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A DIRECT GLOBAL SUPERCONVERGENCE ANALYSIS FOR SOBOLEV AND VISCOELASTICITY TYPE EQUATIONS

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Abstract. In this paper we study the finite element approximations to the Sobolev and viscoelasticity type equations and present a direct analysis for global superconvergence for these problems, without using Ritz projection or its modified forms.

Keywords: Sobolev and viscoelasticity type equations, global superconvergence, direct analysis

MSC 2000: 65B05, 65N30

1. Introduction

Let Ω be a rectangular domain. In order to explain our superconvergence analysis for FEMs succinctly, we only consider the simple Sobolev type equation

$$\begin{cases} -\triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = v & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

and a viscoelasticity type equation

$$\begin{cases} u_{tt} - \triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = v,\ u_t(x,0) = w & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

In (1.1) and (1.2), f, v and w are sufficiently smooth functions.

The problems (1.1) and (1.2) can arise from many physical processes. The numerical approximations to the solutions of these problems have been investigated by Ewing [2, 3], Ford [4], Ford and Ting [5, 6], and Wahlbin [14]. Also Arnold, Douglas and Thomée [1] and Nakao [13] have considered Galerkin approximations to the soluton of the problem (1.1) in a single space dimension with periodic boundary conditions. L^2-error estimates and the interior pointwise superconvergence results have been derived by these authors. In particular, some futher investigations of the finite element methods for the problems (1.1) and (1.2) have been carried out by Lin, Thomée and Wahlbin [12].

According to the conventional error analysis for FEMs of the time-dependent problems, either the Ritz projection initiated by Wheeler [15] or its modifided forms e.g. the so-called Ritz-Volterra projection introduced by Lin et al. [12], have to be used as transitional tools. However, here we will use a new analysis in [7], i.e. an analysis for the "short side" in the FE-right triangle plus the sharp integral estimates of the "hypotenuse", instead of using Ritz projection or its modified forms, to gain the global superconvergence for the problems (1.1) and (1.2) by an interpolation postprocessing technique, rather than the interior pointwise superconvergence by means of the average technique with which numerical analysts are familiar. Our analysis sharpens the results and shortens the proofs of error estimates that appeared in the previous literature under the rectangular mesh assumption.

2. Sobolev type equations

First of all, we discuss the model problem (1.1). Here and below, assume that T^h is a rectangular partition over Ω with mesh size h. The weak form of (1.1) consists in finding $u(.\,,t) \in H_0^1(\Omega)$ for any $t \in [0,T]$ (the Sobolev space) such that

$$\begin{cases} (\nabla u_t, \nabla\varphi) + (\nabla u, \nabla\varphi) = (f,\varphi) \quad \forall\varphi \in H_0^1, \\ u(0) = v. \end{cases} \tag{2.1}$$

Let $S_0^h \subset H_0^1$ consist of piecewise bilinear functions. Thus, a continuous Galerkin approximation $u^h(x,t)\colon\ [0,T] \to S_0^h$ is defined so that

$$\begin{cases} (\nabla u_t^h, \nabla\varphi) + (\nabla u^h, \nabla\varphi) = (f,\varphi) \quad \forall\varphi \in S_0^h, \\ u^h(0) = i_h v, \end{cases} \tag{2.2}$$

where $i_h v \in S_0^h$ is the bilinear interpolation function of v. From (2.1) and (2.2) we get the error equation

$$(\nabla(u_t^h - u_t), \nabla\varphi) + (\nabla(u^h - u), \nabla\varphi) = 0 \quad \forall\varphi \in S_0^h. \tag{2.3}$$

We need the following (see [9])

2.1. Lemma. *For* $\varphi \in S_0^h$,

$$|(\nabla(u - i_h u), \nabla\varphi)| \leqslant \begin{cases} ch^2\|u\|_3|\varphi|_1, \\ ch^2\|u\ |_4\|\varphi\|_0. \end{cases}$$

2.1. Theorem. *For sufficiently smooth* u *and* u_t, *we have*

$$\|u^h - i_h u\|_1 \leqslant ch^2\left[\int_0^t (\|u_t\|_3 + \|u\|_3)^2\,\mathrm{d}s\right]^{1/2}.$$

P r o o f. Let

$$\theta(x,t) = u^h - i_h u.$$

Then, by virtue of (2.3) we have for $\varphi \in S_0^h$

$$(\nabla\theta_t, \nabla\varphi) + (\nabla\theta, \nabla\varphi) = (\nabla(u_t - i_h u_t), \nabla\varphi) + (\nabla(u - i_h u), \nabla\varphi). \tag{2.4}$$

Hence, with $\varphi = \theta_t$ (where θ_t is the classical derivative) and Lemma 2.1,

$$|\theta_t|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 \leqslant ch^2(\|u_t\|_3 + \|u\|_3)|\theta_t|_1$$

or

$$|\theta_t|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 \leqslant ch^4(\|u_t\|_3 + \|u\|_3)^2 + \frac{1}{2}|\theta_t|_1^2.$$

By integration with respect to t, it follows from $\theta(x,0) = 0$ that

$$|\theta|_1 \leqslant ch^2\left[\int_0^t (\|u_t\|_3 + \|u\|_3)^2\,\mathrm{d}s\right]^{1/2}.$$

$\square$

In order to derive L^∞ estimates, we introduce the discrete Green function $G_z^h \in S_0^h$ at any point z$\in \overline{\Omega}$ such that, for $\varphi \in S_0^h$,

$$(\nabla G_z^h, \nabla\varphi) = \varphi(z), \qquad (\nabla D_z G_z^h, \nabla\varphi) = D_z\varphi.$$

We need the following

2.2. Lemma. ([16]) $\|G_z^h\|_0 \leqslant c$, $\|D_z G_z^h\|_0 \leqslant c\Big(\log\dfrac{1}{h}\Big)^{1/2}$.

2.2. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|u_t^h - u_t\|_{0,\infty} \leqslant ch^2(\|u_t\|_{2,\infty} + \|u_t\|_4 + \|u\|_4).$$

P r o o f. Taking $\varphi = G_z^h$ in (2.4), we have by Lemma 2.1 and Lemma 2.2

$$\theta_t(z,t) + \theta(z,t) \leqslant ch^2(\|u_t\|_4 + \|u\|_4), \tag{2.5}$$

and hence, with $\theta(0) = 0$,

$$\theta_t(z,t) + \int_0^t \theta_t(z,s)\,\mathrm{d}s \leqslant ch^2(\|u_t\|_4 + \|u\|_4).$$

Therefore, it follows from Gronwall's Lemma that

$$|\theta_t(z,t)| \leqslant ch^2(\|u_t\|_4 + \|u\|_4).$$

Thus, Theorem 2.2 holds in virtue of the triangle inequality

$$\|u_t^h - u_t\|_{0,\infty} \leqslant \|u_t^h - i_h u_t\|_{0,\infty} + \|i_h u_t - u_t\|_{0,\infty}.$$

□

2.3. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|u^h - u\|_{0,\infty} \leqslant ch^2(\|u\|_{2,\infty} + \|u\|_4 + \|u_t\|_4).$$

P r o o f. From (2.5) and Theorem 2.2, we drive

$$|\theta(z,t)| \leqslant ch^2(\|u_t\|_4 + \|u\|_4),$$

and hence, Theorem 2.3 holds in virtue of the triangle inequality

$$\|u^h - u\|_{0,\infty} \leqslant \|u^h - i_h u\|_{0,\infty} + \|i_h u - u\|_{0,\infty}.$$

□

2.4. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|u_t^h - i_h u_t\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u_t\|_4 + \|u\|_4).$$

P r o o f. Setting $\varphi = D_z G_z^h$ in (2.4), we get by means of Lemma 2.1 and 2.2

$$D_z \theta_t(z,t) + D_z \theta(z,t) \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u_t\|_4 + \|u\|_4). \tag{2.6}$$

And thus, with $D_z\theta(0) = 0$,

$$D_z \theta_t(z,t) + \int_0^t D_z \theta_t(z,s)\,\mathrm{d}s \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u_t\|_4 + \|u\|_4).$$

Then, Theorem 2.4 follows from Gronwall's Lemma. □

2.5. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|u^h - i_h u\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u_t\|_4 + \|u\|_4).$$

P r o o f. From (2.6) and Theorem 2.4 we obtain Theorem 2.5. □

Theorems 2.1, 2.4 and 2.5 play key roles in the analysis of the global superconvergence for the problem (1.1). Now we use an interpolation postprocessing technique from [8] to get the desired results. We assume that T^h has been gained from T^{2h} with mesh size $2h$ by subdividing each element of T^{2h} into four congruent elements. Thus, we can define a nodal biquadratic interpolation operator I_{2h}^2 associated with T^{2h}. It is easy to check that

$$I_{2h}^2 i_h = I_{2h}^2, \quad \|I_{2h}^2 \varphi\|_{1,p} \leqslant c\|\varphi\|_{1,p} \qquad \forall \varphi \in S_0^h \ \ (p = 2, \infty),$$

$$\|I_{2h}^2 \varphi - \varphi\|_{1,p} \leqslant ch^2 \|\varphi\|_{3,p} \quad (p = 2, \infty).$$

And thus, we have the following main results.

2.6. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2 \left\{ \|u\|_3 + \left[\int_0^t (\|u_t\|_3 + \|u\|_3)^2 \,\mathrm{d}s \right]^{1/2} \right\}.$$

Proof. Due to the property of I_{2h}^2, we have

$$I_{2h}^2 u^h - u = I_{2h}^2 (u^h - i_h u) + (I_{2h}^2 u - u).$$

Therefore, it follows from Theorem 2.1 and the interpolation theorem that

$$\begin{aligned} \|I_{2h}^2 u^h - u\|_1 &\leqslant c\|u^h - i_h u\|_1 + ch^2\|u\|_3 \\ &\leqslant ch^2 \left\{ \|u\|_3 + \left[\int_0^t (\|u_t\|_3 + \|u\|_3)^2 \, \mathrm{d}s \right]^{1/2} \right\}. \end{aligned}$$

□

Analogously, by Theorems 2.4 and 2.5 we have the next assertions.

2.7. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|I_{2h}^2 u_t^h - u_t\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u_t\|_{3,\infty} + \|u_t\|_4 + \|u\|_4).$$

2.8. Theorem. *For sufficiently smooth u and u_t, we have*

$$\|I_{2h}^2 u^h - u\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} (\|u\|_{3,\infty} + \|u_t\|_4 + \|u\|_4).$$

3. Viscoelasticity type equations

In this section, we will consider the semidiscrete Galerkin approximation to the problem (1.2). The weak form of (1.2) reads as follows: Find $u(.\,,t) \in H_0^1(\Omega)$ for any fixed $t \in [0,T]$ such that

$$\text{(3.1)} \qquad \begin{cases} (u_{tt}, \varphi) + (\nabla u_t, \nabla \varphi) + (\nabla u, \nabla \varphi) = (f, \varphi) \ \ \forall \varphi \in H_0^1(\Omega), \\ u(0) = v, \qquad u_t(0) = w. \end{cases}$$

Thus, a continuous Galerkin approximation $u^h(x,t)\colon [0,T] \to S_0^h$ is defined so that

$$\text{(3.2)} \qquad \begin{cases} (u_{tt}^h, \varphi) + (\nabla u_t^h, \nabla \varphi) + (\nabla u^h, \nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in S_0^h, \\ u^h(0) = i_h v, \qquad u_t^h(0) = i_h w, \end{cases}$$

where $i_h v$, $i_h w \in S_0^h$ stand for the bilinear interpolation functions of v and w, respectively. Then, we obtain the error equation from (3.1) and (3.2):

$$\text{(3.3)} \qquad (u_{tt} - u_{tt}^h, \varphi) + (\nabla(u_t - u_t^h), \nabla \varphi) + (\nabla(u - u^h), \nabla \varphi) = 0 \qquad \forall \varphi \in S_0^h.$$

3.1. Theorem. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\|u^h - i_h u\|_1 + \|u_t^h - u_t\|_0 \leqslant ch^2 \left\{ \|u_t\|_2 + \left[\int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \,\mathrm{d}s \right]^{1/2} \right\}.$$

P r o o f. Let

$$\theta(x,t) = u^h(x,t) - i_h u(x,t).$$

According to (3.3), we have for $\varphi \in S_0^h$ that

$$\begin{aligned} (\theta_{tt}, \varphi) + (\nabla\theta_t, \nabla\varphi) + (\nabla\theta, \nabla\varphi) &= (u_{tt} - i_h u_{tt}, \varphi) + (\nabla(u_t - i_h u_t), \nabla\varphi) \\ &\quad + (\nabla(u - i_h u), \nabla\varphi), \end{aligned} \tag{3.4}$$

and hence, with $\varphi = \theta_t$ and Lemma 2.1,

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\theta_t\|_0^2 + c\|\theta_t\|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 &\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2 + |\theta_t|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 \\ &\leqslant ch^4(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 + \frac{c}{2}\|\theta_t\|_0^2, \end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta_t\|_0^2 + |\theta|_1^2) \leqslant ch^4(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2.$$

By integration with respect to t, it follows from $\theta(0) = \theta_t(0) = 0$ that

$$\|\theta_t\|_0^2 + |\theta|_1^2 \leqslant ch^4 \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \,\mathrm{d}s,$$

that is

$$\|\theta_t\|_0 + |\theta|_1 \leqslant ch^2 \left[\int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \,\mathrm{d}s \right]^{1/2}.$$

And thus, Theorem 3.1 follows from the triangle inequality

$$\|u_t^h - u_t\|_0 \leqslant \|u_t^h - i_h u_t\|_0 + \|i_h u_t - u_t\|_0.$$

$\square$

3.2. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned} \|u_t^h - i_h u_t\|_1 + \|u_{tt}^h - u_{tt}\|_0 \leqslant ch^2 \{ &\|u_{tt}\|_2 + [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2} \}. \end{aligned}$$

P r o o f. Differentiating (3.4) with respect to t, we get, for $\varphi \in S_0^h$,

$$(\theta_{ttt}, \varphi) + (\nabla\theta_{tt}, \nabla\varphi) + (\nabla\theta_t, \nabla\varphi) = (u_{ttt} - i_h u_{ttt}, \varphi) + (\nabla(u_{tt} - i_h u_{tt}), \nabla\varphi) + (\nabla(u_t - i_h u_t), \nabla\varphi),$$

and hence, with $\varphi = \theta_{tt}$ and Lemma 2.1,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{tt}\|_0^2 + c\|\theta_{tt}\|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta_t|_1^2 &\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{tt}\|_0^2 + |\theta_{tt}|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta_t|_1^2 \\ &\leqslant ch^4(\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 + \frac{c}{2}\|\theta_{tt}\|_0^2,\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta_{tt}\|_0^2 + |\theta_t|_1^2) \leqslant ch^4(\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2.$$

Therefore, by virtue of $\theta_t(0) = 0$, we get

$$\|\theta_{tt}\|_0^2 + |\theta_t|_1^2 \leqslant \|\theta_{tt}(0)\|_0^2 + ch^4\int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s. \tag{3.5}$$

Let t $= 0$ and $\varphi = \theta_{tt}(0)$ in (3.4). Then

$$\|\theta_{tt}(0)\|_0 \leqslant ch^2(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)$$

which, together with (3.5), leads to

$$\|\theta_{tt}\|_0^2 + |\theta_t|_1^2 \leqslant ch^4\Big[(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s\Big],$$

or

$$\|\theta_{tt}\|_0 + \|\theta_t\|_1 \leqslant ch^2\Big[(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s\Big]^{1/2},$$

and Theorem 3.2 follows from the triangle inequality

$$\|u_{tt}^h - u_{tt}\|_0 \leqslant \|u_{tt}^h - i_h u_{tt}\|_0 + \|i_h u_{tt} - u_{tt}\|_0.$$

□

3.3. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|u_t^h - u_t\|_{0,\infty} \leqslant ch^2\Big\{ &\|u_{tt}\|_2 + \|u_t\|_4 + \|u_t\|_{2,\infty} + \|u\|_4 \\ &+ [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

P r o o f. Setting $\varphi = G_z^h$ in (3.4), we have according to Lemmas 2.1 and 2.2

$$(\theta_{tt}, G_z^h) + \theta_t(z,t) + \theta(z,t) \leqslant ch^2(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4), \tag{3.6}$$

or

$$|\theta_t(z,t)| \leqslant \int_0^t |\theta_t(z,s)| \,\mathrm{d}s + c\|\theta_{tt}\|_0 + ch^2(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4).$$

And thus, it follows from Gronwall's Lemma and Theorem 3.2 that

$$\begin{aligned}|\theta_t(z,t)| \leqslant ch^2\Big\{ &\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 \\ &+ [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

Then, Theorem 3.3 holds by virtue of the triangle inequality

$$\|u_t^h - u_t\|_{0,\infty} \leqslant \|u_t^h - i_h u_t\|_{0,\infty} + \|i_h u_t - u_t\|_{0,\infty}.$$

□

3.4. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|u^h - u\|_{0,\infty} \leqslant ch^2\Big\{ &\|u\|_{2,\infty} + \|u\|_4 + +\|u_t\|_4 + \|u_{tt}\|_2 \\ &+ [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

P r o o f. From (3.6), Lemma 2.2 and Theorems 3.2, 3.3 we derive

$$\begin{aligned}|\theta(z,t)| &\leqslant |\theta_t(z,t)| + c\|\theta_{tt}\|_0 + ch^2(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4) \\ &\leqslant ch^2\Big\{ \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 \\ &\quad + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\},\end{aligned}$$

and hence, Theorem 3.4 holds in virtue of the triangle ineqality

$$\|u^h - u\|_{0,\infty} \leqslant \|u^h - i_h u\|_{0,\infty} + \|i_h u - u\|_{0,\infty}.$$

□

3.5. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|u_t^h - i_h u_t\|_{1,\infty} \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big\{&\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4\\ &+ [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2\\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

P r o o f. Taking $\varphi = D_z G_z^h$ in (3.4), we obtain according to Lemmas 2.1 and 2.2

$$(3.7)\quad (\theta_{tt}, D_z G_z^h) + D_z\theta_t(z,t) + D_z\theta(z,t) \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4),$$

or

$$\begin{aligned}|D_z\theta_t(z,t)| \leqslant \int_0^t |D_z\theta_t(z,s)|\,\mathrm{d}s + c\Big(\log\frac{1}{h}\Big)^{1/2}\|\theta_{tt}\|_0\\ + ch^2\Big(\log\frac{1}{h}\Big)^{1/2}(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4).\end{aligned}$$

And thus, Theorem 3.5 follows from Gronwall's Lemma and Theorem 3.2. □

3.6. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|u^h - i_h u\|_{1,\infty} \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big\{&\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4\\ &+ [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2\\ &+ \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

P r o o f. From (3.7) and Theorems 3.2, 3.5 we obtain Theorem 3.6. □

Theorems 3.1, 3.2, 3.5 and 3.6 are essential, by them we can get the global superconvergence for the problem (1.2) instead of the interior pointwise superconvergence. Identically to Section 2, we derive the following main theorems by means of the interpolation postprocessing technique initiated in [8].

3.7. Theorem. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2 \left\{ \|u_t\|_2 + \|u\|_3 + \left[\int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s \right]^{1/2} \right\}.$$

3.8. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|I_{2h}^2 u_t^h - u_t\|_1 \leqslant ch^2 \Big\{ & \|u_{tt}\|_2 + \|u_t\|_3 \\ & + [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ & + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \, \mathrm{d}s]^{1/2} \Big\}.\end{aligned}$$

3.9. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|I_{2h}^2 u_t^h - u_t\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} \Big\{ & \|u_{tt}\|_2 + \|u_t\|_4 + \|u_t\|_{3,\infty} \\ & + \|u\|_4 + [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ & + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \, \mathrm{d}s]^{1/2} \Big\}.\end{aligned}$$

3.10. Theorem. *For sufficiently smooth u, u_t, u_{tt} and u_{ttt}, we have*

$$\begin{aligned}\|I_{2h}^2 u^h - u\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h} \Big)^{1/2} \Big\{ & \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_{3,\infty} \\ & + \|u\|_4 + [(\|u_{tt}(0)\|_2 + \|u_t(0)\|_4 + \|u(0)\|_4)^2 \\ & + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_4 + \|u_t\|_4)^2 \, \mathrm{d}s]^{1/2} \Big\}.\end{aligned}$$

Remark 1. In another paper, we will discuss the case of k ($k \geqslant 2$) which is the order of finite elements for the problems above.

Remark 2. When Ω is a convex quadrilateral domain, the corresponding superconvergent results hold for such problems as above if the quadrilateral meshes are almost uniform and are constructed by connecting the equi-proportional points of two opposite boundaries.

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