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BIQUADRATIC SPLINES INTERPOLATING MEAN VALUES

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Summary. Continuity conditions for a biquadratic spline interpolating given mean values in terms of proper parameters are given. Boundary conditions determining such a spline and the algorithm for computing local parameters for the given data are studied. The notion of the natural spline and its extremal property is mentioned.

Keywords: splines, biquadratic splines, mean value interpolation

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1. Quadratic splines—basic relations

Let us have an increasing sequence of spline knots

$$(\Delta x) := \{x_i\,;\ a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\}, \qquad h_i = x_{i+1} - x_i.$$

We call a function $s(x)$ *a quadratic spline on the knot set* (Δx) if it has the following properties:

1° $s(x) \in \mathbb{C}^1\,\langle a, b\rangle$;

2° $s(x)$ is a quadratic polynomial on every interval $\langle x_i, x_{i+1}\rangle$, $i = 0(1)n$.

Let us denote by $\mathscr{S}(\Delta x)$ the linear space of such splines; we have

$$\dim \mathscr{S}(\Delta x) = n + 3.$$

Quadratic splines interpolating function values, mean values or values of the first derivative were described in [2]–[7]; we recall here only the necessary relations.

Let us denote $s_i = s(x_i)$, $s'_i = s'(x_i)$, $g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x)\,\mathrm{d}x$. We can use various local representations of $s(x)$ on the intervals $\langle x_i, x_{i+1}\rangle$—e.g. with the local parameters s_i, s'_i:

$$(1) \qquad s(x) = s_i + (x - x_i)s'_i + \tfrac{1}{2}(x - x_i)^2 s''_i \quad \text{with } s''_i = \frac{1}{h_i}(s'_{i+1} - s'_i)$$

or

$$(2)\qquad s(x) = (1-q^2)s_i + q^2 s_{i+1} + q(1-q)h_i s_i', \qquad q = (x-x_i)/h_i.$$

We can obtain a representation with parameters s_i, g_i substituting

$$s_i' = \frac{2}{h_i}(3g_i - s_{i+1} - 2s_i), \quad s_{i+1}' = \frac{2}{h_i}(-3g_i + 2s_{i+1} + s_i)$$

or

$$(3)\qquad \tfrac{1}{2}s_i'' = \tfrac{3}{4}(s_{i+1} - 2g_i + s_i).$$

The continuity condition 1° at the knots x_i, $i = 1(1)n$ can be expressed as

$$(4)\qquad \frac{1}{2}(s_{i-1}' + s_i') = \frac{1}{h_{i-1}}(s_i - s_{i-1}) \quad \text{via parameters } s_i,\ s_i';$$

or through parameters s_i, g_i as

$$(5a)\qquad h_i s_{i-1} + 2(h_{i-1} + h_i)s_i + h_{i-1}s_{i+1} = 3[h_i g_{i-1} + h_{i-1}g_i];$$

or in some slightly modified more symmetric form as

$$(5b)\qquad \frac{1}{h_{i-1}}s_{i-1} + 2\Big(\frac{1}{h_{i-1}} + \frac{1}{h_i}\Big) + \frac{1}{h_i}s_{i+1} = 3\Big[\frac{1}{h_{i-1}}g_{i-1} + \frac{1}{h_i}g_i\Big];$$

or by means of the parameters s_i', g_i as

$$(6)\qquad h_{i-1}s_{i-1}' + 2(h_{i-1} + h_i)s_i' + h_i s_{i+1}' = 6(g_i - g_{i-1}).$$

The continuity conditions (4) completed by one (initial) condition can be used to find all local parameters of the spline interpolating given function values or the first derivative values on (Δx) (see [1], [4]).

The continuity conditions (6) were used to calculate local parameters of the spline interpolating given mean values g_i under two additional (boundary) conditions (see [7]). Let us mention the feature of error propagation without damping which is connected with splines interpolating function values or the first derivative values at the knot mesh (Δx), but not with splines interpolating mean values (see [3], [7]). The instability in function values interpolation can be overwhelmed by a proper choice of the points of interpolation t_i different from the knots of the spline x_i. Such quadratic splines with $x_i < t_i < x_{i+1}$ were discussed in [2], [3].

2. Biquadratic spline

Let us have a rectangular domain $D = \{(x,y);\, a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$ with the knot set

$$(\Delta) = (\Delta x) \times (\Delta y) = \{(x_i, y_j);\, i = 0(1)n+1,\ j = 0(1)m+1\},$$

where $(\Delta y) = \{y_j : c = y_0 < y_1 < \ldots < y_m < y_{m+1} = d\}$, $k_j = y_{j+1} - y_j$.

We call $s(x,y)$ *a biquadratic spline on the knot set* (Δ), if

1° $s(x,y) \in \mathrm{C}^{1,1}(D)$ (continuous first derivatives $s^{k,r}$, $s^{k,r} = \partial^{k+r}s/\partial x^k \partial y^r$, $k,r = 0,1$);

2° $s(x,y)$ is a biquadratic polynomial on each subrectangle

$$D_{ij} = \langle x_i, x_{i+1}\rangle \times \langle y_j, y_{j+1}\rangle .$$

Let us denote by $\mathscr{S}(\Delta)$ the linear space of biquadratic splines on (Δ); then

$$\dim \mathscr{S}(\Delta) = (n+3)(m+3).$$

Corollary.

$s(x, y_j)$, $j = 0(1)n+1$ *are quadratic splines on the mesh* (Δx),
$s(x_i, y)$, $i = 0(1)n+1$ *are quadratic splines on the mesh* (Δy).

There are various local representations for $s(x,y)$ on D_{ij}:

$$\text{(T)} \qquad s(x,y) = \sum_{k,r=0}^{2} a_{k,r}^{ij}(x - x_i)^k (y - y_j)^r$$

with

$$\text{(7)} \qquad a_{k,r}^{ij} = \frac{1}{k!r!} s^{k,r}(x_i, y_j) = \frac{1}{k!r!} \frac{\partial^{k+r}}{\partial x^k \partial y^r} s(x_i, y_j)$$

(the Taylor polynomial, $s_{ij} = s_{ij}^{0,0}$);

$$\begin{aligned} s(x,y) = {} & (1-u^2)(1-v^2)s_{ij} + u^2(1-v^2)s_{i+1,j} + (1-u^2)v^2 s_{i,j+1} \\ \text{(SD)} \qquad & + u^2v^2 s_{i+1,j+1} + u(1-u)h_i\big[(1-v^2)s_{ij}^{1,0} + v^2 s_{i,j+1}^{1,0}\big] \\ & + v(1-v)k_j\big[(1-u^2)s_{ij}^{0,1} + u^2 s_{i+1,j}^{0,1}\big] + u(1-u)v(1-v)h_i k_j s_{ij}^{1,1}, \end{aligned}$$

$$\text{(8)} \qquad (x,y) \in D_{ij},\ u = (x - x_i)/h_i,\ v = (y - y_j)/k_j,\ u,v \in \langle 0,1\rangle$$

(the tensor product technique applied to (1), (2)).

Both these representations work with nonsymmetrically distributed local parameters (at one point (x_i, y_j) for (T), at the vertices of D_{ij} for (SD)—see Fig. 1a).

Local parameters (SD) are used for interpolation of function values at knots in [8]; the interpolation at different points $P_{ij} = (x_i + u_i h_i, y_j + v_j k_j)$, $u_i, v_j \in (0,1)$ is studied in [5], where the appropriate choice of local parameters is used (see Fig. 1b) to obtain storage economy (symmetry, common parameters for neighbouring rectangles).

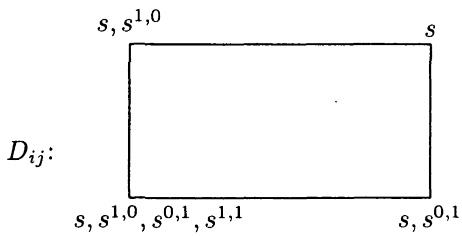

Fig. 1a

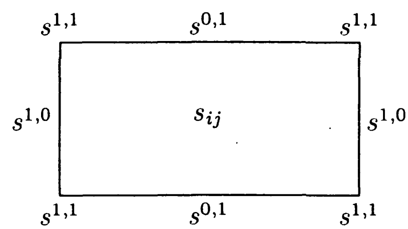

Fig. 1b

In the algorithm for splines interpolating mean values

$$
(9) \qquad g_{ij} = \frac{1}{h_i k_j} \iint_{D_{ij}} s(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{9}(4s_{ij} + 2s_{i+1,j} + 2s_{i,j+1} + s_{i+1,j+1}) + \frac{h_i}{18}(2s_{ij}^{1,0} + s_{i,j+1}^{1,0}) + \frac{k_j}{18}(2s_{ij}^{0,1} + s_{i+1,j}^{0,1}) + \frac{1}{36} h_i k_j s_{ij}^{1,1}
$$

we will use other local parameters (one-dimensional mean values)

$$
(10) \qquad \begin{aligned} g_{ij}^x &= \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x, y_j)\,\mathrm{d}x = \frac{1}{3}(2s_{ij} + s_{i+1,j} + \tfrac{1}{2}h_i s_{ij}^{1,0}) \\ g_{ij}^y &= \frac{1}{k_j}\int_{y_j}^{y_{j+1}} s(x_i, y)\,\mathrm{d}y = \frac{1}{3}(2s_{ij} + s_{i,j+1} + \tfrac{1}{2}k_j s_{ij}^{0,1}) \end{aligned}
$$

(the right-hand sides in (9)–(10) were calculated using (8)).

As the local parameters of the spline $s(x,y)$ on (Δ) we can now use the following nine values shown in Fig. 2a, which are symmetrically dispersed over D_{ij} (common values can be used for neighbouring rectangles).

Theorem 1. *A biquadratic spline $s(x,y)$ is uniquely determined on D_{ij} by the nine parameters shown in Fig. 2a:*

$$
(\mathrm{P}) \qquad s_{ij},\ s_{i+1,j},\ s_{i,j+1},\ s_{i+1,j+1},\ g_{ij}, g_{ij}^y,\ g_{i+1,j}^y,\ g_{ij}^x,\ g_{i,j+1}^x;
$$

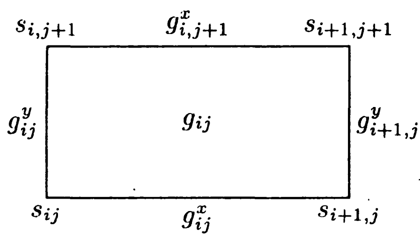

Fig. 2a

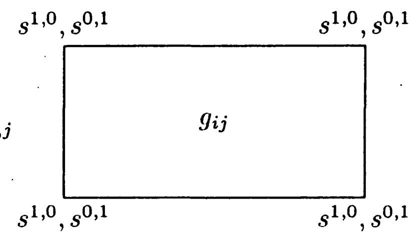

Fig. 2b

the corresponding local representation can be written as

$$
\text{(SG)}\qquad
\begin{aligned}
s(x,y) = {} & (1-u)(1-v)(1-3u-3v+9uv)s_{ij} \\
& + u(1-v)(-2+3u+6v-9uv)s_{i+1,j} \\
& + (1-u)v(-2+6u+3v-9uv)s_{i,j+1} \\
& + uv(4-6u-6v+9uv)s_{i+1,j+1} \\
& + 6(1-u)(1-v)\big[u(1-3v)g^x_{ij} + v(1-3u)g^y_{ij}\big] \\
& + 6uv\big[(1-u)(3v-2)g^x_{i,j+1} + (1-v)(3u-2)g^y_{i+1,j}\big] \\
& + 36u(1-u)v(1-v)g_{ij}
\end{aligned}
$$

with $u = (x - x_i)/h_i$, $v = (y - y_j)/k_j$.

P r o o f. There are four common function values in the representations (SD), (SG). The relations (9), (10) can be viewed as a one-to-one mapping between the local parameters of this two representations with a regular triangular matrix. Substituting (9), (10) into (SD), we obtain the representation (SG). □

R e m a r k s. 1° It is possible to show explicitly the relations between the parameters of representations (T) and (SG) (here we write $a^{ij}_{kr} = a_{kr}$, $h_i = h$, $k_j = k$ for short):

$$
\text{(11)}\qquad
\begin{aligned}
s_{ij} &= a_{00}, \\
s_{i+1,j} &= a_{00} + a_{10}h + a_{20}\tfrac{1}{2}h^2, \quad s_{i,j+1} = a_{00} + a_{01}k + a_{02}\tfrac{1}{2}k^2, \\
s_{i+1,j+1} &= a_{00} + a_{10}h + a_{01}k + \tfrac{1}{2}(a_{20}h^2 + 2a_{11}hk + a_{02}k^2 + a_{21}h^2k \\
&\qquad + a_{12}hk^2 + a_{22}\tfrac{1}{2}h^2k^2), \\
g^x_{ij} &= a_{00} + a_{10}\tfrac{1}{2}h + a_{20}\tfrac{1}{6}h^2, \quad g^y_{ij} = a_{00} + a_{01}\tfrac{1}{2}k + a_{02}\tfrac{1}{6}k^2,
\end{aligned}
$$

$$
\begin{aligned}
g^x_{i,j+1} &= a_{00} + a_{10}\tfrac{1}{2}h + a_{01}k + a_{20}\tfrac{1}{6}h^2 + a_{11}\tfrac{1}{2}hk + a_{02}\tfrac{1}{2}k^2 \\
&\quad + a_{21}\tfrac{1}{6}h^2k + a_{12}\tfrac{1}{4}hk^2 + a_{22}\tfrac{1}{12}h^2k^2, \\
g^y_{i+1,j} &= a_{00} + a_{10}h + a_{01}\tfrac{1}{2}k + a_{20}\tfrac{1}{2}h^2 + a_{11}\tfrac{1}{2}hk + a_{02}\tfrac{1}{6}k^2 \\
&\quad + a_{21}\tfrac{1}{4}h^2k + a_{12}\tfrac{1}{6}hk^2 + a_{22}\tfrac{1}{12}h^2k^2, \\
g_{ij} &= a_{00} + a_{10}\tfrac{1}{2}h + a_{01}\tfrac{1}{2}k + a_{20}\tfrac{1}{6}h^2 + a_{11}\tfrac{1}{4}hk + a_{02}\tfrac{1}{6}k^2 \\
&\quad + a_{21}\tfrac{1}{12}h^2k + a_{12}\tfrac{1}{12}hk^2 + a_{22}\tfrac{1}{36}h^2k^2.
\end{aligned}
$$

The inverse relations can be expressed in the following way:

$$
\begin{aligned}
(12)\qquad a_{00} &= s_{ij}, \\
a_{10} &= \tfrac{2}{h}(3g^x_{ij} - s_{i+1,j} - 2s_{ij}), \quad a_{01} = \tfrac{2}{k}(3g^y_{ij} - s_{i,j+1} - 2s_{ij}), \\
a_{20} &= \tfrac{6}{h^2}(-2g^x_{ij} + s_{i+1,j} + s_{ij}), \quad a_{02} = \tfrac{6}{k^2}(-2g^y_{ij} + s_{i,j+1} + s_{ij}), \\
a_{11} &= \tfrac{4}{hk}(9g_{ij} + 4s_{ij} + 2s_{i+1,j} + 2s_{i,j+1} + s_{i+1,j+1} \\
&\quad - 6g^x_{ij} - 3g^x_{i,j+1} - 6g^y_{ij} - 3g^y_{i+1,j}), \\
a_{21} &= -\tfrac{12}{h^2k}(6g_{ij} + 2s_{ij} + 2s_{i+1,j} + s_{i,j+1} + s_{i+1,j+1} \\
&\quad - 4g^x_{ij} - 2g^x_{i,j+1} - 3g^y_{ij} - 3g^y_{i+1,j}), \\
a_{12} &= -\tfrac{12}{hk^2}(6g_{ij} + 2s_{ij} + s_{i+1,j} + 2s_{i,j+1} + s_{i+1,j+1} \\
&\quad - 3g^x_{ij} - 3g^x_{i,j+1} - 4g^y_{ij} - 2g^y_{i+1,j}), \\
a_{22} &= \tfrac{36}{h^2k^2}(4g_{ij} + s_{ij} + s_{i+1,j} + s_{i,j+1} + s_{i+1,j+1} \\
&\quad - 2g^x_{ij} - 2g^x_{i,j+1} - 2g^y_{ij} - 2g^y_{i+1,j}).
\end{aligned}
$$

2° Every quadratic function $s(x)$ is determined on the interval $\langle x_i, x_{i+1}\rangle$ by the values s'_i, s'_{i+1}, g_i (mean value). However, the biquadratic function $s(x,y)$ is not uniquely determined on the rectangle D_{ij} by the nine parameters (Fig. 2b)

$$
g_{ij},\ s^{1,0}_{ij},\ s^{1,0}_{i,j+1},\ s^{1,0}_{i+1,j},\ s^{1,0}_{i+1,j+1},\ s^{0,1}_{ij},\ s^{0,1}_{i+1,j},\ s^{0,1}_{i,j+1},\ s^{0,1}_{i+1,j+1};
$$

the matrix of the system of the relations between these parameters and the parameters (T) has its rank equal to eight.

Let $s(x,y)$ be a biquadratic spline on (Δ). The functions of x,

$$
\begin{aligned}
(13)\qquad g^y(x,y_j) &= \frac{1}{k_j}\int_{y_j}^{y_{j+1}} s(x,y)\,\mathrm{d}y \\
&= \tfrac{2}{3}\big[(1-u^2)s_{ij} + u^2 s_{i+1,j} + h_i u(1-u)s_{ij}^{1,0}\big] \\
&\quad + \tfrac{1}{3}\big[(1-u^2)s_{i,j+1} + u^2 s_{i+1,j+1} + h_i u(1-u)s_{i,j+1}^{1,0}\big] \\
&\quad + \tfrac{1}{6}k_j\big[(1-u^2)s_{ij}^{0,1} + u^2 s_{i+1,j}^{0,1} + h_i u(1-u)s_{ij}^{1,1}\big],
\end{aligned}
$$

are quadratic splines on (Δx) with function values at $x = x_i$ equal to

$$g^y(x_i,y_j) = \frac{1}{k_j}\int_{y_j}^{y_{j+1}} s(x_i,y)\,\mathrm{d}y = g_{ij}^y$$

(see (10)). Similarly, the functions of y,

$$
\begin{aligned}
(14)\qquad g^x(x_i,y) &= \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x,y)\,\mathrm{d}x \\
&= \tfrac{2}{3}\big[(1-v^2)s_{ij} + v^2 s_{i,j+1} + k_j v(1-v)s_{ij}^{0,1}\big] \\
&\quad + \tfrac{1}{3}\big[(1-v^2)s_{i+1,j} + v^2 s_{i+1,j+1} + k_j v(1-v)s_{i+1,j}^{0,1}\big] \\
&\quad + \tfrac{1}{6}h_i\big[(1-v^2)s_{ij}^{1,0} + v^2 s_{i,j+1}^{1,0} + k_j v(1-v)s_{ij}^{1,1}\big],
\end{aligned}
$$

are quadratic splines on (Δy) with function values at $y = y_j$ equal to

$$g^x(x_i,y_j) = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x,y_j)\,\mathrm{d}x = g_{ij}^x$$

(see (10)). Finally there is a connection with the given mean values g_{ij}:

$$
\begin{aligned}
(15)\qquad g_{ij} &= \frac{1}{h_i k_j}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} s(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{h_i}\int_{x_i}^{x_{i+1}} g^y(x,y_j)\,\mathrm{d}x \\
&= \frac{1}{k_j}\int_{y_j}^{y_{j+1}} g^x(x_i,y)\,\mathrm{d}y
\end{aligned}
$$

—the values g_{ij} can be then considered as one-dimensional mean values of the splines $g^y(x,y_j)$ (with function values g_{ij}^y) or of the splines $g^x(x_i,y)$ (with function values

g^x_{ij}). We can therefore use the continuity conditions (5b) applied to the following subjects shown in Tab. 1

spline function	function values	mean values
$s(x_i, y)$	s_{ij}	g^y_{ij}
$s(x, y_j)$	s_{ij}	g^x_{ij}
$g^y(x, y_j)$	g^y_{ij}	g_{ij}
$g^x(x_i, y)$	g^x_{ij}	g_{ij}

to prove the following theorem.

Theorem 2. *Let us have a biquadratic spline $s(x, y)$ on the knot set (Δ) which interpolates given mean values*

$$g_{ij} = \frac{1}{h_i k_j} \iint_{D_{ij}} s(x, y) \, \mathrm{d}x \, \mathrm{d}y \, .$$

Then the continuity conditions $s(x, y) \in \mathrm{C}^{1,1}(D)$ can be expressed as the following system of linear relations between the parameters s_{ij}, g^x_{ij}, g^y_{ij}, g_{ij} (see (10), (13), (14)):

$$\beta_{j-1} s_{i,j-1} + 2(\beta_{j-1} + \beta_j) s_{ij} + \beta_j s_{i,j+1} = 3(\beta_{j-1} g^y_{i,j-1} + \beta_j g^y_{ij}), \quad i = 0(1)n+1, \; j = 1(1)m; \tag{16}$$

$$\alpha_{i-1} s_{i-1,j} + 2(\alpha_{i-1} + \alpha_i) s_{ij} + \alpha_i s_{i+1,j} = 3(\alpha_{i-1} g^x_{i-1,j} + \alpha_i g^x_{ij}), \quad i = 1(1)n, \; j = 0(1)m+1; \tag{17}$$

$$\alpha_{i-1} g^y_{i-1,j} + 2(\alpha_{i-1} + \alpha_i) g^y_{ij} + \alpha_i g^y_{i+1,j} = 3(\alpha_{i-1} g_{i-1,j} + \alpha_i g_{ij}), \quad i = 1(1)n, \; j = 0(1)m; \tag{18}$$

$$\beta_{j-1} g^x_{i,j-1} + 2(\beta_{j-1} + \beta_j) g^x_{ij} + \beta_j g^x_{i,j+1} = 3(\beta_{j-1} g_{i,j-1} + \beta_j g_{ij}), \quad i = 0(1)n, \; j = 1(1)m, \tag{19}$$

where $\alpha_i = 1/h_i$, $\beta_j = 1/k_j$.

Remarks. 1° The relations (18) {or (19)} form for each $j = 0(1)m$ $\{i = 0(1)n\}$ the system of n $\{m\}$ equations for $n + 2$ $\{m + 2\}$ unknown quantities g^y_{ij} $\{g^x_{ij}\}$ with a band matrix. To complete it to tridiagonal systems we need some boundary conditions for the spline under consideration. With known values g^x_{ij}, g^y_{ij} the relations (16), (17) tighten together the function values s_{ij} on horizontal or vertical lines of the knot set (Δ)—also some additional information is needed for computing all s_{ij}—but only $mn + 2(m+n)$ relations are independent; one can choose all vertical and two boundary horizontal lines—or vice versa (see the algorithm).

2° The partial derivatives

$$
\begin{aligned}
s^{1,0}(x_i, y) &= (1-v^2)s_{ij}^{1,0} + v^2 s_{i,j+1}^{1,0} + k_j v(1-v) s_{ij}^{1,1}, \\
s^{0,1}(x, y_j) &= (1-u^2)s_{ij}^{0,1} + u^2 s_{i+1,j}^{0,1} + h_i u(1-u) s_{ij}^{1,1}
\end{aligned} \tag{20}
$$

are also quadratic splines (in variables y, x) with mean values

$$
\begin{aligned}
\frac{1}{k_j}\int_{y_j}^{y_{j+1}} s^{1,0}(x_i, y)\,\mathrm{d}y &= \tfrac{2}{3}s_{ij}^{1,0} + \tfrac{1}{3}s_{i,j+1}^{1,0} + \tfrac{1}{6}k_j s_{ij}^{1,1}, \\
\frac{1}{h_i}\int_{x_i}^{x_{i+1}} s^{0,1}(x, y_j)\,\mathrm{d}x &= \tfrac{2}{3}s_{ij}^{0,1} + \tfrac{1}{3}s_{i+1,j}^{0,1} + \tfrac{1}{6}h_i s_{ij}^{1,1}.
\end{aligned} \tag{21}
$$

Inserting them into (6) we can obtain continuity conditions for $s(x,y)$ as relations between $s_{ij}^{1,1}$ and $s_{ij}^{1,0}$, $s_{ij}^{0,1}$. We could obtain some other relations between $s^{1,1}$, $s^{1,0}$, $s^{0,1}$ using (4):

$$
\begin{aligned}
\frac{1}{2}(s_{i-1,j}^{1,1} + s_{ij}^{1,1}) &= \frac{1}{h_{i-1}}(s_{ij}^{0,1} - s_{i-1,j}^{0,1}), \\
\frac{1}{2}(s_{i,j-1}^{1,1} + s_{ij}^{1,1}) &= \frac{1}{k_{j-1}}(s_{ij}^{0,1} - s_{i,j-1}^{0,1}).
\end{aligned}
$$

3° The continuity conditions (16)–(19) can be obtained also in an elementary (but more cumbersome) way using the Taylor representation (T).

4. Boundary conditions

For a biquadratic spline $s(x,y)$ on the knot set (Δ) we have altogether $4mn + 6n + 6m + 9$ local parameters in the representation (SG). The given mean values g_{ij} and the continuity conditions (16)–(19) represent together $4mn + 4n + 4m + 1$ linear independent relations between these parameters. Therefore we need additional $2n + 2m + 8 = 2(n+1) + 2(m+1) + 4$ conditions (e.g. boundary conditions) for the unique determination of the local parameters. The type of boundary conditions is to be chosen in such a way as to complete the continuity conditions (16)–(19) to solvable systems of linear equations.

4.1. Mean boundary values. Given the one-dimensional boundary mean values (see (10), (13), (14))

$$\begin{array}{lll} g^x_{i,0},\ g^x_{i,m+1}, & i = 0(1)n \\ g^y_{0,j},\ g^y_{n+1,j}, & j = 0(1)m \end{array} \qquad \{2(n+1) + 2(m+1) \text{ conditions}\}, \tag{22}$$

we can write (18), (19) as a system of $m + 1$ resp. $n + 1$ linear equations with fixed tridiagonal matrices and compute all local parameters g^x_{ij}, g^y_{ij}.

4.1.1. Given further four function values at the vertices of D

$$s_{ij}, \qquad i = 0, n+1,\ j = 0, m+1, \tag{23}$$

(a) we can use these values with (17) {or (16)} for the indices $j = 0, m+1$ $\{i = 0, n+1\}$ to compute the values s_{ij} on all horizontal {vertical} boundary lines by solving two systems with the same tridiagonal matrix;

(b) then we can use (16) {(17)} with all indices $i = 0(1)n+1$ $\{j = 0(1)m+1\}$ to compute the values s_{ij} on all vertical lines $x = x_i$ {horizontal lines $y = y_j$} using the boundary function values from step (a)—altogether $n + 2$ $\{m + 2\}$ systems of equations with the same tridiagonal matrix.

4.1.2. It is possible to prescribe four values of derivatives

$$s^{1,0}_{ij}, \qquad i = 0, n+1,\ j = 0, m+1 \qquad \{\text{or } s^{0,1}_{ij}\}. \tag{24}$$

In this case we use a one-dimensional algorithm which uses (6) (see [7]) to compute

$$s^{1,0}_{i0},\ s^{1,0}_{i,m+1}, \qquad i = 1(1)n,$$

and then

$$s_{i0},\ s_{i,m+1}, \qquad i = 0(1)n+1,$$

e.g.

$$\begin{aligned} s_{ij} &= g^x_{ij} - \tfrac{1}{6}h_i(s^{1,0}_{i+1,j} + 2s^{1,0}_{ij}), \qquad & i = 0(1)n,\ j = 0, m+1, \\ s_{n+1,j} &= g^x_{n0} + \tfrac{1}{6}h_i(s^{1,0}_{nj} + 2s^{1,0}_{n+1,j}), & j = 0, m+1. \end{aligned} \tag{25}$$

Similarly we can determine $s^{0,1}_{0j}$, $s^{0,1}_{n+1,j}$ and s_{0j}, $s_{n+1,j}$.

In this way we can obtain the same boundary values s_{ij} as in the stage (a) of the foregoing algorithm in 4.1.1; we can finish our computation using its stage (b)—solving the systems for the remaining parameters s_{ij}.

4.2. Function boundary values. It is also possible to prescribe the boundary function values

$$(26)\qquad \begin{matrix} s_{i0},\ s_{i,m+1}, & i = 0(1)n+1 \\ s_{0j},\ s_{n+1,j}, & j = 1(1)m \end{matrix} \qquad \{2(n+2)+2m \text{ values}\}.$$

The remaining four values for the unique determination of $s(x,y)$ can be chosen for example from the values of $s^{1,0}$, $s^{0,1}$ at the vertices of D in several combinations ensuring the existence of such a spline. We shall discuss briefly only one of these cases:

$$(27)\qquad \text{given (26) and } s^{1,0}_{00},\ s^{1,0}_{0,m+1},\ s^{0,1}_{00},\ s^{0,1}_{n+1,0},$$

then using (4) we can calculate uniquely from given s_{i0} $\{$or $s_{i,m+1}\}$ and $s^{1,0}_{00}$ $\{s^{1,0}_{0,m+1}\}$ the parameters of the interpolating spline $s(x, y_0)$ $\{s(x, y_{m+1})\}$. Similarly we can obtain splines $s(x_0, y)$, $s(x_{n+1}, y)$. Now we are able to compute explicitly (using (9)) the mean values g^x_{i0}, $g^x_{i,m+1}$ and g^y_{0j}, $g^y_{n+1,j}$. Further we can proceed as in 4.1 to compute all values g^x_{ij}, g^y_{ij}, s_{ij}.

Let us summarize the results obtained in the following theorem.

Theorem 3. *Given a knot set* (Δ), *mean values* $\{g_{ij}\,;\, i = 0(1)n, j = 0(1)m\}$ *and one of the boundary conditions*

- (22) *and* [(23) *or* (24)] *(boundary mean values + corner values),*
- (26) *and* [(27) *or some proper variation of it*] *(boundary function values + corner derivatives).*

Then there exists a unique biquadratic spline $s(x,y)$ *determined by these conditions.*

5. Description of the algorithm

We describe the algorithm for computing the parameters (SG) of the biquadratic spline $s(x,y)$ determined on (Δ) by the prescribed mean values $\{g_{ij}\,;\, i = 0(1)n,\ j = 0(1)m\}$ and the boundary conditions (22), (23) in more detail.

Algorithm BQSIMV.

1° Compute numbers $h_i = x_{i+1} - x_i$, $i = 0(1)n$, $k_j = y_{j+1} - y_j$, $j = 0(1)m$.

2° Calculate the coefficients

$$\alpha_i = 1/h_i,\ i = 0(1)n, \qquad \beta_j = 1/k_j,\ j = 0(1)m$$

and form the tridiagonal matrices

$$A_n = \begin{bmatrix} 2(\alpha_0 + \alpha_1), & \alpha_1 & & \\ \alpha_1, & 2(\alpha_1 + \alpha_2), & \alpha_2 & \\ & \cdots & & \\ & \alpha_{n-1}, & 2(\alpha_{n-1} + \alpha_n) \end{bmatrix},$$

$$B_m = \begin{bmatrix} 2(\beta_0 + \beta_1), & \beta_1 & & \\ \beta_1, & 2(\beta_1 + \beta_2), & \beta_2 & \\ & \cdots & & \\ & \beta_{m-1}, & 2(\beta_{m-1} + \beta_m) \end{bmatrix}.$$

3° Calculate the corresponding vectors of the right-hand sides according to (18), (19) (including the terms with g^x_{i0}, $g^x_{i,m+1}$, g^y_{0j}, $g^y_{n+1,j}$ from the left)

$$\begin{aligned}
&3[\alpha_0 g_{0j} + \alpha_1 g_{1j}] - \alpha_0 g^y_{0j},\\
&3[\beta_0 g_{i0} + \beta_1 g_{i1}] - \beta_0 g^x_{i0},\\
&3[\alpha_{i-1} g_{i-1,j} + \alpha_i g_{ij}], \qquad i = 2(1)n-1,\\
&3[\beta_{j-1} g_{i,j-1} + \beta_j g_{ij}], \qquad j = 2(1)m-1,\\
&3[\alpha_{n-1} g_{n-1,j} + \alpha_n g_{nj}] - \alpha_n g^y_{n+1,j},\\
&3[\beta_{m-1} g_{i,m-1} + \beta_m g_{im}] - \beta_m g^x_{i,m+1}.
\end{aligned}$$

Using the decomposition algorithm for tridiagonal systems, solve

$m+1$ systems with the matrix A_n for the values g^y_{ij},

$n+1$ systems with the matrix B_m for g^x_{ij}.

4° With g^x_{ij} computed in step 3° and function values s_{ij} at vertices form similarly the right-hand sides in (17) for $j = 0, m+1$, solve two systems with the matrix A_n to compute the values s_{ij} on horizontal boundaries.

5° With values g^y_{ij} computed in step 3° and boundary values s_{ij} from step 4°, form the right-hand sides given in (16) and for $i = 0(1)n+1$ solve the systems with the constant matrix B_m to compute all remaining values s_{ij}.

6° When needed, we can pass from the representation (SG) of the spline to the Taylor representation (T) using relations (12), or to the representation (SD) using relations (9), (10) (e.g. for graphic visualization of results).

6. Examples

Example 1. The biquadratic spline corresponding to the discrete data g_{ij} given in Tab. 2 and to the boundary conditions

$$g^y_{0j} = g_{0j},\ g^y_{5j} = g_{5j},\ g^x_{i0} = g^x_{i0},\ g^x_{i6} = g^x_{i6},\ s_{ij} = g_{ij}, \qquad i = 0,5,\ j = 0,5$$

is shown in Fig. 3.

x_i	0	2	3	4	6	9
	0					
		1	2	2.5	1.5	1
	1					
		1.5	2	4	3	2
	2					
y_j		1	2	3	2	3
	4					
		2	1.5	1.5	2	2.5
	5					
		1	1.5	2	2	1.5
	7					
	10	0.5	1	1.5	2	1

g_{ij}

Tab. 2

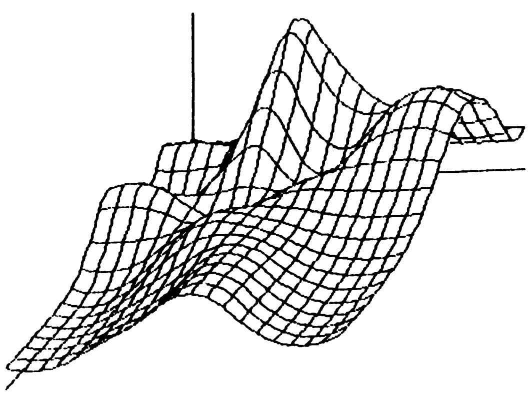

Fig. 3

Example 2. We can see the biquadratic spline approximating the function $f(x,y) = 50xy(1-x)^2(1-y)^2$; $x, y \in [0,1]$, $h_i = k_j = 0.2$ with $g^x_{ij} = g^y_{ij} = s_{ij} = 0$ on the boundary and mean values g_{ij} calculated from $f(x,y)$ in Fig. 4.

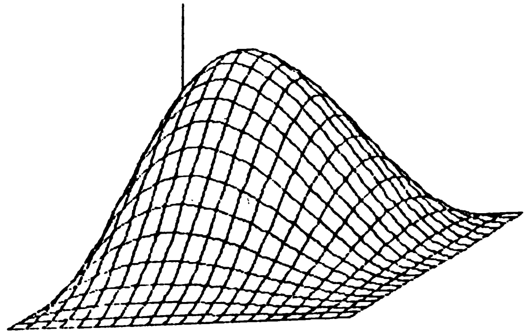

Fig. 4

7. Natural biquadratic spline

It was shown in [7] that the extremal property connected with the "natural" splines $s(x)$ is attained by the spline with zero values s'_0, s'_{n+1}. Taking account of this and of Remarks (2) in Sections 2, 3, we introduce the following definition of the "natural biquadratic spline $s \in \mathscr{S}(\Delta)$ on D".

Definition 1. Let the rectangle D with the knot set (Δ) and mean values $\{g_{ij}\,;\, i = 0(1)n,\ j = 0(1)m\}$ be given. We call $s(x,y) \in \mathscr{S}(\Delta)$ *a natural biquadratic spline interpolating given mean values*, if it fulfils the boundary conditions

$$(31)\qquad \begin{aligned} &s^{1,0}(x_0, y_j) = s^{1,0}(x_{n+1}, y_j) = 0, && j = 0(1)m+1,\\ &s^{0,1}(x_i, y_0) = s^{0,1}(x_i, y_{m+1}) = 0, && i = 0(1)n+1,\\ &s^{1,1}(x_0, y_0) = 0. \end{aligned}$$

Corollary. *For the natural spline $s(x,y) \in \mathscr{S}(\Delta)$ we have*

$$(32)\qquad \begin{aligned} &s^{0,1}(x, y_0) \equiv 0 \equiv s^{0,1}(x, y_{m+1}) \quad \text{for all } x \in \langle a, b\rangle\,,\\ &s^{1,0}(x_0, y) \equiv 0 \equiv s^{1,0}(x_{n+1}, y) \quad \text{for all } y \in \langle c, d\rangle\,. \end{aligned}$$

P r o o f. We have

$$\begin{aligned} s^{1,0}(x_0, y) &= (1 - v^2)s^{1,0}_{0j} + v^2 s^{1,0}_{0,j+1} + v(1-v)k_j s^{1,1}_{0j} \quad \text{for } y \in \langle y_j, y_{j+1}\rangle\,,\\ s^{0,1}(x, y_0) &= (1 - u^2)s^{0,1}_{i0} + u^2 s^{0,1}_{i+1,0} + u(1-u)h_i s^{1,1}_{i0} \quad \text{for } x \in \langle x_i, x_{i+1}\rangle\,. \end{aligned}$$

Hence, (31) implies the vanishing of $s^{1,0}$ on the vertical parts and of $s^{0,1}$ on the horizontal parts of the boundary ∂D.

As a consequence we have

$$s^{1,1}(x,y) \equiv 0 \quad \text{on the whole boundary } \partial D. \tag{33}$$

□

Theorem 4. *The natural biquadratic spline is uniquely determined by the mean values* $\{g_{ij}\,;\, i=0(1)n,\ j=0(1)m\}$ *and the boundary conditions* (31).

Proof. According to (9), (10) we have

$$\begin{aligned} 3g_{i0} &= \tfrac{1}{3}(4s_{i0} + 2s_{i1} + 2s_{i+1,0} + s_{i+1,1}) + \tfrac{1}{6}h_i(2s_{i0}^{1,0} + s_{i1}^{1,0} \\ &\quad + \tfrac{1}{6}k_0(2s_{i0}^{0,1} + s_{i+1,0}^{0,1}) + \tfrac{1}{2}h_i k_0 s_{i0}^{1,1}, \\ 2g_{i0}^x + g_{i1}^x &= \tfrac{1}{3}(4s_{i0} + 2s_{i1} + 2s_{i+1,0} + s_{i+1,1}) + \tfrac{1}{6}h_i(2s_{i0}^{1,0} + s_{i1}^{1,0}). \end{aligned}$$

Taking into account zero boundary conditions (31)–(33), we can write

$$\begin{aligned} & 2g_{i0}^x + g_{i1}^x = 3g_{i0}, \qquad i = 0(1)n \\ \text{and similarly}\quad & 2g_{0j}^y + g_{1j}^y = 3g_{0j}, \qquad j = 0(1)m \quad \text{follows.} \end{aligned} \tag{34}$$

In a similar way we obtain

$$\begin{aligned} g_{im}^x + 2g_{i,m+1}^x &= 3g_{im}, \qquad i = 0(1)n, \\ g_{nj}^y + 2g_{n+1,j}^y &= 3g_{nj}, \qquad j = 0(1)m. \end{aligned} \tag{35}$$

These relations suitably combined complete the continuity conditions (18)–(19) to tridiagonal systems with a diagonally dominating matrix for computing g_{ij}^x, g_{ij}^y. Similarly, we complete the continuity conditions (16)–(17) by the relations

$$\begin{aligned} 2s_{i0} + s_{i1} &= 3g_{i0}^y, \quad i = 0(1)n \\ s_{im} + 2s_{i,m+1} &= 3g_{im}^y \end{aligned} \quad \text{or} \quad \begin{aligned} 2s_{0j} + s_{1j} &= 3g_{0j}^x, \quad j = 0(1)m \\ s_{nj} + 2s_{n+1,j} &= 3g_{nj}^x \end{aligned} \tag{36}$$

which follow from (10) and (31). In such a way we can uniquely determine all local parameters of the spline in the representation (SG). □

It is well known that the construction of a smoothing cubic spline is based on the extremal property of natural cubic splines (see [1], for bicubic spline [9]). Quadratic splines interpolating the function values at spline knots or different points of interpolation generally do not have such simple extremal property. For quadratic splines $s(x)$ interpolating mean values such extremal property with respect to the functional $J(f) = \|f'\|_2^2$ was proved in [7] for splines with $s_0' = s_{m+1}' = 0$ and then used to the construction of the corresponding smoothing spline. We shall show such an extremal property for the natural biquadratic spline described in Section 7.

Theorem 5. *Let the knot set* $(\Delta) = (\Delta x)\times(\Delta y)$ *and mean values* $\{g_{ij}\,;\, i = 0(1)n,$ $j = 0(1)m\}$ *on the rectangle* $D = \mathbb{U}_{i,j} D_{ij}$ *be given. Let us denote*

(37)
$$\mathscr{V} = \Big\{ f(x,y) \in W_2^{1,1}(D)\,;\; h_i k_j g_{ij} = \iint_{D_{ij}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y,\ i = 0(1)n,\ j = 0(1)m \Big\}$$

and introduce the functional

(38)
$$J(f) = \iint_D \left[f^{1,1}(x,y)\right]^2 \mathrm{d}x\,\mathrm{d}y\,.$$

Then the natural biquadratic spline interpolating the mean values g_{ij} *(see Def. 1) minimizes the functional* $J(f)$ *on* $\mathscr{V}$.

P r o o f. For the natural spline $s \in S(\Delta)$ and $f \in \mathscr{V}$ we have

(39)
$$0 \leqslant J(f-s) = J(f) - J(s) - 2\Big\{ \iint_D \left[(f^{1,1} - s^{1,1})s^{1,1}\right](x,y)\,\mathrm{d}x\,\mathrm{d}y \Big\}.$$

Applying repeatedly the integration by parts rule to the last term and using the continuity of the derivatives, we obtain

(40)
$$\begin{aligned}
&\iint_D \left[s^{1,1}(f^{1,1} - s^{1,1})\right](x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{y_0}^{y_{m+1}} \Big\{ \int_{x_0}^{x_{n+1}} \left[s^{1,1}(f^{1,1} - s^{1,1})\right](x,y)\,\mathrm{d}x \Big\}\,\mathrm{d}y \\
&= \int_{y_0}^{y_{m+1}} \Big\{ \left[s^{1,1}(s-f)^{0,1}\right](x_{n+1},y) - \left[s^{1,1}(s-f)^{0,1}\right](x_0,y) \Big\}\,\mathrm{d}y \\
&\quad - \int_{y_0}^{y_{m+1}} \int_{x_0}^{x_{n+1}} \left[s^{2,1}(s-f)^{0,1}\right](x,y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Repeating integration by parts in the last double integral, we obtain (with $s(x+0) = \lim\{s(x+h)\,;\, h \to 0,\ h > 0\}$)

$$\sum_i \Big\{ s^{2,1}(x_i + 0, y_{m+1}) \int_{x_i}^{x_{i+1}} (s - f)(x, y_{m+1})\,\mathrm{d}x$$
$$- s^{2,1}(x_i + 0, y_0) \int_{x_i}^{x_{i+1}} (s - f)(x, y_0)\,\mathrm{d}x \Big\}$$
$$+ \sum_i \sum_j s^{2,2}(x_i + 0, y_j + 0) \iint_{D_{ij}} (s - f)\,\mathrm{d}x\,\mathrm{d}y\,.$$

The first terms and the simple sums vanish because $s^{1,1}(x,y) \equiv 0$ on ∂D. The last double sum vanishes as a consequence of zero value of the double integral (as $s, f \in \mathcal{V}$). The orthogonality relation

$$\iint_D \big[s^{1,1}(f^{1,1} - s^{1,1}\big](x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

implies the inequality

$$J(s) + J(f - s) \leqslant J(f),$$

which proves the theorem. □

Remarks. 1° We have not succeeded in extending the functional (38) analogous to the bicubic case (see [9]).

2° Applications to the construction of the smoothing biquadratic spline will be dealt with in a forthcoming paper.

Example 3. The natural biquadratic spline corresponding to the data g_{ij} from Example 1 is shown in Fig. 5.

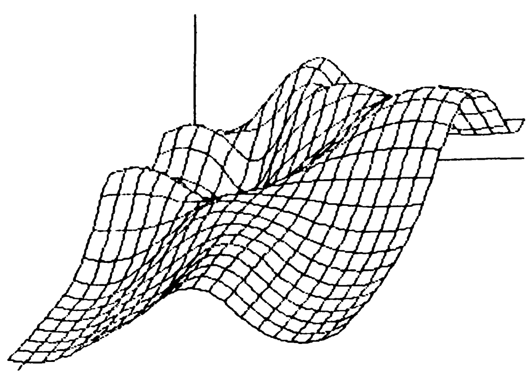

Fig. 5

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