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DENSITIES OF DETERMINANT RATIOS, THEIR MOMENTS AND SOME SIMULTANEOUS CONFIDENCE INTERVALS IN THE MULTIVARIATE GAUSS-MARKOFF MODEL

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Summary. The following three results for the general multivariate Gauss-Markoff model with a singular covariance matrix are given or indicated. 1° determinant ratios as products of independent chi-square distributions, 2° moments for the determinants and 3° the method of obtaining approximate densities of the determinants.

Keywords: products of independent chi-squares, multivariate general linear Gauss-Markoff model, Wishart distribution, singular covariance matrix, set of linear parametric functions, moment of determinant ratio, approximate simultaneous confidence interval

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1. INTRODUCTION

We are interested in the multivariate general linear Gauss-Markoff model MGM

$$(U, XB, \sigma^2\Sigma \otimes V), \tag{1.1}$$

with a known matrix $V \geqslant 0$, where $\otimes$ is the symbol of the Kronecker product of matrices, B is a matrix of unknown fixed parameters, X is a given known design matrix, U is a random matrix of observations with the expected value $\varepsilon(U) = XB$ and with a fixed nonsingular matrix Σ, σ^2 is an unknown positive scalar. Let $T = V + XMX'$ where $M = M'$ is an arbitrary matrix such that $R(X) \subset R(T)$. The symbol $R(X)$ is used to denote the vector space spanned on the columns of the matrix X.

Let $A\tilde{\ }W_p(n, \Sigma)$ denote that a $p \times p$ random matrix A is distributed as the p-dimensional Wishart distribution with n degrees of freedom and a covariance matrix Σ. Determinant ratios as products of independent chi-square distributions under the assumption $V^{*\sim}W_p(n,\Sigma)$, where V^* and $\Sigma > 0$ are $p \times p$ matrices, have been given by Srivastava and Khatri [9] p. 82,3.38; Rao [7] p. 540, X; Muirhead [4] p. 447.

By applying this result and that given by Oktaba and Kieloch [6] for MGM three theorems are established (Sec. 2).

Moments of determinant ratios $|V^*||\Sigma|^{-1}$ under the assumption $V^{*\sim}W_p(n,\Sigma)$ can be found in Anderson [1] or Rao ([7], p. 540). Their forms in the case of the MGM model are deduced by using results of Oktaba and Kieloch (loc. cit) (Sec. 3).

In very many papers and books the confidence intervals are given, e.g. S.N. Roy [9], Anderson [1]; Browkowicz and Linnik [2]; Muirchead [4]; Rao [7], Srivastava and Khatri [9]; Wani and Kabe [10], and some others. Using the procedure of Srivastava and Khatri, the method of approximate density of determinant rations is suggested. In this way we get a generalization of Wani and Kabe results [10] from the standard model $(U, XB, \Sigma \otimes I)$ to the MGM model. It creates a possibility for constructing an approximate simultaneous confidence region for a pair: a set of parametric estimable functions L^*B and the determinant $|\sigma^2\Sigma|$.

2. Densities of determinant ratios in MGM model

The following theorem (Srivastava and Khatri [9] pp. 82, 3.38; Rao [7], p. 540, X; Muirhead [4], p. 447) will be used in this section.

Theorem 2.1. *Let us consider a $p \times p$ matrix $V^{*\sim}W_p(n,\Sigma)$, where Σ is $p \times p$ matrix, $p \times p$ matrix $\Sigma > 0$. Then the determinant*

$$|W| = \frac{|V^*|}{|\Sigma|} = z_1 \cdot z_2 \dots z_p = \chi^2_n \chi^2_{n-1} \cdots \chi^2_{n-p+1}, \tag{2.1}$$

where z_i, $i = 1, \dots, p$ are independently distributed as χ^2_{n-i+1} with $n - i + 1$ degrees of freedom.

Let $r(A)$ denote the rank of the matrix A. The symbol $(A \vdots B)$ is reserved for the matrix involving two submatrices A and B.

The additivity of central p-dimensional Wishart distributions is stated in the following theorem.

Theorem 2.2 (Srivastava and Khatri [9] p. 82; Muirhead [4] p. 446). *If $p \times p$ matrices $V_1^*, \dots, V_r^*$ are stochastically independent and $V_i^{*\sim}W_p(\nu_i, \Sigma)$, $i = 1, \dots, r$, then*

$$\Sigma_{i=1}^r V_i^{*\sim} W_p\left(\sum_{i=1}^n \nu_{i;} \Sigma\right). \tag{2.2}$$

Theorems 2.3, 2.4 and 2.5 of Oktaba and Kieloch [6] will be applied in this section.

Let L^*B be a set of linear estimable parametric functions, where L^* is an $a \times m$ matrix. The following quadratic forms are of interest:

$$S_H = (L^*\hat{B} - \psi)'L^-(L^*\hat{B} - \psi) = (L^*\hat{B} - \psi)'(L^*C_4L^{*'})^-(L^*\hat{B} - \psi), \tag{2.3}$$

$$S_e = U'C_1U, \tag{2.4}$$

where

$$\hat{B} = (X'T^-X)^-X'T^-U = C_2'U = C_3U, \tag{2.5}$$

$$\begin{cases} C_1 & = T^- - T^-X(X'T^-X)^-X'T^- = T^-(I - XC_3) \\ C_2' & = C_3 = (X'T^-X)^-X'T^-, \end{cases} \tag{2.6}$$

$$L = (L^*C_3)V(L^*C_3)' = L^*C_4L^{*'} \tag{2.7}$$

(Oktaba [5], (2.1), 179).

The symbol L^- is reserved for any choice of the g-inverse, i.e. the following relation holds:

$$LL^-L = L. \tag{2.8}$$

We recall that

$$\begin{pmatrix} V & X \\ X' & 0 \end{pmatrix}^- = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}.$$

Theorem 2.3 (Oktaba and Kieloch [6]). *If*

$$U\char`~N_{np}(XB, \sigma^2\Sigma \otimes V), \tag{2.9}$$

where $N_{np}(\,,\,)$ denotes an np-variate normal distribution N_{np} with parameters defined in Introduction, L^*B is a set of estimable linear combinations of parameters and the hypothesis $L^*B = \psi$ is true, then

$$S_H\char`~W_p[r(L), \sigma^2\Sigma]. \tag{2.10}$$

Theorem 2.4 (Oktaba and Kieloch [6]). *Subject to the assumption* (2.9) *we have*

$$S_e\char`~W_p[r(\vdots X) - r(X), \sigma^2\Sigma], \tag{2.11}$$

where S_e is defined in (2.4)

Theorem 2.5 (Oktaba and Kieloch [6]). *If the assumptions of Theorems 2.3 and 2.4 concerning the matrices S_H and S_e are fulfilled, then S_H and S_e are stochastically independent.*

As a consequence of the above five theorems we almost immediately obtain the following three theorems.

Theorem 2.6. *In the model* (1.1) *let the $p \times p$ matrix S_e satisfy*

$$S_e = U'C_1U\char`~W_p(\nu_e, \sigma^2\Sigma), \tag{2.12}$$

where the $p \times p$ matrix Σ and the scalar σ^2 are positive,

$$\nu_e = r(V\vdots X) - r(X), \tag{2.13}$$

C_1 is given as in (2.6). *Then we have*

$$\frac{|S_e|}{|\sigma^2\Sigma|} = \prod_{i=1}^{p} \xi_i, \tag{2.14}$$

where ξ_i are independently distributed as $\chi^2_{\nu_e-i+1}$, $i = 1, \ldots, p$; with $\nu_e - i + 1$ degrees of freedom.

Proof. We observe (2.11) in model (1.1), where S_e and ν_e are as in (2.12) and (2.13). Replacing in Theorem 2.1 V^*, Σ, n and z_i by $S_e, \sigma^2\Sigma, \nu_e$ and ξ_i, respectively, we get (2.14). □

Theorem 2.7. *In the model* (1.1) *let the $p \times p$ matrix S_H (Oktaba and Kieloch* [6]*) in* (2.3) *satisfy*

$$S_H\char`~W_p(\nu_H, \sigma^2\Sigma) \tag{2.15}$$

under the true hypothesis

$$H_0\colon L^*B = \psi, \tag{2.16}$$

where the $p \times p$ matrix Σ and the scalar σ^2 are positive,

$$C_4 = (X', T^-X)^- - M, \quad \nu_H = r(L), \tag{2.17}$$

L, $\tilde{B}$ and C_3 are as in (2.7), (2.5) *and* (2.6), *respectively; the matrices T and M are defined in Introduction.*

Then

$$\frac{|S_H|}{|\sigma^2\Sigma|} \prod_{i=1}^{p} w_i, \tag{2.18}$$

where w_i are independently distributed as $\chi^2_{\nu_H-i+1}$ with $\nu_H - i + 1$ degrees of freedom, $i = 1, \dots, p$.

Proof is analogous to that of Theorem 2.6. □

Theorem 2.8. *Under the true hypothesis* (2.16) *in the model* (1.1) *let the $p \times p$ matrix S_y satisfy*

$$S_y = S_H + S_e \tilde{} W_p(\nu_H + \nu_e, \sigma^2\Sigma), \tag{2.19}$$

where the $p \times p$ matrix Σ and scalar σ^2 are positive. Then

$$\frac{|S_y|}{|\sigma^2\Sigma|} \prod_{i=1}^{p} m_i, \tag{2.20}$$

where m_i are independently distributed as $\chi^2_{\nu_y-i+1}$ with $\nu_y - i + 1 = \nu_h + \nu_e - i + 1$ degrees of freedom, $i = 1, \dots, p$.

Proof. By virtue of (2.10), (2.12) (Oktaba and Kieloch [6]) and the additive property of Wishart distributions given in Theorem 2.2 we conclude that (2.19) holds. Applying Theorem 2.1 where we put $V^* = S_e + S_H = S_y$ and replace n by $\nu_y = \nu_e + \nu_H = r(V\vdots X) - r(X) + r(L)$, we obtain (2.20). □

A particular case of the model (1.1) is the multivariate standard model

$$(U, 1 \cdot \mu', \Sigma \otimes I_n), \tag{2.21}$$

where U is a $n \times p$ matrix of observations, 1—the vector $n \times 1$ involving unities, $\mu' = (\mu_1, \dots, \mu_p)$, the $p \times p$ matrix Σ is positive,

$$S_e = U'\Big(I - \frac{1}{n} 1\, 1'\Big)U = V^* \tilde{} W_p(n-1, \Sigma) \tag{2.22}$$

and

$$|W| = \frac{|S_e|}{|\Sigma|} = \chi^2_{n-p} \cdots \chi^2_{n-2} \cdot \chi^2_{n-1}, \tag{2.23}$$

where $W = S_e\Sigma^{-1}$ (Anderson [1], (17), p. 171).

To prove (2.23) it is sufficient to set in (1.1): $X = 1$, $B = \mu'$, $V = I_n$, $\sigma^2 = 1$, $\nu_e = n - 1$. From (2.14) we get (2.23).

3. Moments of determinant ratios $|S_e|/(|\sigma^2\Sigma|)$ and $|S_H|/(|\sigma^2\Sigma|)$ in the MGM model with singular covariance matrix

For a general p a closed form density of the product of p independent central chi-squares is not available. It is possible to give the h-th moment of the determinant $|W| = |V^*||\Sigma|^{-1}$, where

$$V^* \sim W_p(n, \Sigma). \tag{3.1}$$

Its formula is as follows (Anderson [1]; Srivastava, Khatri [9], p. 83:

$$E(|W|^h) = E\Big(\frac{|V^*|}{|\Sigma|}\Big)^h = 2^{ph}\frac{\Gamma_p\big(\frac{n}{2}+h\big)}{\Gamma_p\big(\frac{n}{2}\big)}, \tag{3.2}$$

where n denotes the number of degrees of freedom of V^* in (3.1),

$$\Gamma_p\Big(\frac{n}{2}\Big) = \Pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\Big(\frac{n-i+1}{2}\Big) \tag{3.3}$$

is the multivariate gamma function (James, [3], p. 483),

$$\Gamma(r) = \int_0^\infty y^{r-1}\mathrm{e}^{-y}\,\mathrm{d}y \tag{3.4}$$

is the complete gamma function.

Theorem 3.1. *The h-th moments of $|S_e|(|\sigma^2\Sigma|)^{-1}$, $|S_H|(|\sigma^2\Sigma|)^{-1}$ and $|S_y| \times (|\sigma^2\Sigma|)^{-1}$ in the MGM model of the form* (1.1) *with a singular covariance matrix are*

$$E\Big(\frac{|S_e|}{|\sigma^2\Sigma|}\Big)^h = 2^{ph}\cdot\frac{\Gamma_p\big(\frac{\nu_e}{2}+h\big)}{\Gamma_p\big(\frac{\nu_e}{2}\big)}, \tag{3.5}$$

where S_e and ν_e are given as in (2.12) *and* (2.13).

$$E\Big(\frac{|S_H|}{|\sigma^2\Sigma|}\Big)^h = 2^{ph}\cdot\frac{\Gamma_p\big(\frac{\nu_H}{2}+h\big)}{\Gamma_p\big(\frac{\nu_H}{2}\big)}, \tag{3.6}$$

where S_H and ν_H are given as in (2.3) *and* (2.17).

Moreover

$$(3.7) \qquad E\Big(\frac{|S_y|}{|\sigma^2\Sigma|}\Big)^h = 2^{ph} \cdot \frac{\Gamma_p\Big(\frac{\nu_y}{2}+h\Big)}{\Gamma_p\Big(\frac{\nu_y}{2}\Big)},$$

where

$$(3.8) \qquad \nu_y = \nu_e + \nu_H, \quad S_y = S_e + S_H.$$

Proof. Results of Oktaba and Kieloch [6] yield that S_e and S_H are distributed as $W_p(\nu_e, \sigma^2\Sigma)$ and $W_p(\nu_H, \sigma^2\Sigma)$, respectively. Formula (3.2), in which we replace V^*, n and Σ by S_e, $\nu_e = r(V\vdots X) - r(X)$ and $\sigma^2\Sigma$ respectively implies (3.5). Similarly, replacing V^*, n and Σ by S_H, $\nu_H = r(L)$ and $\sigma^2\Sigma$, we get (3.6). As a consequence of additivity given in Theorem 2.2 we state that $S_y = S_e + S_H\,\check{}\,W_p(\nu_e + \nu_H, \sigma^2\Sigma)$, so by formula (3.2) we have (3.7). □

4. Simultaneous confidence interval for L^*B and $|\sigma^2\Sigma|$ in multivariate model (1.1) with singular covariance matrix

Let us consider the problem of constructing a simultaneous confidence region in model (1.1) for a set of estimable parametric functions L^*B and a determinant $|\sigma^2\Sigma|$. Applying Oktaba's and Kieloch's [6] results we state that in (1.1) under the assumption

$$(4.1) \qquad U\,\check{}\,N_{np}(XB, \sigma^2\Sigma \otimes V)$$

and the estimable true hypothesis

$$(4.2) \qquad H_0\colon L^*B = \psi,$$

the matrices S_H and S_e in (2.3) and (2.4), respectively, are independently distributed.

From (2.20) we know that the density of the random variable

$$(4.3) \qquad |W_y| = \frac{|S_y|}{|\sigma^2\Sigma|}$$

is the same as the density of p chi-squares independently distributed (cf. Wani and Kabe [10], p. 18).

The density of the random variable $|W_y|$ can be used to construct simultaneous confidence regions for the pair $(L^*B, |\sigma^2|)$ under the assumption of estimability of L^*B in (4.2) (Wani and Kabe, [10], p. 18).

Remark 4.1. The moments of $|W_e| = \frac{|S_e|}{|\sigma^2\Sigma|}$, $|W_H| = \frac{|S_H|}{|\sigma^2\Sigma|}$ and $|W_y| = \frac{|S_y|}{|\sigma^2\Sigma|}$ are given in (3.5), (3.6) and (3.7), so under the assumptions that S_e, S_H and S_y are distributed as $W_p(\nu_e, \sigma^2\Sigma)$, $W_p(\nu_H, \sigma^2\Sigma)$ and $W_p(\nu_y, \sigma^2\Sigma)$ we can obtain an approximate density of variates $|W_e|$, $|W_H|$ and $|W_y|$ applying the method which is described by Srivastava and Khatri ([9], chapter 6.3). It is known that density is determined uniquely by the moments (loc. cit. p. 176).

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