZ_n^i\|_{1,\Omega_n}^2 \leqslant C,$$

*where the constant $C$ is independent of $m$ and $n$.*

Proof. We can treat the equality (4.1) as a special case of equalities (3.6) for $i = 0$. Subtracting (3.6) for $i$ and $i-1$ and setting $v = {}^mS_n^i$, we obtain

$$(4.16)\quad (\Delta\, {}^mS_n^i, {}^mS_n^i)_{H_n} + \frac{1}{\Delta t_n}({}^mZ_n^i, {}^mZ_n^i) + a_n^I({}^mZ_n^i, \Delta\, {}^mZ_n^i) = \frac{1}{\Delta t_n}({}^mf(t_i), {}^mZ_n^i)_n^I.$$

We have

$$(4.17)\qquad (\Delta\, {}^mS_n^i, {}^mS_n^i)_{H_n} = \frac{1}{2}\|S_{nH}^i\|_{0,H_n}^2 - \frac{1}{2}\|S_{nH}^{i-1}\|_{0,H_n}^2 + \frac{1}{2}\|\Delta S_{nH}^i\|_{0,H_n}^2.$$

Using the symmetry $a_n^I(v,w) = a_n^I(w,v)$, we similarly obtain

$$a_n^I({}^mZ_n^i, \Delta\, {}^mZ_n^i) = \frac{1}{2}a_n^I({}^mZ_n^i, {}^mZ_n^i) - \frac{1}{2}a_n^I({}^mZ_n^{i-1}, {}^mZ_n^{i-1}) + \frac{1}{2}a_n^I(\Delta\, {}^mZ_n^i, \Delta\, {}^mZ_n^i).$$

After summing (4.16) from $i = 1$ to $i = j$ and using (3.7), (3.8), (4.16), (4.17), (3.3), (3.4), we find

$$(4.18)\qquad \begin{aligned} &\|{}^mS_n^j\|_{0,H_n}^2 + \frac{1}{\Delta t_n}\sum_{i=1}^{j} \|{}^mZ_n^i\|_{0,P_n}^2 + \|{}^mZ_n^j\|_{1,\Omega_n} + \sum_{i=1}^{j} \|\Delta {}^mZ_n^i\|_{1,\Omega_n}^2 \\ &\leqslant C_0\Big(1 + \frac{1}{\Delta t_n}\sum_{i=1}^{j}(\Delta^m f(t_i), \Delta\, {}^mZ_n^i)_n^I\Big). \end{aligned}$$

Using summation by parts we obtain

$$(4.19)\qquad \begin{aligned} &\sum_{i=1}^{j}(\Delta^m f(t_i), \Delta\, {}^mZ_n^i)_n^I \\ &= (\Delta\, {}^mf(t_j), {}^mZ_n^j)_n^I - \sum_{i=1}^{j-1}(\Delta^2\, {}^mf(t_{i+1}), {}^mZ_n^i)_n^I - (\Delta^m f(t_1), {}^mZ_n^0)_n^I. \end{aligned}$$

From (3.3) and from the mean value theorem we can estimate

$$(4.20)\qquad \begin{aligned} &C_0|(\Delta\, {}^mf^M(t_1), {}^mZ_n^0)_{M_n}^I| \\ &\leqslant C_0|(\Delta\, {}^mf^M(t_1), {}^mZ_n^0)_{M_n}| + C_0|(\Delta\, {}^mf^M(t_1), {}^mZ_n^0)_{M_n}^I - (\Delta\, {}^mf^M(t_1), {}^mZ_n^0)_{M_n}| \\ &\leqslant C\|\Delta\, {}^mf^M(t_1)\|_{1,\infty,\tilde\Omega_M}\|{}^mZ_n^0\|_{1,M_n} + Ch_n\|\Delta\, {}^mf^M(t_1)\|_{1,\infty,\tilde\Omega_M}\|{}^mZ_n^0\|_{1,M_n} \\ &\leqslant C\Delta t_n\|{}^m\dot f^M\|_{C(\bar I, W_\infty^1(\tilde\Omega_M))}\|{}^mZ_n^0\|_{1,M_n} \leqslant C\Delta t_n. \end{aligned}$$

Similarly we obtain

$$C_0|(\Delta\,{}^m f^M(t_j), {}^m Z_n^j)_{M_n}^I| \leqslant C\Delta t_n \|{}^m\dot{f}^M\|_{C(\bar{I}, W_\infty^1(\bar{\Omega}_M))} \|{}^m Z_n^0\|_{j,M_n}$$

and using an elementary inequality

$$|ab| \leqslant \varepsilon a^2/2 + b^2/(2\varepsilon) \qquad \forall a, b \in \mathbb{R} \quad \forall \varepsilon > 0, \tag{4.21}$$

we find

$$C_0|(\Delta\,{}^m f^M(t_j), {}^m Z_n^j)_{M_n}^I| \leqslant \Delta t_n \Big\| \Big(C + \frac{1}{2}\|{}^m Z_n^j\|\Big)\Big\|_{1,M_n}^2. \tag{4.22}$$

Further, we have

$$\begin{aligned} C_0\|\Delta^2\,{}^m f^M(t_{i+1})\|_{1,\infty,\bar{\Omega}_M} &\leqslant C\int_{t_i}^{t_{i+1}}\int_{s-\Delta t_n}^{s} \|{}^m\ddot{f}^M(\sigma)\|_{1,\infty,\bar{\Omega}_M}\,\mathrm{d}\sigma\,\mathrm{d}s \\ &\leqslant C\Delta t_n \int_{t_{i-1}}^{t_{i+1}} \|{}^m\ddot{f}^M(\sigma)\|_{1,\infty,\bar{\Omega}_M}\,\mathrm{d}\sigma \\ &\leqslant C\Delta t_n \Big(2\Delta t_n \int_{t_{i-1}}^{t_{i+1}} \|{}^m\ddot{f}^M(\sigma)\|_{1,\infty,\bar{\Omega}_M}^2\,\mathrm{d}\sigma\Big)^{1/2}. \end{aligned}$$

From (4.22) and (4.21) we derive

$$\begin{aligned} &\frac{1}{\Delta t_n} C_0 \Big|\sum_{i=1}^{j-1} (\Delta^2\,{}^m f^M(t_{i+1}), {}^m Z_n^i)_{M_n}^I\Big| \\ &\leqslant \frac{1}{\Delta t_n} C \sum_{i=1}^{j-1} \|\Delta^2\,{}^m f^M(t_{i+1})\|_{1,\infty,\bar{\Omega}_M} \|{}^m Z_n^i\|_{1,M_n} \\ &\leqslant C\sum_{i=1}^{j-1} \Big(\Delta t_n \int_{t_{i-1}}^{t_{i+1}} \|{}^m\ddot{f}^M(\sigma)\|_{1,\infty,\bar{\Omega}_M}^2\,\mathrm{d}\sigma\Big)^{1/2} \|{}^m Z_n^i\|_{1,M_n} \\ &\leqslant C\|{}^m\ddot{f}^M\|_{L_2(I, W_\infty^1(\bar{\Omega}_M))}^2 + \frac{1}{2}\Delta t_n \sum_{i=1}^{j-1} \|{}^m Z_n^i\|_{1,M_n}^2. \end{aligned} \tag{4.23}$$

Substituting (4.20), (4.22), (4.23) into (4.18) and (4.19) we obtain

$$\begin{aligned} \|{}^m S_n^j\|_{0,H_n}^2 + \frac{1}{\Delta t_n}\sum_{i=1}^{j} \|{}^m Z_n^i\|_{0,P_n}^2 + \frac{1}{2}\|{}^m Z_n^j\|_{1,\Omega_n}^2 + \sum_{i=1}^{j} \|\Delta\,{}^m Z_n^i\|_{1,\Omega_n} \\ \leqslant C + \frac{1}{2}\Delta t_n \sum_{i=1}^{j-1} \|{}^m Z_n^i\|_{1,\Omega_n}^2. \end{aligned} \tag{4.24}$$

We obtain the assertion from (4.24) using the discrete form of Gronwall's inequality.

□

The following lemma can be found in [9, 10]; it is a special case of results proved in [7].

**Lemma 5.2.** *There exists a linear operator* $I_K^{\mathrm{id}}: C(K^{\mathrm{id}}) \to H^1(K^{\mathrm{id}}) \cap C(K^{\mathrm{id}})$ *such that every function* $w \in C(K^{\mathrm{id}})$ *satisfies*

a) $I_K^{\mathrm{id}} w$ *is uniquely determined by the relations*

$$(I_K^{\mathrm{id}} w)(B_i^K) = w(B_i^K) \qquad (i = 1, 2, 3),$$

*where* $B_i^K$ *are the vertices of both* $K$ *and* $K^{\mathrm{id}}$*;*

b) *the function* $I_K^{\mathrm{id}} w$ *is linear along both straight sides* $B_1^K B_2^K$, $B_2^K B_3^K$ *of the curved triangle* $K^{\mathrm{id}}$*;*

c) *if* $w(B_i^K) = 0 \quad (i = 1, 2, 3)$, *where* $B_1^K, B_3^K \in \Gamma$, *then* $I_K^{\mathrm{id}} w = 0$ *on the curved side of* $K^{\mathrm{id}}$*;*

d) *if* $w \in H^2(K^{\mathrm{id}})$ *then*

$$\|w - I_K^{\mathrm{id}} w\|_{k,K^{\mathrm{id}}} \leqslant C h_K^{2-k} \|w\|_{2,K^{\mathrm{id}}} \quad (k = 0, 1),$$

*where the constant* $C$ *does not depend on* $h_K$ *and* $w$.

**Definition 5.2.** Let $w \in X_n$. The function $\overline{w}: \overline{\Omega}_n \cup \overline{\Omega} \to \mathbb{R}$ is called the natural extension of $w$ if

$$\overline{w} = w \quad \text{on } \Omega_n, \qquad \overline{w}\big|_{K^{\mathrm{id}} - K} = p\big|_{K^{\mathrm{id}} - K} \quad \text{if } \mathrm{mes}(K^{\mathrm{id}} - K) > 0,$$

where $p$ is the linear polynomial satisfying $p\big|_K = w\big|_K$.

**Definition 5.3.** Let $w \in C(\overline{\Omega})$. The function $I_n^{\mathrm{id}} w \in H^1(\Omega) \cap C(\overline{\Omega})$ is called the ideal interpolant of $w$ if

$$(I_n^{\mathrm{id}} w)\big|_{K^{\mathrm{id}}} = I_K^{\mathrm{id}} w \qquad \forall K^{\mathrm{id}} \in \mathcal{T}_{n,M}^{\mathrm{id}} - \mathcal{T}_{n,M},$$
$$(I_n^{\mathrm{id}} w)\big|_{K^{\mathrm{id}}} = p_K\big|_K \qquad \forall K^{\mathrm{id}} \in \mathcal{T}_{n,M}^{\mathrm{id}} \cap \mathcal{T}_{n,M},$$

where $p_K$ is the linear polynomial satisfying $p_K(P_i^K) = w(P_i^K)$ for $i = 1, 2, 3$ and $M = H, P$.

If $w \in X_n$ then the function

$$\hat{w} := I_n^{\mathrm{id}} \overline{w} \in H^1(\Omega) \cap C(\overline{\Omega})$$

is called the function associated with $w$.

Let us set

$$\tau_n = \Omega_n - \overline{\Omega}, \qquad \omega_n = \Omega - \overline{\Omega}_n;$$
$$\tau_{nM} = M_n - \overline{\Omega}_M, \qquad \omega_{nM} = \Omega_M - \overline{M}_n.$$

We have

$$\mathrm{mes}(\varepsilon_n) \leqslant Ch_n^2, \qquad \mathrm{mes}(\varepsilon_{nM}) \leqslant Ch_n^2 \quad (\varepsilon = \tau, \omega).$$

By [9, relations (3.12), (3.17), (3.25), (3.42)], we have for k=0,1 that

$$\|\hat{w}_M\|_{k,M} \leqslant C\|w_M\|_{k,M_n}, \quad \|\hat{w}\|_k \leqslant C\|w\|_{k,\Omega_n}, \tag{5.1}$$

$$\|\overline{w}_M\|_{k,\varepsilon_{nM}} \leqslant Ch_n^{1/2}\|w_M\|_{k,M_n}, \quad \|\overline{w}\|_{k,\varepsilon_n} \leqslant Ch_n^{1/2}\|w\|_{k,\Omega_n}, \tag{5.2}$$

$$\|\hat{w}_M - \overline{w}_M\|_{k,M} \leqslant Ch_n\|w_M\|_{k,M_n}, \quad \|\hat{w} - \overline{w}\|_{k,\Omega} \leqslant Ch_n\|w\|_{k,\Omega_n} \tag{5.3}$$

for all $w \in X_n$, where $\varepsilon = \tau, \omega$ and $M = H, P$.

Relations (5.1) and Lemma 4.5 give us the desired form of a priori estimates:

**Lemma 5.4.** *Let the assumptions of Problem 4.3 be satisfied. Then we have*

$$\|{}^m\hat{S}^i_{nH}\|_{0,H} \leqslant C, \tag{5.4}$$

$$\sum_{i=1}^{r_n} \|{}^m\hat{S}^i_{nP}\|_{0,P}\Delta t_n \leqslant C, \quad \sum_{i=1}^{r_n} \|{}^m\hat{Z}^i_n\|_1^2 \leqslant C, \tag{5.5}$$

*where the constant $C$ is independent of $m$ and $n$.*

## 6. The $C(\overline{I}, V)$ convergence

We start with some definitions of the finite element Rothe's functions

$$\left.\begin{aligned} {}^m\hat{U}_n(t) &= {}^m\hat{U}_n^{i-1} + \delta_n\, {}^m\hat{U}_n^i(t - t_{i-1}) \\ {}^m\hat{Z}_{nP}(t) &= {}^m\hat{Z}_{nP}^{i-1} + \delta_n\, {}^m\hat{Z}_{nP}^i(t - t_{i-1}) \end{aligned}\right\} \quad t \in [t_{i-1}, t_i] \quad (i = 1, \ldots, r_n)$$

and the step functions

$$\left.\begin{aligned} {}^m\hat{u}_n(0) &= {}^m\hat{U}_n^0 & \qquad {}^m\hat{u}_n(t) &= {}^m\hat{U}_n^i \\ {}^m\overline{u}_n(0) &= {}^m\overline{U}_n^0 & {}^m\overline{u}_n(t) &= {}^m\overline{U}_n^i \\ {}^m\hat{z}_n(0) &= {}^m\hat{Z}_n^0 & \text{and} \qquad {}^m\hat{z}_n(t) &= {}^m\hat{Z}_n^i \\ {}^m\overline{z}_n(0) &= {}^m\overline{Z}_n^0 & {}^m\overline{z}_n(t) &= {}^m\overline{Z}_n^i \\ {}^m\hat{s}_n(0) &= {}^m\hat{S}_n^0 & {}^m\hat{s}_n(t) &= {}^m\hat{S}_n^i \\ {}^m\overline{s}_n(0) &= {}^m\overline{S}_n^0 & {}^m\overline{s}_n(t) &= {}^m\overline{S}_n^i \end{aligned}\right\} \quad t \in [t_{i-1}, t_i] \quad (i = 1, \ldots, r_n),$$

where bars and hats are the symbols for natural extension and associated function introduced in Section 5. Finally, we put

$$^m f_n^M(t) = {}^m f^M(t_i) \qquad t \in (t_{i-1}, t_i] \quad (i = 1, \dots, r_n), {}^m f_n^M(0) = {}^m f_n^M(t_1).$$

Using standard arguments, we conclude from Lemma 5.4:

**Lemma 6.1.** *Let the assumptions of Problem 4.3 be fulfilled. Then there exists a function* $^m u \in C(\overline{I}, V)$ *such that*

$$(6.1) \qquad {}^m\dot{u} \in L_2(I,V), {}^m\dot{u}_H \in C(\overline{I}, L_2(\Omega_H)), {}^m\ddot{u}_H \in L_2(I, L_2(\Omega_H)),$$
$$^m u(0) = {}^m u^0, \quad {}^m\dot{u}_H(0) = {}^m z_H^0$$

*and a subsequence* $\{U_{n_k}\}$ *of the sequence* $\{U_n\}$ *which we shall further denote briefly by* $\{U_k\}$ *with the properties*

$$(6.2) \qquad {}^m\hat{U}_k \to {}^m u \quad \text{in} \quad C(\overline{I}, L_2(\Omega)),$$

$$(6.3) \qquad {}^m\hat{U}_k \rightharpoonup {}^m u, \quad {}^m\hat{u}_k \rightharpoonup {}^m u \quad \text{weakly in} \quad L_2(I,V),$$

$$(6.4) \qquad {}^m z_k \rightharpoonup {}^m\dot{u} \quad \text{weakly in} \quad L_2(I,V),$$
$$^m\hat{Z}_{kH} \to {}^m\dot{u}_H \quad \text{in} \quad C(\overline{I}, L_2(\Omega_H)),$$

$$(6.5) \qquad {}^m\hat{Z}_{kH} \rightharpoonup {}^m\dot{u}_H \quad \text{weakly in} \quad L_2(I, V_H),$$

$$(6.6) \qquad {}^m\hat{s}_{kH} \rightharpoonup {}^m\ddot{u}_H \quad \text{weakly in} \quad L_2(I, L_2(\Omega_H)).$$

**Lemma 6.2.** *The function* $^m u$ *from Lemma 6.1 is the unique function satisfying*

$$(6.7) \qquad \int_0^T \{({}^m\ddot{u}(\tau), v(\tau))_H + ({}^m\dot{u}(\tau), v(\tau))_P + a({}^m u(\tau), v(\tau))\}\,\mathrm{d}\tau$$
$$= \int_0^T ({}^m f(\tau), v(\tau))\,\mathrm{d}\tau \qquad \forall v \in L_2(I,V).$$

P r o o f. A) Let $v \in L_2(I,V)$ be any function. The symbol $\tilde{v} \in L_2(I, H^1(\mathbb{R}^2))$ denotes the Calderon extension of the function $v$ (see [9, Lemma 3.9] ), that is

$$\|\tilde{v}\|_{L_2(I,H^1(\mathbb{R}^2))} \leqslant C\|v\|_{L_2(I,V)},$$
$$\tilde{v}\big|_\Omega = v.$$

Similarly as in [10, 31.4 Theorem] we can construct a sequence $\{v_n\}$ of functions $v_n \in L_2(I, V_n)$ with the property

$$(6.8) \qquad \lim_{n\to\infty} \|\tilde{v} - v_n\|_{L_2(I,H^1(\Omega_n))} = 0.$$

B) First we rewrite the equation (4.13) to the form

$$(6.9)\qquad \begin{aligned}&\left({}^m\hat{s}_k(t),\hat{v}_k(t)\right)_H+\left({}^m\hat{z}_k(t),\hat{v}_k(t)\right)_P+a\left({}^m\hat{u}_k(t),\hat{v}_k(t)\right)\\&\quad+\sum_{l=1}^{3}{}^mA_k^l(t)+\sum_{l=1}^{3}{}^mB_k^l(t)+\sum_{l=1}^{4}{}^mC_k^l(t)\\&=\left({}^mf_k(t),{}^m\hat{v}_k(t)\right)+\sum_{l=1}^{3}{}^mD_k^l(t)\qquad\forall t\in I,\end{aligned}$$

where the forms ${}^mA_k^l$, ${}^mB_k^l$, ${}^mC_k^l$, ${}^mD_k^l$ are defined as follows:

$$(6.10)\qquad {}^mA_k^1(t)=\left({}^m\hat{s}_k(t),\overline{v}_k(t)\right)_H-\left({}^m\hat{s}_k(t),\hat{v}_k(t)\right)_H,$$

$$(6.11)\qquad \begin{aligned}{}^mA_k^2(t)&=\left({}^m\overline{s}_k(t),\overline{v}_k(t)\right)_H-\left({}^m\hat{s}_k(t),\overline{v}_k(t)\right)_H,\\{}^mA_k^3(t)&=\left({}^m\overline{s}_k(t),\overline{v}_k(t)\right)_{\tau_kH}-\left({}^m\overline{s}_k(t),\overline{v}_k(t)\right)_{\omega_kH},\end{aligned}$$

$$(6.12)\qquad \begin{aligned}{}^mB_k^1(t)&=\left({}^m\hat{z}_k(t),\overline{v}_k(t)\right)_H-\left({}^m\hat{z}_k(t),\hat{v}_k(t)\right)_H,\\{}^mB_k^2(t)&=\left({}^m\overline{z}_k(t),\overline{v}_k(t)\right)_H-\left({}^m\hat{z}_k(t),\overline{v}_k(t)\right)_H,\\{}^mB_k^3(t)&=\left({}^m\overline{z}_k(t),\overline{v}_k(t)\right)_{\tau_kH}-\left({}^m\overline{z}_k(t),\overline{v}_k(t)\right)_{\omega_kH},\\{}^mC_{kM}^1(t)&=a_M\left({}^m\hat{u}_k(t),\overline{v}_k(t)\right)-a_M\left({}^m\hat{u}_k(t),\hat{v}_k(t)\right),\\{}^mC_{kM}^2(t)&=a_M\left({}^m\overline{u}_k(t),\overline{v}_k(t)\right)-a_M\left({}^m\hat{u}_k(t),\overline{v}_k(t)\right),\\{}^mC_{kM}^3(t)&=a_{\tau_kM}\left({}^m\overline{u}_k(t),\overline{v}_k(t)\right)-a_{\omega_kM}\left({}^m\overline{u}_k(t),\overline{v}_k(t)\right),\\{}^mC_{kM}^4(t)&=a_{M_n}^I\left(\overline{u}_k(t),\overline{v}_k(t)\right)-a_{M_n}\left(\overline{u}_k(t),\overline{v}_k(t)\right),\\{}^mC_k^l(t)&=\sum_{M=H,P}C_{kM}^l,\qquad l=1,2,3,4,\\{}^mD_{kM}^1(t)&=\left({}^mf_k^M(t),\overline{v}_{kM}(t)\right)-\left({}^mf_k^M(t),\hat{v}_{kM}(t)\right),\\{}^mD_{kM}^2(t)&=\left({}^mf_k^M(t),\overline{v}_{kM}(t)\right)_{\tau_kM}-\left({}^mf_k^M(t),\overline{v}_{kM}(t)\right)_{\omega_kM},\\{}^mD_{kM}^3(t)&=\left({}^mf_k^M(t),v_{kM}(t)\right)_{M_k}^I-\left({}^mf_k^M(t),v_{kM}(t)\right)_{M_k},\\{}^mD_k^l(t)&=\sum_{M=H,P}D_{kM}^l,\qquad l=1,2,3.\end{aligned}$$

C) Now, we shall prove the relations

$$(6.13)\qquad \text{a)}\ \lim_{k\to\infty}\int_0^T{}^mA_k^l(t)=0,\qquad l=1,2,3,$$

$$(6.14)\qquad \text{b)}\ \lim_{k\to\infty}\int_0^T{}^mB_k^l(t)=0,\qquad l=1,2,3,$$

$$\text{(6.15)} \qquad \text{c)} \quad \lim_{k\to\infty} \int_0^T {}^mC_k^l(t) = 0, \qquad l = 1,2,3,4,$$

$$\text{(6.16)} \qquad \text{d)} \quad \lim_{k\to\infty} \int_0^T {}^mD_k^l(t) = 0, \qquad l = 1,2,3.$$

a) From (5.4), (6.8), (5.1)–(5.3) we estimate

$$\begin{aligned}
\Big|\int_0^T {}^mA_k^1(t)\Big| &\leqslant \int_0^T \|{}^m\hat{s}_k(t)\|_{0,H}\|{}^m\overline{v}_k(t) - \hat{v}_k(t)\|_{0,H}\,\mathrm{d}t \\
&\leqslant C\int_0^T \|\overline{v}_k(t) - \hat{v}_k(t)\|_{0,H}\,\mathrm{d}t \leqslant Ch_k\int_0^T \|v_k\|_{0,H_k} \leqslant Ch_k, \\
\Big|\int_0^T {}^mA_k^2(t)\Big| &\leqslant \int_0^T \|{}^m\overline{v}_k(t) - {}^m\hat{v}_k(t)\|_{0,H}\|\overline{v}_k(t)\|_{0,H}\,\mathrm{d}t \\
&\leqslant Ch_k\int_0^T \|{}^m\overline{s}_k(t)\|_{0,H_k}\,\mathrm{d}t \leqslant Ch_k, \\
\Big|\int_0^T {}^mA_k^3(t)\Big| &\leqslant \sum_{\varepsilon=\tau,\omega}\int_0^T \|\overline{s}_k(t)\|_{0,\varepsilon_{kH}}\|\overline{v}_k(t)\|_{0,\varepsilon_{kH}}\,\mathrm{d}t \\
&\leqslant Ch_k\sum_{\varepsilon=\tau,\omega}\int_0^T \|\overline{s}_k(t)\|_{H_k}\|v_k(t)\|_{0,H_k}\,\mathrm{d}t \leqslant Ch_k.
\end{aligned}$$

b) We can prove the relation (6.14) analogously.

c) Similarly we obtain

$$\text{(6.17)} \qquad \Big|\int_0^T {}^mC_{kM}^1(t)\Big| \leqslant C\int_0^T \|\overline{v}_k(t) - \hat{v}_k(t)\|_{1,M} \leqslant Ch_k,$$

$$\text{(6.18)} \qquad \Big|\int_0^T {}^mC_{kM}^2(t)\Big| \leqslant C\int_0^T \|\overline{u}_k(t) - \hat{u}_k(t)\|_{1,M} \leqslant Ch_k,$$

$$\begin{aligned}
\Big|\int_0^T {}^mC_{kM}^3(t)\Big| &\leqslant \sum_{\varepsilon=\tau,\omega}\int_0^T \|\overline{u}_k(t)\|_{1,\varepsilon_{kM}}\|\overline{v}_k(t)\|_{1,\varepsilon_{kM}}\,\mathrm{d}t \\
\text{(6.19)} \qquad &\leqslant Ch_k\sum_{\varepsilon=\tau,\omega}\int_0^T \|\overline{u}_k(t)\|_{1,M_k}\|v_k(t)\|_{1,\varepsilon_{M_k}}\,\mathrm{d}t \leqslant Ch_k
\end{aligned}$$

and from (3.2) we have

$$\text{(6.20)} \qquad \Big|\int_0^T {}^mC_{kM}^4(t)\Big| \leqslant Ch_k\int_0^T \|{}^m\overline{u}_k(t)\|_{1,M_k}\|v_k(t)\|_{1,M_k}\,\mathrm{d}t \leqslant Ch_k.$$

The relations (6.17)–(6.20) prove (6.15).

d) It can be proved analogously.

D) Integrating the relation (6.9) from 0 to $T$, passing to the limit for $k \to \infty$ and using (6.2)–(6.6), (6.13)–(6.16) and the inequality

$$\|f_n^M - f^M\|_{L_2\left(I, W^1_\infty(\tilde\Omega_M)\right)} \leqslant \Delta t_n \|\dot f^m\|_{L_2\left(I, W^1_\infty(\tilde\Omega_M)\right)} \tag{6.21}$$

we obtain (6.7).

E) Let ${}^m u_1, {}^m u_2$ be two functions satisfying (6.1), (6.7), then for the difference ${}^m u := {}^m u_1 - {}^m u_2$ we obtain, substituting $v = \chi \dot u \in L_2(I, V)$ in (6.7), where $\chi$ is the characteristic function of the interval $(0, t)$, the equation

$$\begin{aligned}
&\int_0^t \{({}^m\ddot u(\tau), {}^m\dot u(\tau))_H + ({}^m\dot u(\tau), {}^m\dot u(\tau))_P + a({}^m u(\tau), {}^m u(\tau))\}\,\mathrm{d}\tau \\
&= \frac{1}{2}\|{}^m\dot u(t)\|^2_{0,H} + \int_0^t \|{}^m\dot u(\tau)\|^2_{0,P}\,\mathrm{d}\tau + \frac{1}{2}a({}^m u(t), {}^m u(t)) = 0.
\end{aligned}$$

This relation and the V-ellipticity of the form $a(.,.)$ imply

$$^m u(t) = 0 \qquad \forall t \in \bar I.$$

□

**Lemma 6.3.** *Let the assumptions of Problem 4.3 be fulfilled. Then the sequence of finite element Rothe's functions $\{{}^m\hat U_n(t)\}_{n=1}^\infty$ fulfils*

$$^m\hat U_n \to {}^m u \quad in \quad C(\bar I, V). \tag{6.22}$$

Proof. A) From the uniqueness in Lemma 6.2 we easily find the equality $\{n_k\} = \{n\}$ in Lemma 6.1. Integrating (6.9) from 0 to $t$ with $k = n$, we find

$$\begin{aligned}
&\int_0^t \{({}^m\hat s_n(\tau), \hat v_n(\tau))_H + ({}^m\hat z_n(\tau), \hat v_n(\tau))_P + a({}^m\hat u_n(\tau), \hat v_n(\tau))\}\,\mathrm{d}\tau \\
&= \int_0^t ({}^m f_n(\tau), \hat v_n(\tau))\,\mathrm{d}\tau + \int_0^t {}^m E_n(\tau)\,\mathrm{d}\tau \qquad \forall v_n \in L_2(I, V_n),
\end{aligned} \tag{6.23}$$

where

$$^m E_n(t) = \sum_{l=1}^3 {}^m D_n^l(t) - \sum_{l=1}^3 {}^m A_n^l(t) - \sum_{l=1}^3 {}^m B_n^l(t) - \sum_{l=1}^4 {}^m C_n^l(t). \tag{6.24}$$

There exists a sequence $\{{}^m w_n\}_{n=1}^{\infty}$, ${}^m w_n \in L_2(I, V_n)$ such that

$$\lim_{n\to\infty} \|{}^m w_n - {}^m\tilde{w}\|_{L_2(I,H^1(\Omega_n))} = 0, \tag{6.25}$$

where ${}^m w = {}^m\dot{u}$ and ${}^m\tilde{w}$ denotes its Calderon extension.

Let us put

$${}^m v_n := {}^m z_n - {}^m w_n \in L_2(I, V_n).$$

Setting $v_n = {}^m v_n$ in (6.23) we obtain

$$\lim_{n\to\infty} \int_0^T |{}^m E_n(t)|\,\mathrm{d}t = 0, \tag{6.26}$$

since the sequence $\{\|{}^m v_n\|_{L_2(I,V_n)}\}_{n=1}^{\infty}$ is bounded.

B) From (6.23) with $v_n = {}^m v_n$ let us subtract the equation

$$\int_0^t \{({}^m\ddot{u}(\tau), {}^m\hat{v}_n(\tau))_H + ({}^m\dot{u}(\tau), {}^m\hat{v}_n(\tau))_P + a({}^m u(\tau), {}^m\hat{v}_n(\tau))\}\,\mathrm{d}\tau$$
$$= \int_0^t ({}^m f(\tau), {}^m\hat{v}_n(\tau))\,\mathrm{d}\tau,$$

which we obtain from (6.7) by substituting $v = \chi^m\hat{v}_n$. After elementary transformations we find

$$\begin{aligned}
&\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{Z}_n(s) - {}^m\dot{u}(s)), {}^m\hat{Z}_n(s) - {}^m\dot{u}(s)\Big)_H \mathrm{d}s \\
&\quad + \int_0^t ({}^m\hat{z}_n(s) - {}^m\dot{u}(s), {}^m\hat{z}_n(s) - {}^m\dot{u}(s))_P\,\mathrm{d}s \\
&\quad + \int_0^t a\Big({}^m\hat{U}_n(s) - {}^m u(s), \frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{U}_n(s) - {}^m u(s))\Big)\,\mathrm{d}s \\
&= \int_0^t {}^m E_n(s)\,\mathrm{d}s - \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{Z}_n(s) - {}^m\dot{u}(s)), {}^m\dot{u}(s) - {}^m\hat{w}_n(s)\Big)_H \mathrm{d}s \\
&\quad - \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{Z}_n(s) - {}^m\dot{u}(s)), {}^m\hat{z}_n(s) - {}^m\hat{Z}_n(s)\Big)_H \mathrm{d}s \\
&\quad - \int_0^t ({}^m\hat{z}_n(s) - {}^m\dot{u}(s), {}^m\dot{u}(s) - {}^m\hat{w}_n(s))_P\,\mathrm{d}s \\
&\quad - \int_0^t a({}^m\hat{u}_n(s) - {}^m\hat{U}_n(s), {}^m\hat{z}_n(s) - {}^m\dot{u}(s))\,\mathrm{d}s \\
&\quad - \int_0^t a({}^m\hat{u}_n(s) - {}^m u(s), {}^m\dot{u}(s) - {}^m\hat{w}_n(s))\,\mathrm{d}s \\
&\quad + \int_0^t ({}^m f_n(s) - {}^m f(s), {}^m\hat{z}_n(s) - {}^m\hat{w}_n(s))\,\mathrm{d}s\,.
\end{aligned} \tag{6.27}$$

Now we estimate the terms on the right-hand side. By (6.6) and (5.4) we have

$$(6.28)\qquad \int_0^t \left|\left(\frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{Z}_n(s) - {}^m\dot{u}(s)), {}^m\hat{z}_n(s) - {}^m\hat{Z}_n(s)\right)_H\right| \mathrm{d}s \leqslant \|{}^m\hat{s}_n - {}^m\ddot{u}\|_{L_2(I,L_2(\Omega_H))} \|{}^m\hat{z}_n - {}^m\hat{Z}_n\|_{L_2(I,L_2(\Omega_H))} \leqslant C\Delta t_n^{1/2}$$

and analogously

$$(6.29)\qquad \int_0^t \left|\left(\frac{\mathrm{d}}{\mathrm{d}s}({}^m\hat{Z}_n(s) - {}^m\dot{u}(s)), {}^m\dot{u}(s) - {}^m\hat{w}_n(s)\right)_H\right| \mathrm{d}s \leqslant C\|{}^m\dot{u} - {}^m\hat{w}_n\|_{L_2(I,L_2(\Omega_H))}.$$

By (6.4),

$$(6.30)\qquad \int_0^t |({}^m\hat{z}_n(s) - {}^m\dot{u}(s), {}^m\dot{u}(s) - {}^m w_n(s))_P| \,\mathrm{d}s \leqslant C\|{}^m\dot{u} - {}^m w_n(s)\|_{L_2(I,L_2(\Omega_P))}.$$

From (6.4), (5.5) it follows that

$$(6.31)\qquad \int_0^t |a({}^m\hat{u}_n(s) - {}^m\hat{U}_n(s), {}^m z_n(s) - {}^m\dot{u}(s))| \,\mathrm{d}s \leqslant C\Delta t_n^{1/2}$$

and by (6.3) we have

$$(6.32)\qquad \int_0^t |a({}^m\hat{U}_n(s) - {}^m u(s), {}^m\dot{u}(s) - {}^m w_n(s))| \,\mathrm{d}s \leqslant C\|{}^m\dot{u} - {}^m\hat{w}_n\|.$$

Finally, by (6.21) we have

$$\int_0^t |({}^m f_n^M(s) - {}^m f^M(s), {}^m\hat{z}_n(s) - {}^m\hat{w}_n(s))_M| \leqslant C\|{}^m f_n^M - {}^m f^M\|_{L_2(I,W^1_\infty(\bar{\Omega}_M))} \leqslant C\Delta t_n \quad (M = H, P).$$

C) Using the integration by parts on the left-hand side of (6.27), we obtain using (6.28)–(6.32)

$$(6.33)\qquad \|{}^m\hat{Z}_n(t) - {}^m\dot{u}(t)\|^2_{0,H} + \int_0^T \|{}^m\hat{z}_n(\tau) - {}^m\dot{u}(\tau)\|^2_{0,P} \,\mathrm{d}\tau + \|{}^m\hat{U}_n(t) - {}^m u(t)\|_1^2 \leqslant C\Big\{ \|{}^m\hat{Z}_n^0 - {}^m\hat{z}^0\|^2_{0,P} + \|{}^m\hat{U}_n^0 - {}^m u^0\|_1^2 + \Delta t_n^{1/2} + \int_0^T |{}^m E_n(\tau)| \,\mathrm{d}\tau + \|{}^m\dot{u} - {}^m\hat{w}_n\|_{L_2(I,V)} \Big\} \quad \forall t \in \bar{I}.$$

By (4.14), (4.11), (4.12), (4.9), (6.25), (6.26), the result follows, since all the terms on the right-hand side of (6.33) tend to zero. □

Finally, we shall use the following standard inequality (see e.g. [4]):

**Lemma 6.4.** *For all natural numbers $r$, $s$ and for all $t \in \overline{I}$ we have*

$$\begin{aligned}(6.34)\qquad & \|{}^r\dot{u}(t) - {}^s\dot{u}(t)\|_{0,H}^2 + \int_0^T \|{}^r\dot{u}(\tau) - {}^s\dot{u}(\tau)\|_{0,P}^2 \,\mathrm{d}\tau \\ & + \|{}^r u(t) - {}^s u(t)\|_1^2 \\ & \leqslant C\Big\{ \|{}^r U^0 - {}^s U^0\|_1^2 + \|{}^r Z^0 - {}^s Z^0\|_{0,P}^2 \\ & + \sum_{M=H,P} \|{}^r f^M - {}^s f^M\|_{L_2(I, W_\infty^1(\tilde{\Omega}_M))} \Big\},\end{aligned}$$

*where ${}^r u$ and ${}^s u$ are the functions from Lemmas 6.1–6.3 for $m = r$ and $m = s$.*

In what follows we shall need the following definitions of Rothe's functions:

$$\begin{aligned}\hat{U}_n(t) &= \hat{U}_n^{i-1} + \delta_n \hat{U}_n^i (t - t_{i-1}) \qquad t \in [t_{i-1}, t_i] \quad (i = 1, \ldots, r_n), \\ \hat{Z}_{nP}(t) &= \hat{Z}_{nP}^{i-1} + \delta_n \hat{Z}_{nP}^i (t - t_{i-1}) \qquad t \in [t_{i-1}, t_i] \quad (i = 1, \ldots, r_n),\end{aligned}$$

where $U_n^i$ is the solution of Problem 3.2 and $Z_n^i := \delta_n U_n^i$.

Now, we can formulate the main result of this paper:

**Theorem 6.5.** *Let $u^0 \in V$ and $z_H^0 \in L_2(\Omega_H)$, $f^M$, $\dot{f}^M \in L_2(I, W_\infty^1(\tilde{\Omega}_M))$ ($M = H, P$). Then there exists a unique solution $u \in C(\overline{I}, V)$ of Problem 2.1 and it satisfies*

$$(6.35)\qquad \hat{U}_n \to u \quad \text{in} \quad C(\overline{I}, V),$$

$$(6.36)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_n \rightharpoonup \dot{u} \quad \text{weakly in} \quad L_2(I, L_2(\Omega)),$$

$$(6.37)\qquad \hat{Z}_{nH} \to \dot{u}_H \quad \text{in} \quad C(\overline{I}, L_2(\Omega_H)).$$

Proof. A) By (6.34) there exist functions $u \in C(\overline{I}, V)$ and $z \in L_2(I, L_2(\Omega))$ with its H-component $z_H := z_{|\Omega_H} \in C(\overline{I}, L_2(\Omega_H))$ such that

$$(6.38)\qquad {}^m u \to u \quad \text{in} \quad C(\overline{I}, V),$$

$$(6.39)\qquad {}^m\dot{u} \to z \quad \text{in} \quad L_2(I, L_2(\Omega)),$$

$$(6.40)\qquad {}^m\dot{u}_H \to z_H \quad \text{in} \quad C(\overline{I}, L_2(\Omega_H)).$$

We have

$$(6.41)\qquad {}^m u(t) = {}^m u(0) + \int_0^t {}^m\dot{u}(\tau)\,\mathrm{d}\tau \qquad \forall t \in \overline{I}.$$

Passing to the limit in (6.41), we obtain

$$u(t) = u(0) + \int_0^t z(\tau)\,\mathrm{d}\tau \qquad \forall t \in \bar{I}.$$

Thus, we have

$$\dot{u}(t) = z(t) \qquad \text{almost everywhere in} \quad I.$$

From (6.7) we can derive similarly as in [9, Theorem 3.10] that

$$\|{}^m\ddot{u}_H\|_{L_2(I,V_H^*)} \leqslant C \quad \forall m.$$

By the reflexivity of $L_2(I, V_H^*)$ there exist a subsequence $\{{}^k\ddot{u}_H\}$ of $\{{}^m\ddot{u}_H\}$ and a function $g \in L_2(I, V_H^*)$ such that

$${}^k\ddot{u}_H \rightharpoonup g \qquad \text{weakly in} \quad L_2(I, V_H^*).$$

Let us write (6.7) only for the subsequence $\{k\}$ and let us pass to the limit for $k \to \infty$. Then we obtain

$$\int_0^T \{\langle g(\tau), v(\tau)\rangle_H + \big(\dot{u}(\tau), v(\tau)\big)_P + a\big(u(\tau), v(\tau)\big)\}\,\mathrm{d}\tau$$
$$= \int_0^T \big(f(\tau), v(\tau)\big)\,\mathrm{d}\tau \quad \forall v \in L_2(I, V).$$

We can write

$$({}^k\dot{u}_H(t), v_H)_H - ({}^kz_H^0(t), v_H)_H = \int_0^t ({}^k\ddot{u}_H(\tau), v_H)_H\,\mathrm{d}\tau \qquad \forall v_H \in V_H \quad \forall t \in \bar{I}.$$

Passing to the limit for $k \to \infty$, we obtain

$$\langle \dot{u}_H(t), v_H\rangle_H - \big\langle z_H^0(t), v_H\big\rangle_H$$
$$= \int_0^t \langle g(\tau), v_H\rangle_H\,\mathrm{d}\tau \qquad \forall v_H \in V_H \quad \forall t \in \bar{I}.$$

This fact and the equality $\int_0^t \langle g(\tau), v_H\rangle_H\,\mathrm{d}\tau = \big\langle \int_0^t g(\tau), v_H\big\rangle_H\,\mathrm{d}\tau$ imply

(6.42) $$\dot{u}_H(t) = z_H^0 + \int_0^t g(\tau)\,\mathrm{d}\tau \qquad \forall t \in \bar{I}.$$

Differentiating (6.42), we find

$$\ddot{u}_H(t) = g(t) \qquad \text{a.e. in} \quad I.$$

The existence of the solution of Problem 2.1 is proved.

B) Now we shall prove the uniqueness. Let $u_1$, $u_2$ be two solutions of Problem 2.1. Then $u := u_1 - u_2$ fulfils

$$(6.43)\int_0^T \{\langle \ddot{u}(t), v(t)\rangle_H + \big(\dot{u}(t), v(\tau)\big)_P + a\big(u(t), v(t)\big)\}\,\mathrm{d}t = 0 \quad \forall v \in L_2(I, V),$$
$$u(0) = 0,$$
$$\dot{u}_H(0) = 0.$$

Let us choose $s \in \overline{I}$ arbitrarily and let us set

$$v_s(t) = \begin{cases} \int_s^t u(\tau)\,\mathrm{d}\tau & 0 \leqslant t \leqslant s, \\ 0 & s \leqslant t \leqslant T. \end{cases}$$

Then

$$(6.44)\qquad \int_0^T a\big(u(\tau), v_s(\tau)\big)\,\mathrm{d}\tau = \int_0^s a\big(\dot{v}_s(\tau), v_s(\tau)\big)\,\mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}\tau} a\big(v_s(\tau), v_s(\tau)\big)\,\mathrm{d}\tau = -\frac{1}{2}a\big(v_s(0), v_s(0)\big),$$

$$(6.45)\qquad \int_0^T \big(\dot{u}_H(\tau), \dot{v}_{sH}(\tau)\big)_H\,\mathrm{d}\tau = \int_0^s \big(\dot{u}_H(\tau), u_H(\tau)\big)_H\,\mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}\tau}\big(u_H(\tau), u_H(\tau)\big)_H\,\mathrm{d}\tau = \frac{1}{2}\|u_H(s)\|_{0,H}^2,$$

$$(6.46)\int_0^T \big(\dot{u}_P(\tau), v_s(\tau)\big)_P\,\mathrm{d}\tau = [(u_P(\tau), v_{sP}(\tau))_P]_0^T - \int_0^T \big(u_P(\tau), \dot{v}_{sP}(\tau)\big)_P\,\mathrm{d}\tau$$
$$= -\int_0^T \|u_P(\tau)\|_{0,P}^2\,\mathrm{d}\tau\,.$$

Let us set $v = v_s$ in (6.43). Then according to (6.44)–(6.46), we obtain

$$\frac{1}{2}a\big(v_s(0), v_s(0)\big) + \frac{1}{2}\|u_H(t)\|_{0,H}^2 + \int_0^T \|u_P(\tau)\|_{0,P}^2\,\mathrm{d}\tau = 0.$$

Since $s \in \overline{I}$ was chosen arbitrarily, we have

$$a\big(v_s(0), v_s(0)\big) = 0 \qquad \forall s \in \overline{I}.$$

Thus

$$u(t) = 0 \qquad \text{a.e. in} \quad I.$$

C) We shall prove the relations (6.35)–(6.37). Let ${}^mU_n^i$ be the solution of Problem 4.3 and let $U_n^i$ be the solution of Problem 3.2. Similarly as in Lemma 4.5 we obtain the inequality

$$\|{}^mZ_n^j - Z_n^j\|_{0,H_n}^2 + \Delta t_n \sum_{i=1}^{j} \|{}^mZ_n^j - Z_n^j\|_{0,P_n}^2 + \|{}^mU_n^j - U_n^j\|_{1,\Omega_n}^2$$
$$\leqslant C\Big\{\|{}^mU_n^0 - U_n^0\|_{1,\Omega_n}^2 + \|{}^mZ_n^0 - Z_n^0\|_{0,H_n}^2$$
$$+ \sum_{M=H,P} \Big(\|{}^mf^M - f^M\|_{C(\bar{I},W_\infty^1(\bar{\Omega}_M))} + \|{}^m\dot{f}^M - \dot{f}^M\|_{L_2(I,W_\infty^1(\bar{\Omega}_M))}\Big)\Big\}.$$

By virtue of (5.1)

$$\begin{aligned} &\|{}^m\hat{Z}_{nH}(t) - \hat{Z}_{nH}(t)\|_{0,H} \leqslant \alpha_n + \beta_m, \\ &\int_0^T \|{}^mz_{nP}(t) - z_{nP}(t)\|_{0,P}^2 \leqslant \alpha_n + \beta_m, \\ &\|{}^m\hat{U}_n(t) - \hat{U}_n(t)\|_1 \leqslant \alpha_n + \beta_m, \end{aligned} \tag{6.47}$$

where

$$\beta_m = C\Big\{\|{}^mU^0 - u^0\|_1 + \|{}^mZ_H^0 - z_H^0\|_{0,H}$$
$$+ \sum_{M=H,P} \Big(\|{}^mf^M - f^M\|_{C(\bar{I},W_\infty^1(\bar{\Omega}_M))} + \|{}^m\dot{f}^M - \dot{f}^M\|_{L_2(I,W_\infty^1(\bar{\Omega}_M))}\Big)\Big\} \quad \forall m \in \mathbb{N}$$

and

$$\alpha_n = \alpha_n(m) = C\{\|{}^mU_n^0 - {}^mU_C^0\|_{1,\Omega_n} + \|u_C^0 - U_n^0\|_{1,\Omega_n}$$
$$+ \|{}^mZ_{nH}^0 - {}^m\tilde{Z}_{nH}^0\|_{0,H_n} + \|{}^m\tilde{z}_{nH}^0 - {}^mz_{nH}^0\|_{0,H_n}\} \quad \forall m, n \in \mathbb{N}.$$

Here we denote by ${}^mU_C^0$, $u_C^0$ the Calderon extensions of the functions ${}^mU^0$, $u^0 \in V$ and by ${}^m\tilde{Z}_H^0$, $\tilde{z}_H^0$ the extensions by zero of the functions ${}^mZ_H^0$, $z_H^0 \in L_2(\Omega_H)$. Now (4.2)–(4.5) yield

$$\lim_{m\to\infty} \beta_m = 0 \tag{6.48}$$

and for every fixed $m$ by (4.14), (4.15) we have

$$\lim_{n\to\infty} \alpha_n = 0.$$

Passing to the limit in (6.34) and using (4.2)–(4.5), (6.38)–(6.40), we obtain

$$\|{}^m\dot{u}_H(t) - \dot{u}_H(t)\|_{0,H} \leqslant \beta_m \qquad \forall t \in \bar{I} \qquad \forall m \in \mathbb{N},$$
$$\int_0^T \|{}^m\dot{u}_P(t) - \dot{u}_P(t)\|_{0,P}^2 \, \mathrm{d}t \leqslant \beta_m \qquad \forall m \in \mathbb{N},$$
$$\|{}^m u(t) - u(t)\|_1 \leqslant \beta_m \qquad \forall t \in \bar{I} \qquad \forall m \in \mathbb{N}. \tag{6.49}$$

Let us prove (6.35): Let $\varepsilon > 0$ be chosen arbitrarily. First we find, according to (6.49), such a natural number $m_1$ that

$$\|{}^m u(t) - u(t)\|_1 \leqslant \frac{\varepsilon}{3} \qquad \forall t \in \bar{I} \qquad \forall m \geqslant m_1. \tag{6.50}$$

Futher, let $m_0 \geqslant m_1$ and $n_1 \in \mathbb{N}$ be such numbers that by (6.48), (6.47) we have

$$\|{}^{m_0}\hat{U}_n(t) - \hat{U}_n(t)\|_1 \leqslant \frac{\varepsilon}{3} \qquad \forall t \in \bar{I} \qquad \forall n \geqslant n_1. \tag{6.51}$$

Finally, by (6.22) we can find $n_0 \geqslant n_1$ such that

$$\|{}^{m_0}\hat{U}_n(t) - {}^{m_0}u(t)\|_1 \leqslant \frac{\varepsilon}{3} \qquad \forall t \in \bar{I} \qquad \forall n \geqslant n_0. \tag{6.52}$$

Thus according to (6.50)–(6.52) we obtain

$$\begin{aligned}\|\hat{U}_n(t) - u(t)\|_1 &\leqslant \|\hat{U}_n(t) - {}^{m_0}\hat{U}_n(t)\|_1 \\ &\quad + \|{}^{m_0}\hat{U}_n(t) - {}^{m_0}u(t)\|_1 + \|{}^{m_0}u(t) - u(t)\|_1 \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \qquad \forall t \in \bar{I} \qquad \forall n \geqslant n_0.\end{aligned}$$

The relations (6.36) and (6.37) can be proved analogously. □

*References*


[1] *Demerdash, N.A., Gillot, D.H.*: A new approach for determination of eddy current and flux penetration in nonlinear ferromagnetic materials. IEEE Trans. MAG-10 (1974), 682–685.

[2] *Gajewski, H., Gröger, K., Zacharias, K.*: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie, Berlin, 1974.

[3] *Hannalla, A.Y., Macdonald, D.C.*: Numerical analysis of transient field problems in electrical machines. Proc. IEE *123* (1976), 893–898.

[4] *Kačur, J.*: Method of Rothe in Evolution Equations. Teubner-Texte zur Mathematik, Band 80. Leipzig, 1985.

[5] *Lütke-Daldrup, B.*: Numerische Lösung zweidimensionaler nichtlinearer instantionärer Feld- und Wirbelstromprobleme. Archiv für Elektrotechnik *68* (1985), 223–228.

[6] *Melkes, F., Zlámal M.*: Numerical solution of nonlinear quasistationary magnetic fields. Internat. J. Numer. Methods Engineering *19* (1983), 1053–1062.
[7] *Zlámal, M.*: Curved elements in the finite element method. I. SIAM J. Numer. Anal. *10* (1973), 229–240.
[8] *Zlámal, M.*: Finite element solution of quasistationary nonlinear magnetic field. RAIRO Anal. Numér. *16* (1982), 161–191.
[9] *Ženíšek A.*: Finite element variational crimes in parabolic-elliptic problems. Numer. Math. *55* (1989), 343–376.
[10] *Ženíšek, A.*: Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations. Academic Press, London, 1990.

*Author's address*: Rudolf Hlavička, Technical University Brno, Technická 2, 616 69 Brno, Czech Republic.