María Dolores Esteban

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# A GENERAL CLASS OF ENTROPY STATISTICS

M. D. Esteban, Madrid

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*Abstract.* To study the asymptotic properties of entropy estimates, we use a unified expression, called the $H_{h,v}^{\varphi_1,\varphi_2}$-entropy. Asymptotic distributions for these statistics are given in several cases when maximum likelihood estimators are considered, so they can be used to construct confidence intervals and to test statistical hypotheses based on one or more samples. These results can also be applied to multinomial populations.

*Keywords*: entropy, asymptotic distribution, maximum likelihood estimators, testing statistical hypotheses

*MSC 2000*: 62B10, 62E20

## 1. Introduction

To study a majority of entropy measures cited in literature, a general mathematical expression is proposed in this paper. In favour of this mathematical tool is the fact that any entropy measure can be obtained as a particular case of the $H_{h,v}^{\varphi_1,\varphi_2}$-entropy functional, and therefore, all properties which are proved for the functional are also true for any entropy measure. Entropy estimates are obtained by replacing parameters by their corresponding maximum likelihood estimates and their asymptotic distributions are obtained, too. Applications to test statistical hypotheses and to build confidence intervals are also given.

Let $(\mathfrak{X}, \beta_{\mathfrak{X}}, P_\theta)_{\theta\in\Theta}$ be a statistical space, where $\mathfrak{X}$ is the sample space and $\Theta$ is an open subset of $\mathbb{R}^M$. We shall assume that there exists a probability density function (p.d.f) $f_\theta(x)$ for the distribution $P_\theta$ with respect to a $\sigma$-finite measure $\mu$.

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In this context, Esteban, M.D. [4] defined the $H_{h,v}^{\varphi_1,\varphi_2}$-entropy by the following expression:

$$H_{h,v}^{\varphi_1,\varphi_2}(\theta) = h\left(\frac{\int_{\mathfrak{X}} v(x)\varphi_1(f_\theta(x))\,\mathrm{d}\mu}{\int_{\mathfrak{X}} v(x)\varphi_2(f_\theta(x))\,\mathrm{d}\mu}\right),$$

where $v(x)$ is a weight function and we suppose that $v\colon \mathfrak{X} \to [0,\infty)$, $\varphi_1\colon [0,\infty) \to \mathbb{R}$, $\varphi_2\colon [0,\infty) \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are any of the triples of functions appearing in Table 1.

In Table 1, functions $v(x)$, $h(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are given for the following entropy measures: (1) Shannon [13], (2) Renyi [11], (3) Aczel-Daróczy [1], (4) Aczel-Daróczy [1], (5) Aczel-Daróczy [1], (6) Varma [18], (7) Varma [18], (8) Kapur [8], (9) Havrda-Charvat [7], (10) Arimoto [2], (11) Sharma-Mittal [14], (12) Sharma-Mittal [14], (13) Taneja [17], (14) Sharma-Taneja [15], (15) Sharma-Taneja [16], (16) Ferreri [5], (17) Sant'anna-Taneja [12], (18) Sant'anna-Taneja [12], (19) Belis-Guiasu [3] and Gil [6], (20) Picard [9], (21) Picard [9], (22) Picard [9] and (23) Picard [9].

| Measure | $h(x)$ | $\varphi_1(x)$ | $\varphi_2(x)$ | $v(x)$ |
|---|---|---|---|---|
| 1 | $x$ | $-x\log x$ | $x$ | $v$ |
| 2 | $(1-r)^{-1}\log x$ | $x^r$ | $x$ | $v$ |
| 3 | $x$ | $-x^r\log x$ | $x^r$ | $v$ |
| 4 | $(s-r)^{-1}\log x$ | $x^r$ | $x^s$ | $v$ |
| 5 | $(1/s)\arctan x$ | $x^r\sin(s\log x)$ | $x^r\cos(s\log x)$ | $v$ |
| 6 | $(m-r)^{-1}\log x$ | $x^{r-m+1}$ | $x$ | $v$ |
| 7 | $(m(m-r))^{-1}\log x$ | $x^{r/m}$ | $x$ | $v$ |
| 8 | $(1-t)^{-1}\log x$ | $x^{t+s-1}$ | $x^s$ | $v$ |
| 9 | $(1-s)^{-1}(x-1)$ | $x^s$ | $x$ | $v$ |
| 10 | $(t-1)^{-1}(x^t-1)$ | $x^{1/t}$ | $x$ | $v$ |
| 11 | $(1-s)^{-1}(e^x-1)$ | $(s-1)x\log x$ | $x$ | $v$ |
| 12 | $(1-s)^{-1}(x^{\frac{s-1}{r-1}}-1)$ | $x^r$ | $x$ | $v$ |
| 13 | $x$ | $-x^r\log x$ | $x$ | $v$ |
| 14 | $(s-r)^{-1}x$ | $x^r-x^s$ | $x$ | $v$ |
| 15 | $(\sin s)^{-1}x$ | $-x^r\sin(s\log x)$ | $x$ | $v$ |
| 16 | $\left(1+\frac{1}{\lambda}\right)\log(1+\lambda)-\frac{x}{\lambda}$ | $(1+\lambda x)\log(1+\lambda x)$ | $x$ | $v$ |
| 17 | $x$ | $-x\log\left(\frac{\sin(sx)}{2\sin(s/2)}\right)$ | $x$ | $v$ |
| 18 | $x$ | $-\frac{\sin(xs)}{2\sin(s/2)}\log\left(\frac{\sin(sx)}{2\sin(s/2)}\right)$ | $x$ | $v$ |
| 19 | $x$ | $-x\log x$ | $x$ | $w(x)$ |
| 20 | $x$ | $-\log x$ | $1$ | $v(x)$ |
| 21 | $(1-r)^{-1}\log x$ | $x^{r-1}$ | $1$ | $v(x)$ |
| 22 | $(1-s)^{-1}(e^x-1)$ | $(s-1)\log x$ | $1$ | $v(x)$ |
| 23 | $(1-s)^{-1}(x^{\frac{r-1}{s-1}}-1)$ | $x^{r-1}$ | $1$ | $v(x)$ |

Table 1

## 2. Asymptotic distribution of $H_{h,v}^{\varphi_1,\varphi_2}$-statistics

We suppose that the following regularity assumptions hold:

1. For all $\theta_1 \neq \theta_2 \in \Theta$,
$$\mu\left(\{x \in \mathfrak{X} / f_{\theta_1}(x) \neq f_{\theta_2}(x)\}\right) > 0.$$
2. For all $\theta_1, \theta_2 \in \Theta$,
$$\int_{A(\theta_1)} f_{\theta_2}(x) d\mu = 1,$$
where $A(\theta) = \{x \in \mathfrak{X} / f_\theta(x) > 0\}$.
3. For almost every $x \in \mathfrak{X}$ there exists a neighbourhood $U$ of the true value of the parameter such that for any $\theta \in \overline{U}$ the following derivatives exist and are finite:
$$\frac{\partial f_\theta(x)}{\partial \theta_i}, \frac{\partial^2 f_\theta(x)}{\partial \theta_i \partial \theta_j}, \frac{\partial^3 f_\theta(x)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \qquad i, j, k = 1, \ldots, M.$$
4. For almost every $x \in \mathfrak{X}$ and for all $\theta \in \overline{U}$,
$$\left|\frac{\partial f_\theta(x)}{\partial \theta_i}\right| < F(x), \qquad \left|\frac{\partial^2 f_\theta(x)}{\partial \theta_i \partial \theta_j}\right| < F(x),$$
$$\left|\frac{\partial^3 f_\theta(x)}{\partial \theta_i \partial \theta_j \partial \theta_k}\right| < H(x), \qquad i, j, k = 1, \ldots, M,$$
where
$$\int_{\mathfrak{X}} F(x)\, \mathrm{d}\mu < \infty \qquad \text{and} \qquad \int_{\mathfrak{X}} H(x) f_\theta(x)\, \mathrm{d}\mu < \eta,$$
and $\eta > 0$ is independent of $\theta$.
5. For all $\theta \in \overline{U}$, Fisher's information matrix
$$I_F(\theta) = \left(\int_{\mathfrak{X}} \frac{\partial \log f_\theta(x)}{\partial \theta_i} \frac{\partial \log f_\theta(x)}{\partial \theta_j} f_\theta(x)\, \mathrm{d}\mu\right)_{i,j=1,\ldots,M}$$
is finite and positive definite.

In this section we obtain the asymptotic distribution of $H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta})$ where $\widehat{\theta}$ is the maximum likelihood estimator of $\theta$ based on a simple random sample $X_1, \ldots, X_n$ of $f_\theta$.

We write $f \in C^i(A)$ to denote that the real valued function $f$ has a continuous derivative of $i-th$ order in the set $A$. We write $\xrightarrow[n,m\uparrow\infty]{L}$ to denote convergence in law.

**Theorem 2.1.** *Assume the regularity conditions* 1–5. *Moreover suppose that* $h \in C^1(\mathbb{R})$, $\varphi_1 \in C^1([0,\infty))$, $\varphi_2 \in C^1([0,\infty))$,

$$\left| v(x)\varphi_s'(f_\theta(x)) \frac{\partial f_\theta(x)}{\partial \theta_j} \right| < F(x), \qquad j = 1, \ldots, M, \qquad s = 1, 2,$$

*with* $F(x)$ *finitely integrable. Then*

$$n^{1/2}[H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - H_{h,v}^{\varphi_1,\varphi_2}(\theta)] \xrightarrow[n,m\uparrow\infty]{L} \mathcal{N}(0,\sigma^2)$$

*where* $\sigma^2 = T^t I_F^{-1}(\theta) T > 0$, $T^t = (t_1, \ldots, t_M)$ *and*

$$t_i = h'\left(\frac{\int_{\mathfrak{X}} v(x)\varphi_1(f_\theta(x))\,\mathrm{d}\mu}{\int_{\mathfrak{X}} v(x)\varphi_2(f_\theta(x))\,\mathrm{d}\mu}\right) \left[\int_{\mathfrak{X}} v(x)\varphi_1'(f_\theta(x)) \frac{\partial f_\theta(x)}{\partial \theta_i}\,\mathrm{d}\mu \int_{\mathfrak{X}} v(x)\varphi_2(f_\theta(x))\,\mathrm{d}\mu \right.$$
$$\left. - \int_{\mathfrak{X}} v(x)\varphi_1(f_\theta(x))\,\mathrm{d}\mu \int_{\mathfrak{X}} v(x)\varphi_2'(f_\theta(x)) \frac{\partial f_\theta(x)}{\partial \theta_i}\,\mathrm{d}\mu \right] \left[\int_{\mathfrak{X}} v(x)\varphi_2(f_\theta(x))\,\mathrm{d}\mu\right]^{-2}.$$

P r o o f. By the mean value theorem we have

$$H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) = H_{h,v}^{\varphi_1,\varphi_2}(\theta) + \sum_{i=1}^{M} \frac{\partial H_{h,v}^{\varphi_1,\varphi_2}(\tilde{\theta})}{\partial \theta_i}(\widehat{\theta}_i - \theta_i),$$

where $\| \tilde{\theta} - \theta \|_2 < \| \widehat{\theta} - \theta \|_2$. Hence the random variables

$$\sqrt{n}[H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - H_{h,v}^{\varphi_1,\varphi_2}(\theta)] \qquad \text{and} \qquad \sqrt{n}T^t(\widehat{\theta} - \theta)$$

have asymptotically the same distribution.

Since

$$\sqrt{n}T^t(\widehat{\theta} - \theta) \xrightarrow[n,m\uparrow\infty]{L} \mathcal{N}(0, T^t I_F^{-1}(\theta) T),$$

we conclude that

$$\sqrt{n}[H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - H_{h,v}^{\varphi_1,\varphi_2}(\theta)] \xrightarrow[n,m\uparrow\infty]{L} \mathcal{N}(0,\sigma^2).$$

□

**Proposition 2.1.** *Assume conditions of Theorem* 2.1. *If* $S_n = n^{1/2}T^t(\widehat{\theta} - \theta)$, *then*

$$S_n = 0 \quad \text{a.s.} \quad \forall n \in \mathbb{N} \qquad \textit{if and only if} \qquad \sigma^2 = 0.$$

P r o o f. If $S_n = 0$ a.s. $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} V[S_n] = \sigma^2 = 0$.
If $\sigma^2 = T^t I_F^{-1}(\theta) T = 0$, then $T \equiv 0$ because $I_F(\theta)$ is positive definite. Therefore $S_n = 0$ a.s. $\forall n \in \mathbb{N}$. □

**Theorem 2.2.** *Assume conditions of Theorem 2.1. Moreover suppose that* $h \in C^2(\mathbb{R})$, $\varphi_1, \varphi_2 \in C^2([0,\infty))$,

$$\left| v(x)\varphi_s'(f_\theta(x)) \frac{\partial f_\theta(x)}{\partial \theta_j} \right| < F(x),$$

$$\left| v(x)\varphi_s''(f_\theta(x)) \frac{\partial f_\theta(x)}{\partial \theta_i} \frac{\partial f_\theta(x)}{\partial \theta_j} \right| < F(x) \qquad \text{and}$$

$$\left| v(x)\varphi_s'(f_\theta(x)) \frac{\partial^2 f_\theta(x)}{\partial \theta_i \partial \theta_j} \right| < F(x), \qquad i,j = 1,\dots,M, \qquad s = 1,2,$$

*with* $F(x)$ *finitely integrable. If* $T^t I_F^{-1} T = 0$, *then*

$$2n[H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - H_{h,v}^{\varphi_1,\varphi_2}(\theta)] \xrightarrow[n,m\uparrow\infty]{L} \sum_{i=1}^{M} \beta_i \chi_{1,i}^2,$$

*where the* $\chi_{1,i}^2$*'s are independent and the* $\beta_i$*'s are the nonnull eigenvalues of the matrix* $AI_F^{-1}(\theta)$ *with*

$$A = (a_{ij})_{i,j=1,\dots,M} = \left( \frac{\partial^2 H_{h,v}^{\varphi_1,\varphi_2}(\theta)}{\partial \theta_i \partial \theta_j} \right)_{i,j=1,\dots,M}.$$

P r o o f. By Proposition 2.1 and the mean value theorem we have

$$H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) = H_{h,v}^{\varphi_1,\varphi_2}(\theta) + \frac{1}{2}(\widehat{\theta} - \theta)^t \left( \frac{\partial^2 H_{h,v}^{\varphi_1,\varphi_2}(\tilde{\theta})}{\partial \theta_i \partial \theta_j} \right)_{i,j=1\dots,M} (\widehat{\theta} - \theta),$$

where $\| \tilde{\theta} - \theta \|_2 < \| \widehat{\theta} - \theta \|_2$.

So, we conclude that

$$2n[H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - H_{h,v}^{\varphi_1,\varphi_2}(\theta)] \qquad \text{and} \qquad \sqrt{n}(\widehat{\theta} - \theta)^t A \sqrt{n}(\widehat{\theta} - \theta)$$

have asymptotically the same distribution (cf. Rao [10], p. 385).

On the other hand, since $\sqrt{n}(\widehat{\theta} - \theta)$ have aymptotically a zero mean normal distribution with a variance-covariance matrix $I_F^{-1}(\theta)$, we have that

$$n(\widehat{\theta} - \theta)^t A(\widehat{\theta} - \theta) \xrightarrow[n,m\uparrow\infty]{L} \sum_{i=1}^{M} \beta_i \chi_{1,i}^2,$$

where the $\beta_i$'s are the eigenvalues of the matrix $AI_F^{-1}(\theta)$ and the $\chi_{1,i}^2$'s are independent. So, the result follows. □

The previous results giving the asymptotic distribution of the $H_{h,v}^{\varphi_1,\varphi_2}$-entropy statistics can be used in various settings to construct confidence intervals and to test statistical hypotheses based on one or more samples.

**(a) Test for a predicted value of the population entropy.**

To test $H_0\colon H_{h,v}^{\varphi_1,\varphi_2}(\theta) = D_0$ against $H_1\colon H_{h,v}^{\varphi_1,\varphi_2}(\theta) \neq D_0$, we reject the null hypothesis if

$$|T_a| = \left| \frac{n^{1/2}\left(H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - D_0\right)}{\widehat{\sigma}} \right| > z_{\alpha/2},$$

where $\widehat{\sigma}$ is obtained from $\sigma$ in Theorem 2.1 when $\theta$ is replaced by $\widehat{\theta}$ and $z_\alpha$ is the $(1-\alpha)$-quantile of the standard normal distribution. In this context an approximate $1-\alpha$ level confidence interval for $H_{h,v}^{\varphi_1,\varphi_2}(\theta)$ is given by

$$\left( H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) - \frac{\widehat{\sigma} z_{\alpha/2}}{n^{1/2}}, H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}) + \frac{\widehat{\sigma} z_{\alpha/2}}{n^{1/2}} \right).$$

Furthermore, the minimum sample size giving the maximum error $\varepsilon$ at a confidence level $1-\alpha$ is

$$n = \left[ \frac{\widehat{\sigma}^2 z_{\alpha/2}^2}{\varepsilon^2} \right] + 1.$$

**(b) Test for a common predicted value of $r$ population entropies.**

To test $H_0\colon H_{h,v}^{\varphi_1,\varphi_2}(\theta_1) = \ldots = H_{h,v}^{\varphi_1,\varphi_2}(\theta_r) = D_0$, we reject the null hypotheses if

$$T_b = \sum_{j=1}^{r} \frac{n_j \left( H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}_j) - D_0 \right)^2}{\widehat{\sigma}_j^2} > \chi_{r,\alpha}^2,$$

where $n_j$ is the size of the sample in the $jth$ population, $\widehat{\sigma}_j$'s are obtained from $\sigma$ when $\theta_j$ is replaced in Theorem 2.1 by $\widehat{\theta}_j$, $j = 1, \ldots, r$, and $\chi_{r,\alpha}^2$ is the $(1-\alpha)$-quantile of the chi-square distribution with $r$ degrees of freedom.

In this context an approximate $1-\alpha$ confidence interval for the difference of entropies corresponding to independent populations is given by

$$H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}_1) - H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}_2) \pm z_{\alpha/2} \sqrt{\frac{\widehat{\sigma}_1^2}{n_1} + \frac{\widehat{\sigma}_2^2}{n_2}}.$$

Furthermore, for $n = n_1 = n_2$, the minimum sample size giving the maximum error $\varepsilon$ at a confidence level $1-\alpha$ is

$$n = \left[ \frac{(\widehat{\sigma}_1^2 + \widehat{\sigma}_2^2) z_{\alpha/2}^2}{\varepsilon^2} \right] + 1.$$

**(c) Test for the equality of $r$ population entropies.**

To test $H_0\colon\ H_{h,v}^{\varphi_1,\varphi_2}(\theta_1) = \ldots = H_{h,v}^{\varphi_1,\varphi_2}(\theta_r)$, we reject the null hypotheses if

$$T_c = \sum_{j=1}^{r} \frac{n_j \left( H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}_j) - \overline{H} \right)^2}{\widehat{\sigma}_j^2} > \chi^2_{r-1,\alpha},$$

where

$$\overline{H} = \frac{\sum\limits_{j=1}^{r} \dfrac{n_j H_{h,v}^{\varphi_1,\varphi_2}(\widehat{\theta}_j)}{\widehat{\sigma}_j^2}}{\sum\limits_{j=1}^{r} \dfrac{n_j}{\widehat{\sigma}_j^2}},$$

and $n_j$ and $\widehat{\sigma}_j$ are defined above.

**(d) Tests for parameters.**

For the cases where the $H_{h,v}^{\varphi_1,\varphi_2}$-entropy is a bijective function of the parameters, testing the hypotheses

$$\begin{aligned}
&H_0\colon\ \theta = \theta_0 \\
&H_0\colon\ \theta_1 = \theta_2 = \ldots = \theta_r = \theta_0 \\
&H_0\colon\ \theta_1 = \theta_2 = \ldots = \theta_r
\end{aligned}$$

is equivalent to test the hypotheses

$$\begin{aligned}
&H_0\colon\ H_{h,v}^{\varphi_1,\varphi_2}(\theta) = H_{h,v}^{\varphi_1,\varphi_2}(\theta_0) \\
&H_0\colon\ H_{h,v}^{\varphi_1,\varphi_2}(\theta_1) = H_{h,v}^{\varphi_1,\varphi_2}(\theta_2) = \ldots = H_{h,v}^{\varphi_1,\varphi_2}(\theta_r) = H_{h,v}^{\varphi_1,\varphi_2}(\theta_0) \\
&H_0\colon\ H_{h,v}^{\varphi_1,\varphi_2}(\theta_1) = H_{h,v}^{\varphi_1,\varphi_2}(\theta_2) = \ldots = H_{h,v}^{\varphi_1,\varphi_2}(\theta_r).
\end{aligned}$$

There are many entropy and certainty measures which are bijective functions of the parameters. To illustrate this fact it is enough to analyze the expression $\int_{\mathfrak{X}} f(x)^r \, \mathrm{d}\mu$ appearing in the Sharma-Mittal entropy (this entropy measure is a monotone transformation of the previous expression) for the probability distributions

| Distribution | $\int_{\mathfrak{X}} f(x)^r \, d\mu$ |
|---|---|
| Uniform $(a,b)$ | $(b-a)^{1-r}$ |
| Gamma $(a,p)$ | $\dfrac{\Gamma(rp-r+1)}{\Gamma(p)^r a^{1-r} r^{rp-r+1}}, \qquad rp-r+1>0$ |
| Laplace $(b)$ | $2^{1-r} b^{1-r} r^{-1}$ |
| Normal $(\mu,\sigma)$ | $\sigma^{1-r} r^{-\frac{1}{2}} (2\pi)^{\frac{1-r}{2}}$ |
| Pareto $(a,k)$ | $a^r k^{1-r}(ra+r-1)^{-1}$ |
| Beta $(a,b)$ | $\dfrac{\Gamma(ra+1-r)\Gamma(rb+1-r)}{B(a,b)^r\Gamma(ra+rb-2-2r)}, \qquad a>\dfrac{r-1}{r}, \qquad b>\dfrac{r-1}{r}$ |
| Weibull $(a,b)$ | $\dfrac{b^{r-1}}{ra^{r-1}}\left(\dfrac{a}{r}\right)^{\frac{(r-1)(b-1)}{b}} \Gamma\left(\dfrac{br-r+1}{b}\right)$ |
| Gumbel $(b)$ | $r^{-2}\beta^{1-r}$ |

To conclude we give an example of testing the equality of parameters of $r$ exponential distributions based on Shannon's entropy. The expression of this entropy measure for an exponential distribution of the parameter $\theta$ is

$$\varphi(\theta) = \int_0^{\infty} -f_\theta(x) \ln f_\theta(x) \, dx = 1 - \ln\theta.$$

As Shannon's entropy is a bijective function of the parameter, we can use this measure to test

$$H_0\colon\ \theta_1 = \ldots = \theta_r,$$

with $T_c$ statistic given in (c). In this case we have $v(x) = v$, $h(x) = x$, $\varphi_1(x) = -x\ln x$ and $\varphi_2(x) = x$, $\forall x \in \mathbb{R}$, hence

$$\begin{aligned}
T &= \frac{\partial\varphi(\theta)}{\partial\theta} = -\frac{1}{\theta},\\
I_F(\theta) &= \int_0^{\infty} \frac{\partial^2 \ln f_\theta}{\partial\theta^2} f_\theta(x) \, dx = \frac{1}{\theta^2},\\
\sigma^2 &= T^2 I_F(\theta) = \theta^{-4},\\
\overline{H} &= \frac{\sum_{j=1}^r n_j \widehat{\theta}_j^4 \ln\widehat{\theta}_j}{\sum_{j=1}^r n_j \widehat{\theta}_j^4} - 1,
\end{aligned}$$

and the test statistic is

$$T_c = \sum_{j=1}^r n_j \widehat{\theta}_j^4 \ln^2 \widehat{\theta}_j - \frac{\left(\sum_{j=1}^r n_j \widehat{\theta}_j^4 \ln\widehat{\theta}_j\right)^2}{\sum_{j=1}^r n_j \widehat{\theta}_j^4},$$

where $\widehat{\theta}_i = \frac{1}{\overline{X}_i}$, $\overline{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$ and $(X_{i1}, X_{i2}, \dots X_{in_i})$, $i = 1, \dots, r$, are independent simple random samples of size $n_i$ from the exponential distribution of the parameter $\theta_i$. So, we reject the null hypothesis at a level $\alpha$ if

$$T_c > \chi^2_{r-1,\alpha}.$$

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*Author's address*: *M.D. Esteban*, Departamento de Estadística e I.O., Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 – Madrid, Spain.