Lubomír Kubáček

On a linearization of regression models

Applications of Mathematics, Vol. 40 (1995), No. 1, 61–78

Persistent URL: http://dml.cz/dmlcz/134279

ON A LINEARIZATION OF REGRESSION MODELS

LUBOMÍR KUBÁČEK,[1] Bratislava

(Received September 2, 1993)

Summary. An approximate value of a parameter in a nonlinear regression model is known in many cases. In such situation a linearization of the model is possible however it is important to recognize, whether the difference between the actual value of the parameter and the approximate value does not cause significant changes, e.g., in the bias of the estimator or in its variance, etc. Some rules suitable for a solution of this problem are given in the paper.

AMS classification: 62J02, 62J05

Keywords: nonlinear regression model, linearization, parameter effect curvature, intrinsic curvature

INTRODUCTION

There exists many papers and also books (cf., e.g., [5]) on processing experimental data when linear statistical models can be used. They are applied even in nonlinear cases since the theory of linear procedures is thoroughly elaborated and the methods are simple in a comparison with nonlinear methods. However, in a nonlinear case a statistician must be convinced that the model can be linearized; cf., e.g., the procedure given in [3], p. 45.

The aim of the paper is to find a simple way how to attain a decision whether the model can or cannot be linearized.

[1] Supported by the Alexander von Humboldt-Stiftung and the grant No. 366 of the Slovak Academy of Sciences

1 Notations and preliminaries

Let a nonlinear regression model $Y \sim N_n[f(\beta), \Sigma]$ be considered. Here Y is an n-dimensional random vector normally distributed with mean value $E(Y|\beta) = f(\beta), \beta$ is an unknown k-dimensional parameter which can be any element of $\mathbb{R}^k$ (k-dimensional Euclidean space), $f(.)\colon \mathbb{R}^k \to \mathbb{R}^n$ is an n-dimensional vector function and Σ is known covariance matrix of the vector Y, i.e., $\mathrm{Var}(Y) = \Sigma$. The aim is to find an estimator $\hat{\beta}(Y)$ of the parameter β in the case when it is known that $\beta \in \mathcal{O}(\beta_0)$, where $\beta_0 \in \mathbb{R}^k$ and its open neigbourhood $\mathcal{O}(\beta_0)$ are given. As the problem, how to choose β_0 is not solved here, the following consideration cannot be applied within situations with unknown β_0.

It has to be stated in andvance also that the models with a low nonlinearity characterized mainly by the curvature of them (see Definition 2.2) are studied only.

The problem of existence and uniqueness of the least squares estimator (in more detail cf. [3], p. 101) is neglected here as well, since this is not necessary with respect to the problems solved.

Let

$$\kappa_{\delta\beta} = \begin{pmatrix} \delta\beta' H_1 \delta\beta \\ \vdots \\ \delta\beta' H_n \delta\beta \end{pmatrix},$$

where $\delta\beta = \beta - \beta_0, H_i = \partial^2 f_i(\beta)/\partial\beta\partial\beta'|_{\beta=\beta_0}$, $i = 1, \dots, n$, $f_i(.)$ is the i-th component of the function $f(.)$ and β_0 is the mentioned element of $\mathbb{R}^k$.

If, for example, β_0 is so close to the actual β that the vector $\kappa_{\delta\beta}$ can be neglected, then the commonly used procedure leads (under some condition of regularity) to the estimator

$$\hat{\beta} = \beta_0 + \widehat{\delta\beta}, \quad \widehat{\delta\beta} = (F'\Sigma^{-1}F)^{-1}F'\Sigma^{-1}(Y - f_0),$$

where

$$F = \partial f(\beta)/\partial\beta'|_{\beta=\beta_0}, \quad f_0 = f(\beta_0).$$

If a function $h(\beta)$, $\beta \in \mathbb{R}^k$, is to be estimated and the term

$$\frac{1}{2}\delta\beta' \left(\partial^2 h(\beta)/\partial\beta\partial\beta'|_{\beta=\beta_0}\right)\delta\beta$$

can be neglected, then

$$\widehat{h(\beta)} = h(\beta_0) + \left(\partial h(\beta)/\partial\beta'|_{\beta=\beta_0}\right)\widehat{\delta\beta}$$

is the minimum variance linear unbiased estimator.

The problem is how to recognize that the vector $\kappa_{\delta\beta}$ and the value

$$\frac{1}{2}\delta\beta' \left(\partial^2 h(\beta)/\partial\beta\partial\beta'\big|_{\beta=\beta_0}\right)\delta\beta,$$

respectively, can be neglected. For the sake of simplicity in the following only a linear function $h(.)$ is considered and thus the first problem is investigated only.

Assumption 1.1. Let the covariance matrix Σ be positively definite (p.d.) and $f(.)$ have the following properties:

(i) the rank of the $n \times k$ matrix $F(\beta) = \partial f(\beta)/\partial\beta$ is $r[F(\beta)] = k < n$ for any $\beta \in \mathcal{O}(\beta_0)$, where $\mathcal{O}(\beta_0)$ is an open neighbourhood of $\beta_0 \in \mathbb{R}^k$,

(ii) the second derivatives

$$\partial^2 f_i(\beta)/\partial\beta_j\partial\beta_l, \quad i = 1,\ldots,n, \quad \text{and} \quad j,l = 1,\ldots,k,$$

are continuous at any $\beta \in \mathcal{O}(\beta_0)$,
and

(iii) the terms

$$\left(\partial^3 f_i(\beta)/\partial\beta_j\partial\beta_l\partial\beta_s\big|_{\beta=\beta_0}\right)\delta\beta_j\delta\beta_l\delta\beta_s, \quad i = 1,\ldots,n \quad \text{and} \quad j,l,s = 1,\ldots,k,$$

can be neglected for all $\beta = \beta_0 + \delta\beta \in \mathcal{O}(\beta_0)$.

Furthemore the following notation will be used:

$$\kappa_h = (h'H_1h,\ldots,h'H_nh)',$$

where h is any vector from $\mathbb{R}^k$,

$$H_i = \partial^2 f_i(\beta)/\partial\beta\partial\beta'\big|_{\beta=\beta_0}, \quad i = 1,\ldots,n, \quad \Delta = \frac{1}{2}\begin{pmatrix} \delta\beta' H_1 \\ \vdots \\ \delta\beta' H_n \end{pmatrix},$$

i.e., $\frac{1}{2}\kappa_{\delta\beta} = \Delta\delta\beta$, and $C = F'\Sigma^{-1}F$.

The mean value $E(Y|\beta)$ of the vector Y, under Assumption 1.1, is

$$E(Y|\beta) = f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta} = f_0 + (F+\Delta)\delta\beta, \tag{1.1}$$

The BLUE (best linear unbiased estimator) of $\delta\beta$ in the model $Y - f_0 \sim N_n(F\delta\beta, \Sigma)$ is

$$\widehat{\delta\beta}(Y,0) = (F'\Sigma^{-1}F)^{-1}F'\Sigma^{-1}(Y - f_0)$$

and the BLUE of $\delta\beta$ in the model $Y - f_0 \sim N_n[(F+\Delta)\delta\beta, \Sigma]$, where the matrix Δ is known and satisfies the condition $r(F+\Delta) = k$, is

$$\widehat{\delta\beta}(Y, \delta\beta) = [(F+\Delta)'\Sigma^{-1}(F+\Delta)]^{-1}(F+\Delta)'\Sigma^{-1}(Y - f_0)$$

(in more detail cf. [5]).

The statistic

$$\begin{aligned} R_0^2 &= \min\{(Y - f_0 - F\delta\beta)'\Sigma^{-1}(Y - f_0 - F\delta\beta) \colon \delta\beta \in \mathbb{R}^k\} \\ &= (Y - f_0 - F\widehat{\delta\beta})'\Sigma^{-1}(Y - f_0 - F\widehat{\delta\beta}), \end{aligned}$$

where $\widehat{\delta\beta} = (F'\Sigma^{-1}F)^{-1}F'\Sigma^{-1}(Y - f_0)$ (cf. [5]), enables us to verify whether the model $Y = f_0 + F\delta\beta + \varepsilon$ (without the term $\frac{1}{2}\kappa_{\delta\beta}$) is adequate to the measured data or not (whether $\delta\beta = \beta - \beta_0$ is such small that the term $\frac{1}{2}\kappa_{\delta\beta}$ can be neglected).

The statistic R_0^2 can be written in the form

$$R_0^2 = (Y - f_0)'(M_F\Sigma M_F)^+(Y - f_0),$$

more suitable for the following consideration; here

$$M_F = I - F(F'F)^{-1}F'$$

and $(M_F\Sigma M_F)^+$ is the Moore-Penrose inverse (cf. [5]) of the matrix $M_F\Sigma M_F$.

The relation

$$(M_F\Sigma M_F)^+ = \Sigma^{-1} - \Sigma^{-1}F(F'\Sigma^{-1}F)^{-1}F'\Sigma^{-1}$$

can be easily proved.

If

$$E(Y - f_0 | \beta) = F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \quad \kappa_{\delta\beta} \neq 0,$$

then $R_0^2 \sim \chi^2_{n-k}(\delta)$, where the noncentrality parameter

$$\delta = \frac{1}{4}\kappa'_{\delta\beta}(M_F\Sigma M_F)^+\kappa_{\delta\beta}$$

and the probability of the rejection of the adequacy is

$$\gamma = P\{\chi^2_{n-k}(\delta) \geqslant \chi^2_{n-k}(0, 1-\alpha)\},$$

where $\chi^2_{n-k}(0, 1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with $n - k$ degrees of freedom.

With respect to [2], p. 27 the approximation of the noncentral $\chi_f^2(\delta)$ by the random variable

$$\frac{f+2\delta}{f+\delta}\chi^2_{\frac{(f+\delta)^2}{f+2\delta}}(0),$$

where $\chi^2_{\frac{(f+\delta)^2}{f+2\delta}}(0)$ is the central chi-square random variable with $\frac{(f+\delta)^2}{f+2\delta}$ degrees of freedom (they need not be integers),

$$\gamma = P\{\chi^2_{\frac{(f+\delta)^2}{f+2\delta}}(0) \geqslant \frac{f+\delta}{f+2\delta}\chi_f^2(0,1-\alpha)\}$$

where $f = n - k$.

Regarding this consideration the model (1.1) can be linearized at the point β_0 if the value of the term $\frac{1}{2}\kappa_{\delta\beta}$ (i.e., the noncentrality parameter δ) does not influence significantly the value $\gamma = \alpha$ (for $\delta = 0$).

Definition 1.2. The model (1.1) is $d\alpha$-linearizable with respect to its adequacy (to measured data) if

(a)

$$\left|\frac{(f+\delta)^2}{f+2\delta} - f\right| < 0.5 \quad \text{(the degrees of freedom do not change)}$$

and

(b)

$$\Delta\gamma = \left|P\{\chi_f^2(0) \geqslant \frac{f+\delta}{f+2\delta}\chi_f^2(0,1-\alpha)\} - \alpha\right| < d\alpha.$$

Let h be any k-dimensional vector and let

$$\begin{aligned}
b_h^*(\delta\beta) &= E[h'\widehat{\delta\beta}(Y,0)|\delta\beta] - h'\delta\beta, \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),\\
d_h^*(\delta\beta) &= \mathrm{Var}[h'\widehat{\delta\beta}(Y,\delta\beta)|\Sigma] - \mathrm{Var}[h'\widehat{\delta\beta}(Y,0)|\Sigma], \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),\\
U_h^*(\delta\beta) &= h'\widehat{\delta\beta}(Y,\delta\beta) - h'\widehat{\delta\beta}(Y,0), \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),\\
u_h^*(\delta\beta) &= h'\widehat{\delta\beta}(y,\delta\beta) - h'\widehat{\delta\beta}(y,0), \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),
\end{aligned}$$

where y is a realization of the observation vector Y.

Let

$$\begin{aligned}
d_h(\delta\beta) &= \frac{\partial d_h^*(\delta\beta)}{\partial(\delta\beta')}\delta\beta, \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),\\
U_h(\delta\beta) &= \frac{\partial U_h^*(\delta\beta)}{\partial(\delta\beta')}\delta\beta, \beta_0 + \delta\beta \in \mathcal{O}(\beta_0),\\
u_h(\delta\beta) &= \frac{\partial u_h^*(\delta\beta)}{\partial(\delta\beta')}\delta\beta, \beta_0 + \delta\beta \in \mathcal{O}(\beta_0).
\end{aligned}$$

Let $c_b(h)$ (> 0), $c_d^2(h)$, $C_U^2(h)$ and $c_u(h)$ (> 0) be such constants and $\mathcal{O}_b(\beta_0)$, $\mathcal{O}_d(\beta_0)$, $\mathcal{O}_U(\beta_0)$ and $\mathcal{O}_u(\beta_0)$ such neighbourhoods of β_0 that

$$
\begin{aligned}
&(b) && |b_h^*(\delta\beta)| \leqslant c_b(h)\sqrt{h'C^{-1}h}, \beta_0 + \delta\beta \in \mathcal{O}_b(\beta_0),\\
&(d) && |d_h(\delta\beta)| \leqslant c_d^2(h)h'C^{-1}h, \beta_0 + \delta\beta \in \mathcal{O}_d(\beta_0),\\
&(U) && \operatorname{Var}[U_h(\delta\beta)|\Sigma] \leqslant C_U^2(h)h'C^{-1}h, \beta_0 + \delta\beta \in \mathcal{O}_U(\beta_0),\\
&(u) && |u_h(\delta\beta)| \leqslant c_u(h)\sqrt{h'C^{-1}h}, \beta_0 + \delta\beta \in \mathcal{O}_u(\beta_0),
\end{aligned}
$$

respectively.

Definition 1.3. The model (1.1) is c_b-, c_d-, C_U- and c_u-linearizable with respect to a function $h(.)$ in the set $\mathcal{O}_b(\beta_0)$, $\mathcal{O}_d(\beta_0)$, $\mathcal{O}_U(\beta_0)$, and $\mathcal{O}_u(\beta_0)$, if the inequality (b), (d), (U) and (u), respectively, is satisfied.

2 Criteria of a linearization

In the first step the problem of the adequacy of the model (1.1) (cf. Definition 1.2) will be investigated.

Let

$$
\begin{aligned}
f_1(\delta) &= \frac{f+\delta}{f+2\delta}, \ \delta \in (0,\infty),\\
f_2(\delta) &= \frac{(f+\delta)^2}{f+2\delta}, \ \delta \in (0,\infty).
\end{aligned}
$$

If $\delta \ll f$ (only this case is taken into account), then

$$f_1(\delta) \approx 1 - \frac{\delta}{f}, \quad f_2(\delta) \approx f + \frac{\delta^2}{f},$$

in a consequence of which the condition (a) of Definition 1.2 can be written in the form $0 \leqslant \delta \leqslant \sqrt{\frac{f}{2}}$. It will be shown (cf. Example 2.4) that for any reasonable $d\alpha$ in (b) of Definition 1.2 the condition (a) is satisfied in each case when (b) is satisfied.

Let $\chi_f^2(0, 1-\alpha) = q$ and $\frac{f+\delta}{f+2\delta}q = q + \mathrm{d}q$. With respect to the approximation $\frac{f+\delta}{f+2\delta} \approx 1 - \frac{\delta}{f}$, we have $\mathrm{d}q = -\frac{\delta}{f}q$.

Now the condition (b) can be written in the form

$$\Delta\gamma = |P\{\chi_f^2(0) \geqslant q + \mathrm{d}q\} - P\{\chi_f^2(0) \geqslant q\}| < d\alpha.$$

Let $h_f(u) = u^{\frac{f}{2}-1}\mathrm{e}^{-\frac{u}{2}}/[2^{\frac{f}{2}}\Gamma(\frac{f}{2})], u > 0$, and $H_f(q) = \int_0^q h_f(u)\,\mathrm{d}u$. If the approximation

$$\Delta\gamma = |H_f(q + \mathrm{d}q) - H_f(q)| \approx \frac{\mathrm{d}H_f(q)}{\mathrm{d}q}\,\mathrm{d}q = h_f(q)\,\mathrm{d}q < d\alpha$$

is used, and $\mathrm{d}q = -\frac{\delta}{f}q$ is taken into account, then

$$h_f(q)\frac{\delta}{f}q < d\alpha$$

what implies

$$\left(\frac{1}{4}\kappa'_{\delta\beta}(M_F\Sigma M_F)^+\kappa_{\delta\beta} =\right) \quad \delta < \frac{f}{qh_f(q)}d\alpha.$$

By this the following lemma is proved.

Lemma 2.1. *The model* (1.1) *is $d\alpha$-linearizable if*

$$\kappa'_{\delta\beta}(M_F\Sigma M_F)^+\kappa_{\delta\beta} < \frac{4f2^{\frac{f}{2}}\Gamma(\frac{f}{2})}{q^{\frac{f}{2}}\mathrm{e}^{-\frac{q}{2}}}d\alpha.$$

Definition 2.2. The Bates and Watts [1] parameter effect curvature at the point $f(\beta_0)$ in the model (1.1) is

$$K^{(\mathrm{par})} = \sup\{K_s^{(\mathrm{par})} : s \in \mathbf{R}^k\},$$

where

$$K_s^{(\mathrm{par})} = \sqrt{\frac{(P_F^{\Sigma^{-1}}\kappa_s)'\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_s}{(s'F'\Sigma^{-1}Fs)^2}}$$

is the parameter effect curvature at the same point in the direction of the vector $s \in \mathbf{R}^k$; here $P_F^{\Sigma^{-1}} = F(F'\Sigma^{-1}F)^{-1}F'\Sigma^{-1}$ is the projection matrix in the norm $\|x\|_{\Sigma^{-1}} = \sqrt{x'\Sigma^{-1}x}, x \in \mathbf{R}^n$, on the column space $\mathcal{M}(F)$ (tangential space of the mean value surface $f(\beta), \beta \in \mathbf{R}^k$, at the point $f(\beta_0)$) of the matrix F.

The Bates and Watts intrinsic curvature is

$$K^{(\mathrm{int})} = \sup\{K_s^{(\mathrm{int})} : s \in \mathbf{R}^k\},$$

where

$$K_s^{(\mathrm{int})} = \sqrt{\frac{(M_F^{\Sigma^{-1}}\kappa_s)'\Sigma^{-1}M_F^{\Sigma^{-1}}\kappa_s}{(s'F'\Sigma^{-1}Fs)^2}}$$

is the intrinsic curvature at the same point in the direction of the vector $s \in \mathbb{R}^k$; here $M_F^{\Sigma^{-1}} = I - P_F^{\Sigma^{-1}}$.

As $K_{\delta\beta}^{(\mathrm{int})} \leqslant K^{(\mathrm{int})}$ and

$$\kappa'_{\delta\beta}(M_F\Sigma M_F)^+\kappa_{\delta\beta} \leqslant [(\delta\beta)'C\delta\beta]^2[K^{(\mathrm{int})}]^2,$$

the following theorem is a direct consequence of Lemma 2.1 and Definition 2.2.

Theorem 2.3. *If*

$$(\delta\beta)'C\delta\beta \leqslant \frac{r(f,\alpha,d\alpha)}{K^{(\mathrm{int})}},$$

where $r(f,\alpha,d\alpha) = 2\sqrt{f2^{\frac{f}{2}}\Gamma(\frac{f}{2})e^{\frac{q}{2}}q^{-\frac{f}{2}}d\alpha}$, $q = \chi_f^2(0,1-\alpha)$, *the model* (1.1) *is $d\alpha$-linearizable at the point* β_0.

Example 2.4. Let $\alpha = 0.05$. Then the values of $r(f;0.05;d\alpha)$ for $d\alpha =$ 0.01 and 0.05, are given in the following table.

f	$d\alpha = 0.01$	$d\alpha = 0.05$
2	0.7308	1.6342
10	1.1880	2.6565
20	1.4506	3.2437
100	2.2576	5.0482

For the first orientation the following relations can be applied

$$r(f;\alpha = 0.05; d\alpha = 0.01) \approx 1 + 0.013f, \quad f = 2,\dots,100,$$
$$r(f;\alpha = 0.05; d\alpha = 0.05) \approx 2.1 + 0.030f, \quad f = 2,\dots,100.$$

Remark 2.5. Let $\sum\limits_{i=1}^{k} \lambda_i f_i f_i'$, $\lambda_1 \geqslant \ldots \geqslant \lambda_k > 0$ be the spectral decomposition of the matrix $C^{-1} = \mathrm{Var}[\widehat{\delta\beta}(Y,0)|\Sigma]$. Thus the region where the actual value of the vector β $(= \beta_0 + \delta\beta)$ must be located is the ellipsoide

$$\mathcal{E} = \left\{\beta\colon \beta = \beta_0 + \delta\beta, (\delta\beta)'C\delta\beta = \sum_{i=1}^{k}\frac{1}{\lambda_i}(f_i\delta\beta)^2 \leqslant \frac{r(f,\alpha,d\alpha)}{K^{(\mathrm{int})}}\right\}$$

and the values of the semi-axes of it are $\sqrt{\frac{\lambda_i r(f,\alpha,d\alpha)}{K^{(\mathrm{int})}}}$, $i = 1,\dots,k$. The variances $\mathrm{Var}[\widehat{\delta\beta}(Y,0)|\Sigma]$, $i = 1,\dots,k$, occur in the interval $[\lambda_k,\lambda_1]$, the values of the function $r(.;\alpha = 0.05; d\alpha = 0.01)$ are in the interval $[0.5; 2.5]$ (for $f = 2,\dots,100$) and thus $K^{(\mathrm{int})}$ has to be significantly less than 1 in order the semi-axes of the ellipsoide $\mathcal{E}$ to be significantly greater than the standard deviations $\sqrt{\mathrm{Var}[\widehat{\delta\beta}(Y,0)|\Sigma]}$ of the linear estimators. The situation may appear to be less restrictive if $d\alpha$, and the value 0.5 in the condition (a) of Definition 1.2 are enlarged.

Lemma 2.6. *In the model* (1.1) *the bias*

$$b(\delta\beta) = E[\widehat{\delta\beta}(Y,0)|\delta\beta] - \delta\beta$$

can be expressed as follows

$$b(\delta\beta) = E[\widehat{\delta\beta}(Y,0)|\delta\beta] - \delta\beta = C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta}.$$

Proof is obvious. □

Remark 2.7. If in the model (1.1) $\delta\beta \neq 0$, then the bias $b(\delta\beta)$ can be considered as nonsignificant if it is covered by the covariance matrix C^{-1} of the estimator $\widehat{\delta\beta}(Y,0)$ in the following sense:

$$b'(\delta\beta)Cb(\delta\beta)(\text{Mahalanobis distance}) \leqslant \chi^2_k(1-\alpha)\gamma_b^2,$$

where γ_b $(0 < \gamma_b < 1)$ is a constant chosen by a statistician, $\chi^2_k(1-\alpha)$ is the $(1-\alpha)$-quantile of the chi-square distribution with k degrees of freedom and α is the level of the significance chosen also by a statistician.

Lemma 2.8. *Let W be $k \times k$ p.d. matrix. Then*

$$\forall\{h \in \mathbf{R}^k\}\ |h'y| \leqslant |c|\sqrt{y'Wy} \quad \Leftrightarrow \quad y'W^{-1}y \leqslant c^2.$$

Proof. Cf. [6], p. 69. □

Theorem 2.9. *Let $c_b = \gamma_b\sqrt{\chi^2_k(1-\alpha)}$ (cf. Remark 2.7). If*

$$\delta\beta'F'\Sigma^{-1}F\delta\beta \leqslant \frac{2c_b}{K^{(\text{par})}},$$

then

$$\forall\{h \in \mathbf{R}^k\}\ |b_h^*(\delta\beta)| \leqslant c_b\sqrt{h'C^{-1}h}.$$

Furthemore

$$\mathcal{O}_b(\beta_0) = \left\{\beta\colon \beta = \beta_0 + \delta\beta, \delta\beta'C\delta\beta \leqslant 2\frac{c_b}{K^{(\text{par})}}\right\}.$$

Proof. With respect to Lemma 2.6

$$\begin{aligned} &b'(\delta\beta)F'\Sigma^{-1}Fb(\delta\beta) \\ &= \frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}FC^{-1}F'\Sigma^{-1}FC^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} \\ &= \frac{1}{4}(P_F^{\Sigma^{-1}}\kappa_{\delta\beta})'\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta}. \end{aligned}$$

Since

$$K^{(\mathrm{par})} \geqslant K^{(\mathrm{par})}_{\delta\beta} = \sqrt{\frac{(P_F^{\Sigma^{-1}}\kappa_{\delta\beta})'\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta}}{(\delta\beta' F'\Sigma^{-1}F\delta\beta)^2}},$$

$$\begin{aligned}(P_F^{\Sigma^{-1}}\kappa_{\delta\beta})'\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta} &= 4b'(\delta\beta)F'\Sigma^{-1}Fb(\delta\beta)\\ &\leqslant (\delta\beta' F'\Sigma^{-1}F\delta\beta)^2(K^{(\mathrm{par})})^2.\end{aligned}$$

If

$$(\delta\beta' F'\Sigma^{-1}F\delta\beta)^2(K^{(\mathrm{par})})^2 \leqslant 4c_b^2,$$

then

$$b'(\delta\beta)F'\Sigma^{-1}Fb(\delta\beta) \leqslant c_b^2.$$

With respect to Lemma 2.8

$$\begin{aligned}&b'(\delta\beta)F'\Sigma^{-1}Fb(\delta\beta) \leqslant c_b^2 \quad \Leftrightarrow\\ \Leftrightarrow \quad &\forall\{h \in \mathbb{R}^k\}\ |h'b(\delta\beta)| = |b_h(\delta\beta)| \leqslant c_b\sqrt{h'C^{-1}h}.\end{aligned}$$

□

Remark 2.10. If only one function $h(\delta\beta) = h'\delta\beta$, $\delta\beta \in \mathbb{R}^k$, is taken into account, then obviously

$$|b_h^*(\delta\beta)| \leqslant c_b\sqrt{h'C^{-1}h}$$

if and only if

$$\beta \in \mathcal{O}_b(\beta_0) = \left\{\beta\colon \beta = \beta_0 + \delta\beta, \left|\delta\beta' \sum_{i=1}^{k}\{L_h\}_i\frac{1}{2}H_i\delta\beta\right| \leqslant c_b\sqrt{h'C^{-1}h}\right\},$$

where $L_h' = h'C^{-1}F'\Sigma^{-1}$.

Let

$$H_i^* = \begin{pmatrix} e_i'H_1 \\ \vdots \\ e_i'H_n \end{pmatrix}, \quad i = 1, \ldots, k,$$

where $e_i \in \mathbb{R}^k$, $e_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)'$,

$$K_1^{(h)} = \begin{pmatrix} h'C^{-1}\frac{1}{2}(H_1^*)'\Sigma^{-1} \\ \vdots \\ h'C^{-1}\frac{1}{2}(H_k^*)'\Sigma^{-1} \end{pmatrix} \quad \text{and} \quad K_2^{(h)} = \begin{pmatrix} \frac{1}{2}L_h'H_1^* \\ \vdots \\ \frac{1}{2}L_h'H_k^* \end{pmatrix}.$$

Lemma 2.11. *Let in the model* (1.1) $v = Y - f_0 - F\widehat{\delta\beta}(Y,0)$, *where* $\widehat{\delta\beta}(Y,0) = C^{-1}F'\Sigma^{-1}(Y - f_0)$. *Then*

$$\delta\beta' \frac{\partial h'\widehat{\delta\beta}(Y,\delta\beta)}{\partial(\delta\beta)}\Big|_{\delta\beta=0} = \delta\beta'[K_1^{(h)}v - K_2^{(h)}\widehat{\delta\beta}(Y,0)].$$

P r o o f. Let $C(\Delta) = (F+\Delta)'\Sigma^{-1}(F+\Delta)$. Then

$$\begin{aligned}\frac{\partial[h'\widehat{\delta\beta}(Y,\delta\beta)]}{\partial(\delta\beta_i)}\Big|_{\delta\beta=0} &= h'\frac{\partial}{\partial(\delta\beta_i)}\left[C^{-1}(\Delta)(F+\Delta)'\Sigma^{-1}(Y-f_0)\right]\Big|_{\delta\beta=0} \\ &= h'\left[\frac{\partial C^{-1}(\Delta)}{\partial(\delta\beta_i)}(F+\Delta)'\right]\Big|_{\delta\beta=0}\Sigma^{-1}(Y-f_0) \\ &\quad + h'\left[C^{-1}(\Delta)\frac{\partial(F+\Delta)'}{\partial(\delta\beta_i)}\right]\Big|_{\delta\beta=0}\Sigma^{-1}(Y-f_0).\end{aligned}$$

Since

$$\frac{\partial\Delta}{\partial(\delta\beta_i)} = \frac{1}{2}\begin{pmatrix} e_i'H_1 \\ \vdots \\ e_i'H_n\end{pmatrix} = \frac{1}{2}H_i^*, \quad i = 1,\dots,k,$$

$$\frac{\partial C(\Delta)}{\partial(\delta\beta_i)} = \frac{1}{2}[(H_i^*)'\Sigma^{-1}(F+\Delta) + (F+\Delta)'\Sigma^{-1}H_i^*],$$

$$\frac{\partial C^{-1}(\Delta)}{\partial(\delta\beta_i)} = -C^{-1}(\Delta)\frac{\partial C(\Delta)}{\partial(\delta\beta_i)}C^{-1}(\Delta),$$

$$C(\Delta)\big|_{\delta\beta=0} = C \quad \text{and} \quad \Delta\big|_{\delta\beta=0} = 0,$$

it is obvious how to finish the proof. □

Lemma 2.12. *In the model* (1.1)

$$\delta\beta'\frac{\partial\{\mathrm{Var}[h'\widehat{\delta\beta}(Y,\delta\beta)|\Sigma]\}}{\partial\delta\beta}\Big|_{\delta\beta=0} = -\delta\beta'(K_1^{(h)}F + K_2^{(h)})C^{-1}h.$$

P r o o f can be performed analogously as the proof of Lemma 2.11. □

Theorem 2.13. *If in the model* (1.1) $\beta \in \mathcal{O}_d(\beta_0)$, *where*

$$\mathcal{O}_d(\beta_0) = \left\{\delta\beta'\delta\beta \leqslant c_d^2\frac{h'C^{-1}h}{\sqrt{h'C^{-1}(K_1^{(h)}F + K_2^{(h)})'(K_1^{(h)}F + K_2^{(h)})C^{-1}h}}\right\},$$

then $|d_h(\delta\beta)| \leqslant c_d^2 h'C^{-1}h$.

P r o o f. With respect to Lemma 2.12 the quantity $|d_h(\delta\beta)|$ attains its greatest value if $\delta\beta$ is paralel to the vector $(K_1^{(h)}F + K_2^{(h)})C^{-1}h$. Since this value must be less than $c_d^2h'C^{-1}h$, it is obvious how to finish the proof. □

Lemma 2.14. *If the power of components of the vector $\delta\beta$ greater than two is neglected, then in the model* (1.1)

$$U_h(\delta\beta) \sim N_1\left\{h'b(\delta\beta), \delta\beta' W^{(h)}\delta\beta\right\},$$

where

$$W^{(h)} = \left[K_1^{(h)}(\Sigma - FC^{-1}F')(K_1^{(h)})' + K_2^{(h)}C^{-1}(K_2^{(h)})'\right].$$

Proof. It is implied by Lemma 2.11 and by the stochastical independence of the vectors $v = Y - f_0 - F\widehat{\delta\beta}(Y,0)$ and $\widehat{\delta\beta}(Y,0) = C^{-1}F'\Sigma^{-1}(Y - f_0)$. □

Theorem 2.15. *Let the notation $W^{(h)}$ from Lemma* 2.14 *be used.*

If

$$\beta \in \mathcal{O}_U(\beta_0) = \{\beta\colon \beta = \beta_0 + \delta\beta, \delta\beta' W^{(h)}\delta\beta \leqslant C_U^2 h'C^{-1}h\},$$

then $\sqrt{\mathrm{Var}[U_h(\delta\beta)|\Sigma]} \leqslant C_U\sqrt{h'C^{-1}h}$.

Proof. It is a direct consequence of Lemma 2.14. □

Corollary 2.16. *If the criterion from Theorem* 2.15 *is too restrictive for some realization $\widehat{\delta\beta}(y,0)$ of the random variable $\widehat{\delta\beta}(Y,0)$, i.e., if a realization v_{real} of the rezidual v and the vector $\widehat{\delta\beta}(y,0)$ (Lemma* 2.11*) makes the value $u_h(\delta\beta) = \delta\beta'[K_1^{(h)}v_{\mathrm{real}} - K_2^{(h)}\widehat{\delta\beta}(y,0)]$ small, then it is reasonable to calculate the value*

$$\frac{c_u\sqrt{h'C^{-1}h}}{\sqrt{[K_1^{(h)}v_{\mathrm{real}} - K_2^{(h)}\widehat{\delta\beta}(y,0)]'[K_1^{(h)}v_{\mathrm{real}} - K_2^{(h)}\widehat{\delta\beta}(y,0)]}} = T.$$

If $\sqrt{\delta\beta'\delta\beta} \leqslant T$, then $|u_h(\delta\beta)| \leqslant c_u\sqrt{h'C^{-1}h}$.

If the region

$$\{\beta\colon \beta = \beta_0 + \delta\beta, \delta\beta'\delta\beta \leqslant T^2\}$$

covers the region

$$\{\beta\colon \beta = \beta_0 + \delta\beta, \delta\beta' W^{(h)}\delta\beta \leqslant C_U^2 h'C^{-1}h\},$$

then in the actual case the value T is to be preferred.

3. An application and comments

Example 3.1. Let

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3 \left[\begin{pmatrix} \beta_1 t_1 + \beta_1\beta_2 t_1^2 \\ \beta_1 t_2 + \beta_1\beta_2 t_2^2 \\ \beta_1 t_3 + \beta_1\beta_2 t_3^2 \end{pmatrix}, \sigma^2 I \right].$$

In this case the mean value surface $\mathcal{M} = \left\{ f(\beta_1, \beta_2) \colon \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$ is the two-dimensional subspace generated by the vectors $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ and $\begin{pmatrix} t_1^2 \\ t_2^2 \\ t_3^2 \end{pmatrix}$, which causes that the intrinsic curvature $K^{(\mathrm{int})}$ is zero. Nevertheless the model is non-linear and thus the parameter effect curvature is non-zero.

Let the design of experiment be characterized by $t_1 = -1, t_2 = 1, t_3 = 2$ and let $\beta_0 = (\beta_{1,0}, \beta_{2,0})' = (1, \frac{\sqrt{70}-4}{9})'$ $(\frac{\sqrt{70}-4}{9} = 0.485)$.

In such a case

$$f_0 = \begin{pmatrix} -0.515 \\ 1.485 \\ 3.940 \end{pmatrix}, \quad F = \frac{1}{9} \begin{pmatrix} -4.633; & 9 \\ 13.367; & 9 \\ 35.468; & 36 \end{pmatrix};$$

$$C = \sigma^{-2} \begin{pmatrix} 18; & 16.734 \\ 16.734; & 18 \end{pmatrix},$$

$$C^{-1} = \sigma^2 \begin{pmatrix} 0.409; & -0.380 \\ -0.380; & 0.409 \end{pmatrix} = \mathrm{Var}[\widehat{\delta\beta}(Y, 0) | \Sigma],$$

$$P_F^{\Sigma^{-1}} = \frac{1}{11} \begin{pmatrix} 10; & -3; & 1 \\ -3; & 2; & 3 \\ 1; & 3; & 10 \end{pmatrix},$$

$$\hat{\beta} = \begin{pmatrix} 0 \\ 0.641 \end{pmatrix} + \begin{pmatrix} -0.591Y_1 + 0.227Y_2 + 0.091Y_3 \\ 0.605Y_1 - 0.156Y_2 + 0.138Y_3 \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 0; & 1 \\ 1; & 0 \end{pmatrix} = H_2, \quad H_3 = \begin{pmatrix} 0; & 4 \\ 4; & 0 \end{pmatrix}.$$

Criterion for the bias (Theorem 2.9): In this case the restriction on $\delta\beta$ is characterized by the inequality

$$\delta\beta' C \delta\beta \leqslant 2 \frac{c_b}{K^{(\mathrm{par})}}.$$

For our input data

$$K_{\delta\beta}^{(\mathrm{par})} = \sigma \frac{|\delta\beta_1 \delta\beta_2| \sqrt{72}}{D},$$

where

$$D = (6 + 16\beta_{2,0} + 18\beta_{2,0}^2)(\delta\beta_1)^2 + (16\beta_{1,0} + 36\beta_{1,0}\beta_{2,0})\delta\beta_1\delta\beta_2 + 18\beta_{1,0}^2(\delta\beta_2)^2.$$

For $\beta_{1,0} = 1, \beta_{2,0} = 0.485$ we have

$$6 + 16\beta_{2,0} + 18\beta_{2,0}^2 = 18\beta_{1,0}^2$$

and the maximum of $K_{\delta\beta}^{(\mathrm{par})}$ is attained for $\delta\beta = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, $\alpha = -\pi/4$.

Thus

$$K^{(\mathrm{par})} = \sup\left\{K_{\delta\beta}^{(\mathrm{par})} : \delta\beta \in \mathbb{R}^2\right\} = 3.351\sigma,$$

what, with respect to [1], can be considered as an extremaly great value.

The considered restriction can be now rewritten as follows

$$\delta\beta' \frac{1}{c_b\sigma} \begin{pmatrix} 30.159; & 28.038 \\ 28.038; & 30.159 \end{pmatrix} \delta\beta \leqslant 1.$$

The domain $\mathcal{O}_b(\beta_0)$ in the parametric space $\mathbb{R}^2$, characterized by this relationship, is the ellipse with the centre at the point $(1; 0.485)'$ and with the minor semi-axis equals to $\sqrt{c_b\sigma}0.131$ in the direction of the vector $(1/\sqrt{2}; 1/\sqrt{2})'$ and the major semi-axis equals to $\sqrt{c_b\sigma}0.687$ in the direction $(-1/\sqrt{2}; 1/\sqrt{2})'$. Thus, with respect to relatively large value of the parametric effect curvature, this domain is small.

Nevertheless the situation need not be so pesimistic in the case of a single function (Remark 2.10).

Let $h(\beta) = \beta_1$, $\beta \in \mathbb{R}^2$, i.e., the vector $h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In such case

$$L_h' = h'C^{-1}F'\Sigma^{-1} = \frac{1}{198}(-117; 45; 18)$$

and

$$\sum_{i=1}^{2} \{L_h\}_i \frac{1}{2} H_i = 0_{2\times 2};$$

therefore no restrictions occur.

In the case of the function $h(\beta) = \beta_2$, $\beta \in \mathbb{R}^2$, i.e., $h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain

$$\frac{1}{396}\left|\delta\beta' \begin{pmatrix} 0; & 198.238 \\ 198.238; & 0 \end{pmatrix} \delta\beta\right| \leqslant c_b\sqrt{\{C^{-1}\}_{2,2}} = \sigma c_b\sqrt{\frac{9}{22}},$$

or equivalently

$$|\delta\beta_1\delta\beta_2| \leqslant \sigma c_b 0.639.$$

It is obvious that the estimator for $\delta\beta_2$ is significantly more sensitive to the nonlinearity of the model than the estimator of $\delta\beta_1$.

As far as the variance is concerned, we obtain the restriction on the vector $\delta\beta$ for $h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the form

$$\sqrt{h'C^{-1}(K_1^{(h)}F + K_2^{(h)})'(K_1^{(h)}F + K_2^{(h)})C^{-1}h\delta\beta'\delta\beta} \leqslant c_d^2 h'C^{-1}h.$$

As

$$K_1^{(1,0)'} = \frac{1}{44}\begin{pmatrix} -8.367; & -8.367; & -33.468 \\ 9; & 9; & 36 \end{pmatrix}$$

and

$$K_2^{(1,0)'} = \begin{pmatrix} 0; & 0 \\ 0; & 0 \end{pmatrix},$$

we obtain

$$(1;0)C^{-1}(K_1^{(h)}F)'K_1^{(h)}FC^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

and again no restrictions on $\delta\beta$ regarding the variance of the estimator of the first parameter β_1 occur.

For the other parameter $h = (0;1),'$

$$K_1^{(0,1)'} = \frac{1}{44}\begin{pmatrix} 9; & 9; & 36 \\ -8.367; & -8.367; & -33.468 \end{pmatrix}$$

and

$$K_2^{(0,1)'} = \frac{1}{44}\begin{pmatrix} 0; & 197.880 \\ 197.880; & 0 \end{pmatrix}.$$

Thus the restriction is

$$\delta\beta'\delta\beta \leqslant c_d^2 2.32$$

(relatively rigorous; it is the similar situation as in the case of the bias for $\delta\beta$).

Regarding the criterion (U) for the function $h(\beta) = \beta_1$, $\beta \in \mathbb{R}^2$, we obtain (Theorem 2.15)

$$W^{(1,0)'} = \sigma^2\begin{pmatrix} 0.0222; & -0.0239 \\ -0.0239; & 0.0257 \end{pmatrix} = \sigma^2 V^{(1,0)'}$$

(since $K_2^{(1,0)'} = 0$) and thus the region $\mathcal{O}_U(\beta_0)$ is characterized by the relation

$$\delta\beta' V^{(1,0)'}\delta\beta \leqslant C_U^2 0.409.$$

This region is a degenerate ellipse (the determinant of $V^{(1,0)'}$ is zero) with the minor semi-axis of the value $C_U 2.922$ in the direction of the vector $(-0.681; 0.732)'$ and the other semi-axis is infinite.

In the case of the function $h(\beta) = \beta_2$, $\beta \in \mathbb{R}^2$, we obtain

$$W^{(0,1)'} = \sigma^2 \begin{pmatrix} 0.1278; & -0.1189 \\ -0.1189; & 0.1244 \end{pmatrix}$$

and thus the $\mathcal{O}_U(\beta_0)$ is characterized by the ellipse with the minor semi-axis equal to $C_U 1.292$ in the direction of the vector $(-0.712; 0.702)'$ and the major semi-axis equal to $C_U 2.383$.

Remark 3.2. As the model from Example 3.1 is of the zero intrinsic curvature, it is useful to reparametrize the model (cf. [5], [6]). A natural reparametrization seems to be $\theta_1 = \beta_1$, $\theta_2 = \beta_1\beta_2$ and thus

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3 \left[\begin{pmatrix} -1; & 1 \\ 1; & 1 \\ 2; & 4 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \sigma^2 I \right].$$

The BLUE of $(\theta_1; \theta_2)'$ is

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} -0.591Y_1 + 0.227Y_2 + 0.091Y_3 \\ 0.318Y_1 - 0.046Y_2 + 0.182Y_3 \end{pmatrix}$$

and the variance matrix is

$$\operatorname{Var}(\hat{\theta}|\Sigma) = \sigma^2 \begin{pmatrix} 0.409; & -0.182 \\ -0.182; & 0.136 \end{pmatrix}.$$

We can see that $\hat{\theta}_1 = \hat{\beta}_1$. If in the reparametrized model the parameter β_2 is estimated by the statistic

$$\tilde{\beta}_2 = \frac{\hat{\theta}_2}{\hat{\theta}_1} = \frac{0.318Y_1 - 0.045Y_2 + 0.182Y_3}{-0.591Y_1 + 0.227Y_2 + 0.091Y_3},$$

then $\tilde{\beta}_2 \neq \hat{\beta}_2 = 0.641 + 0.605Y_1 - 0.156Y_2 + 0.138Y_3$. Nevertheless, if at the point $\begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} = \begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.485 \end{pmatrix}$ the approximate formula for the variance of the statistic $\tilde{\beta}_2$ is used, we obtain

$$\begin{aligned} \operatorname{Var}(\tilde{\beta}_2|\Sigma) = \sigma^2 & \left(\frac{\partial\left(\frac{\theta_2}{\theta_1}\right)}{\partial\theta_1}, \frac{\partial\left(\frac{\theta_2}{\theta_1}\right)}{\partial\theta_2} \right)\Bigg|_{\theta_0} \begin{pmatrix} 0.409; & -0.182 \\ -0.182; & 0.136 \end{pmatrix} \\ & \times \left[\left(\frac{\partial\left(\frac{\theta_2}{\theta_1}\right)}{\partial\theta_1}, \frac{\partial\left(\frac{\theta_2}{\theta_1}\right)}{\partial\theta_2} \right)\Bigg|_{\theta_0} \right]' \\ & = \sigma^2 0.409 = \operatorname{Var}(\hat{\beta}_2|\Sigma). \end{aligned}$$

R e m a r k 3.3. The design of experiment, characterized in our case by the points $t_1 = -1$, $t_2 = 1$, $t_3 = 2$, can influence the curvature of the model significantly. Let us change the design as follows: $t_1 = -10$, $t_2 = 10$, $t_3 = 20$ and let $\beta_{1,0}$, $\beta_{2,0}$ be the same as before.

The mean value surface in the new model is unchanged and thus the intrinsic curvature remains zero. However the parameter effect curvature is changed drastically, what has a great influence on the restriction on $\delta\beta$ in the linearization.

For the new design we have (the upper index "2" means the new design)

$$H_1^{(2)} = \begin{pmatrix} 0; & 100 \\ 100; & 0 \end{pmatrix} = H_2^{(2)}; \quad H_3^{(2)} = \begin{pmatrix} 0; & 400 \\ 400; & 0 \end{pmatrix};$$
$$\kappa_{\delta\beta}^{(2)} = 100\kappa_{\delta\beta}, (P_F^{\Sigma^{-1}})^{(2)} = P_F^{\Sigma^{-1}}.$$

Thus

$$\left(K_{\delta\beta}^{(\mathrm{par})}\right)^{(2)} = \sigma \frac{10\sqrt{72}|\delta\beta_1\delta\beta_2|}{50700.5(\delta\beta_1)^2 + 190600\delta\beta_1\delta\beta_2 + 180000(\delta\beta_2)^2},$$

and

$$\left(K^{(\mathrm{par})}\right)^{(2)} = \sup\left\{\left(K_{\delta\beta}^{(\mathrm{par})}\right)^{(2)} : \delta\beta \in \mathbb{R}^2\right\} = \sigma 0.184$$

is attained for $\delta\beta = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} -0.8833 \\ 0.4688 \end{pmatrix}$. The new curvature is thus 18.210-times smaller than the original one.

The aim of the consideration of this section is to demonstrate that even in the case of a relative great non-linearity there exist functions for which the linearization is possible without any rigorous requirement on the a priori information on the value of the vector parameter β.

Acknowledgement. Author thanks to the director of the Institute of Geoscience of University Stuttgart, Prof. Dr.-Ing. habil. Tekn. Dr. h.c. E. Grafarend for fruitful discussions during the preparation of the manuscript.

References

[1] *Bates, D. M. and Watts, D. G.*: Relative curvature measures of nonlinearity. J. Roy. Stat. Soc. B *42* (1980), 1–25.

[2] *Janko,J.*: Statistické tabulky. Praha, Academia, 1958.

[3] *Pázman, A.*: Nonlinear Statistical Models. Kluwer Academic Publishers, Dordrecht-Boston-London and Ister Science Press, Bratislava, 1993.

[4] *Potocký, R., To Van Ban*: Confidence regions in nonlinear regression models. Appl. of Math. *37* (1992), 29–39.

[5] *Rao, C.R.*: Linear Statistical Inference and Its Applications (2nd Edition). J. Wiley, New York, 1973.

[6] *Scheffé, H.*: The Analysis of Variance (fifth printing). J. Wiley, New York-London-Sydney, 1967.

Author's address: *Lubomír Kubáček*, Department of Mathematical Analysis and Numerical Methods, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc.