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ON CAUSTICS ASSOCIATED WITH THE LINEARIZED VORTICITY EQUATION

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Abstract. The linearized vorticity equation serves to model a number of wave phenomena in geophysical fluid dynamics. One technique that has been applied to this equation is the geometrical optics, or multi-dimensional WKB technique. Near caustics, this technique does not apply. A related technique that does apply near caustics is the Lagrange Manifold Formalism. Here we apply the Lagrange Manifold Formalism to determine an asymptotic solution of the linearized vorticity equation and to study associated wave phenomena on the caustic curve.

Keywords: Linearized vorticity equation, caustics, turning points, WKB

MSC 2000: 34E20

1. Introduction

An important equation in the mathematical analysis of large-scale atmospheric flow processes is the Rossby wave equation

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \beta(y)\frac{\partial\psi}{\partial x} = 0. \tag{1}$$

In this equation t is the time, x and y are spatial coordinates, $\psi(x,y,t)$ is the streamline and $\beta(y)$ is a vorticity parameter. While this family of equations is a useful model for a variety of geophysical wave phenomena, it is a subset of a broader class of equations known as the linearized vorticity equations

$$\Bigl(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\Bigr)(\nabla^2\psi - F\psi) + (B_1+\beta_1)\frac{\partial\psi}{\partial x} - (B_2+\beta_2)\frac{\partial\psi}{\partial y} = 0. \tag{2}$$

In atmospheric dynamics, $U(y,t)$ and $V(x,t)$ are, respectively, the eastward and northward components of the basic flow. U is a function of the latitude and of time and V is a function of the longitude and of time. F is the Froud number, a parameter inversely proportional to the static instability of the atmosphere [1,p. 60], $B_1(y,t)$ and $B_2(x,t)$ are parameters related to the stability of the basic flow and $\beta_1(x,y)$ and $\beta_2(x,y)$ are vorticity parameters related to the topography [2]. Neither equation (1) nor equation (2) can be solved exactly. Consequently, various techniques, each valid under situation-specific assumptions, have been developed to obtain approximate solutions and to study the phenomena the equations model. One such technique is the multi-dimensional WKB, or geometrical optics, approach developed by Keller and his students [3]. Karoly and Hoskins [4] and Yang [5], among others, have employed and extended this approach to study Rossby-type equations, i.e., variations of equation (1), and Yang has applied it to study the vorticity equation as well. Near turning or caustic points, the classical WKB technique is not valid [6], e.g., physically, near the "critical layers" where the phase velocity of the wave coincides with the velocity of the large-scale current [7,8] or near sharp topographies [2]. A related approach that is valid at caustics is the Lagrange Manifold formalism developed by Arnol'd [9] and Maslov [10]. This approach has been applied to determine the asymptotic solution of equation (1) and to study the associated wave phenomena along the caustic curve [11]. Here we apply the Lagrange Manifold formalism to obtain asymptotic solutions of the vorticity equation at caustic points. We also use this approach to study wave phenomena associated with the vorticity equation at caustics in a manner analogous to the use of the classical WKB technique in studying wave phenomena at off-caustic points. For completeness, we include a summary of the basic algorithm.

2. Formalism

Since Yang's treatment is noteworthy for its clarity, we parallel his development. We first re-scale our independent variables to "slower" variables

$$\varepsilon\bar{r} = \varepsilon(x,y) \to (x,y), \quad \varepsilon t \to t$$

where ε is a small parameter, physically the ratio of the characteristic horizontal scale of the wave phenomena to the average radius of the earth. Then equation (2) becomes

$$\varepsilon^3\Big(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\Big)\Big(\nabla^2\Psi - \frac{1}{\varepsilon^2}F\Psi\Big) + \varepsilon(b_1 + \widehat{\beta}_1)\frac{\partial \Psi}{\partial x} + \varepsilon(b_2 + \widehat{\beta}_2)\frac{\partial \Psi}{\partial y} = 0, \tag{3}$$

where the lower case variables are the re-scaled upper case variables and $\widehat{\beta}_i(x, y) = \beta_i(\varepsilon x, \varepsilon y)$. Near caustics, we assume equation (3) has a solution of the form

$$\Psi(x, y, t) = \int A(\bar{r}, \bar{p}, t, \varepsilon)\mathrm{e}^{\frac{i\Phi}{\varepsilon}}\,\mathrm{d}\bar{p}. \tag{4}$$

In equation (4), $\bar{r} = (x, y)$ and $\bar{p} = (p_x, p_y)$ may be regarded as a wavevector. The amplitude

$$A(\bar{r}, \bar{p}, t, \varepsilon) \sim \sum_{k=0} A_k(\bar{r}, \bar{p}, t)(i/\varepsilon)^k$$

and its derivatives are assumed bounded and

$$\Phi(\bar{r}, t, \bar{p}, \omega) = \bar{r} \cdot \bar{p} - \omega t - S(\bar{p}, \omega)$$

where $S(\bar{p}, \omega)$ may be regarded as a phase. Then substituting equation (4) into equation (3), introducing wavevector $\bar{p}$ and frequency ω

$$\bar{p} = \nabla\Phi, \quad \omega = -\frac{\partial\Phi}{\partial t} \tag{5}$$

leads to

$$\begin{aligned}
\int \Big\{&[(b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y + ((\bar{p} \cdot \bar{p}) + F)(\omega - up_x - vp_y)]A \\
&+ i\varepsilon\Big[u(F - 3p_x^2 - p_y^2) + 2p_x(\omega - 2vp_y) + b_1 + \widehat{\beta}_1)\frac{\partial A}{\partial x} \\
&+ (v(F - 3p_y^2 - p_x^2) + 2p_y(\omega - 2up_x) - b_2 - \widehat{\beta}_2)\frac{\partial A}{\partial y} + (F - \bar{p} \cdot \bar{p})\frac{\partial A}{\partial t}\Big] \\
&+ (i\varepsilon)^2\Big[(3up_x + vp_y - \omega)\frac{\partial^2 A}{\partial x^2} + (3vp_y + up_x - \omega)\frac{\partial A}{\partial y^2} \\
&+ 2(up_y + vp_x)\frac{\partial^2 A}{\partial x \partial y} + 2\Big(p_x\frac{\partial^2 A}{\partial t \partial x} + p_y\frac{\partial^2 A}{\partial t \partial x}\Big)\Big] \\
&+ (i\varepsilon)^3\Big[u\Big(\frac{\partial^3 A}{\partial x^3} + \frac{\partial^3 A}{\partial x \partial y^2}\Big) + v\Big(\frac{\partial^3 A}{\partial y^3} + \frac{\partial^3 A}{\partial y \partial x^2}\Big) \\
&+ \Big(\frac{\partial^3 A}{\partial t \partial x^2} + \frac{\partial^3 A}{\partial t \partial y^2}\Big)\Big]\Big\}\mathrm{e}^{\frac{i\Phi}{\varepsilon}}\,\mathrm{d}\bar{p} \sim 0.
\end{aligned} \tag{6}$$

The coefficient of the $(i\varepsilon)^0$ term is Maslov's Hamiltonian

$$H = (b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y + ((\bar{p} \cdot \bar{p}) + F)(\omega - up_x - vp_y). \tag{7}$$

The field at any caustic point $\bar{r} = (x, y)$ is determined by the stationary phase $(\nabla_p\Phi = 0)$ evaluation of the integral in equation (6), which turns Maslov's Hamiltonian into an eikonal equation

$$(b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y + ((\bar{p} \cdot \bar{p}) + F)(\omega - up_x - vp_y) = 0 \tag{8}$$

and determines the Lagrange manifold

$$\bar{r} = \nabla_p S(\bar{p}, \omega), \tag{9}$$

once $S(\bar{p}, \omega)$ is known. To determine $S(\bar{p}, \omega)$, we first apply Hamilton's equations to obtain the trajectories

$$\begin{aligned} \bar{r} &= \bar{r}(\gamma, \bar{\sigma}) \qquad & \bar{p} &= \bar{p}(\gamma, \bar{\sigma}) \\ t &= t(\gamma, \bar{\sigma}) \qquad & \omega &= \omega(\gamma, \bar{\sigma}), \end{aligned}$$

where γ is a raypath parameter and $\bar{\sigma}$ is a parametrized initial condition, e.g., direction cosines. Then inversion of the frequency, time and wave vector transformations, followed by substitution into the coordinate space map, determines the Lagrange manifold explicitly,

$$\bar{r} = \bar{r}(\gamma(\bar{p}, \omega), \bar{\sigma}(\bar{p}, \omega)) = \nabla_p S(\bar{p}, \omega). \tag{10}$$

Integration along the trajectories determines

$$S(\bar{p}, \omega) = \int_{\bar{p}_0}^{\bar{p}} \bar{r} \cdot \mathrm{d}\bar{p} \tag{11}$$

and hence the phase

$$\Phi(\bar{r}, t, \bar{p}, \omega) = \bar{r} \cdot \bar{p} - \omega t - S(\bar{p}, \omega). \tag{12}$$

To obtain a transport equation for the amplitudes we Taylor expand the Hamiltonian near the Lagrange manifold

$$\begin{aligned} &(b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y + ((\bar{p} \cdot \bar{p}) + F)(\omega - up_x - vp_y) \\ &\quad = \Big(b_1\Big(\frac{\partial S}{\partial p_y}, t\Big) + \widehat{\beta}_1(\nabla_p S)\Big)p_x - \Big(b_2\Big(\frac{\partial S}{\partial p_x}, t\Big) - \widehat{\beta}_2(\nabla_p S)\Big) \\ &\qquad + (\bar{p} \cdot \bar{p} + F)\Big(\omega - u\Big(\frac{\partial S}{\partial p_y}, t\Big)p_x - v\Big(\frac{\partial S}{\partial p_x}, t\Big)p_y\Big) + \overline{D} \cdot (\bar{r} - \nabla_p S) \\ &\quad = \overline{D} \cdot (\bar{r} - \nabla_p S), \end{aligned} \tag{13}$$

where

$$\overline{D} = \int_0^1 \nabla_r H(\xi(\bar{r} - \nabla_p S) + \nabla_p S, \bar{p}, \omega, t)\, \mathrm{d}\xi, \tag{14}$$

that is, the remainder of the Taylor series. Then substituting equation (13) into equation (6) and performing a partial integration leads to

$$
\begin{aligned}
(15)\quad \int \Big\{ & i\varepsilon\Big[-\overline{D}\cdot\nabla_p A - A\nabla_p\cdot\overline{D} + \big(u(F-3p_x^2-p_y^2) \\
& + 2p_x(\omega-2vp_y) + b_1 + \widehat{\beta}_1\big)\frac{\partial A}{\partial x} \\
& + \big(v(F-3p_y^2-p_x^2) + 2p_y(\omega-2up_x) - b_2 - \widehat{\beta}_2\big)\frac{\partial A}{\partial y} + (F-\bar{p}\cdot\bar{p})\frac{\partial A}{\partial t}\Big] \\
& + (i\varepsilon)^2\Big[(3up_x+vp_y-\omega)\frac{\partial^2 A}{\partial x^2} + (3vp_y+up_x-\omega)\frac{\partial^2 A}{\partial y^2} \\
& + 2(up_y+vp_x)\frac{\partial^2 A}{\partial x\partial y} + 2\Big(p_x\frac{\partial^2 A}{\partial t\partial x} + p_y\frac{\partial^2 A}{\partial t\partial x}\Big)\Big] \\
& + (i\varepsilon)^3\Big[u\Big(\frac{\partial^3 A}{\partial x^3} + \frac{\partial^3 A}{\partial x\partial y^2}\Big) + v\Big(\frac{\partial^3 A}{\partial y^3} + \frac{\partial^3 A}{\partial y\partial x^2}\Big) \\
& + \Big(\frac{\partial^3 A}{\partial t\partial x^2} + \frac{\partial^3 A}{\partial t\partial y^2}\Big)\Big]\Big\}\mathrm{e}^{\frac{i\Phi}{\varepsilon}}\,\mathrm{d}\bar{p} \sim 0.
\end{aligned}
$$

Finally, introducing into this integral the non-Hamiltonian flow

$$
\begin{aligned}
\dot{\bar{r}} &= (2\omega p_x - 3up_x^2 - up_y^2 - 2vp_xp_y + uF + b_1 + \widehat{\beta}_1, \\
&\qquad 2\omega p_y - 3vp_y^2 - vp_x^2 - 2up_xp_y + vF - b_2 - \widehat{\beta}_2) \\
\dot{\bar{p}} &= -\overline{D} \\
\dot{t} &= -(F + \bar{p}\cdot\bar{p})
\end{aligned}
$$

where the differentiation is with respect to the raypath parameter, determines a transport equation in a neighborhood of the Lagrange manifold

$$
\begin{aligned}
\frac{\mathrm{d}A_k}{\mathrm{d}t} & - (\nabla_p\cdot\overline{D})A_k - \omega\Big(\frac{\partial^2 A_{k-1}}{\partial x^2} + \frac{\partial^2 A_{k-1}}{\partial y^2}\Big) + 2\Big(p_x\frac{\partial^2 A_{k-1}}{\partial t\partial x} + p_y\frac{\partial^2 A_{k-1}}{\partial t\partial y}\Big) \\
& + up_x\Big(\frac{\partial^2 A_{k-1}}{\partial x^2} + \frac{\partial^2 A_{k-1}}{\partial y^2}\Big) + 2u\Big(p_x\frac{\partial^2 A_{k-1}}{\partial x^2} + p_y\frac{\partial^2 A_{k-1}}{\partial y^2}\Big) \\
& + vp_y\Big(\frac{\partial^2 A_{k-1}}{\partial x^2} + \frac{\partial^2 A_{k-1}}{\partial y^2}\Big) + 2v\Big(p_x\frac{\partial^2 A_{k-1}}{\partial y\partial x} + p_y\frac{\partial^2 A_{k-1}}{\partial y^2}\Big) \\
& + \frac{\partial^3 A_{k-2}}{\partial t\partial x^2} + \frac{\partial^3 A_{k-2}}{\partial t\partial y^2} + u\Big(\frac{\partial^3 A_{k-2}}{\partial x^3} + \frac{\partial^3 A_{k-2}}{\partial x\partial y^2}\Big) + v\Big(\frac{\partial^3 A_{k-2}}{\partial y^3} + \frac{\partial^3 A_{k-2}}{\partial y\partial x^2}\Big) = 0.
\end{aligned}
$$

An expanded treatment of this algorithm, along with an example, appears in [11].

3. Analysis

Yang has successfully employed the WKB technique to study the physical phenomenology associated with various equation of geophysical fluid dynamics away from caustic points. In [11], the Lagrange Manifold Formalism enabled an analysis of the corresponding phenomena at the caustics associated with equation (1). The analysis led to equations for phenomena on the caustic identical to those determined by Yang away from the caustic. (While the Lagrange Manifold is usually applied near the caustic curve, it applies away from the caustic curve as well. Essentially, it may be regarded as an integral interpretation of the WKB technique.) Similarly, we may develop equations at the caustic associated with the linearized vorticity equation identical to those determined by Yang away from the caustic.

If the eikonal equation is solved for ω, we obtain the same dispersion equation on the caustic that Yang determines away from the caustic

$$\omega = up_x + vp_y - \frac{(b_1 + \widehat{\beta}_1)}{K^2} + \frac{(b_2 + \widehat{\beta}_2)}{K^2}, \tag{17}$$

where $K^2 = p_x^2 + p_y^2 + F$. From equation (17) we determine identical phase velocities

$$\begin{aligned} c_{px} &= \frac{\omega}{p_x} = u + \frac{vp_y}{p_x} - \frac{(b_1 + \widehat{\beta}_1)}{K^2} + \frac{(b_2 + \widehat{\beta}_2)p_y}{K^2 p_x} \\ c_{py} &= \frac{\omega}{p_y} = \frac{up_x}{p_y} + v - \frac{(b_1 + \widehat{\beta}_1)p_x}{K^2 p_y} + \frac{(b_2 + \widehat{\beta}_2)}{K^2} \end{aligned} \tag{18}$$

and group velocities

$$\begin{aligned} c_{gx} &= \frac{\partial\omega}{\partial p_x} = u - K^{-4}\{(b_1 + \widehat{\beta}_1)K^2 - 2p_x[(b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y]\} \\ c_{gy} &= \frac{\partial\omega}{\partial p_y} = v + K^{-4}\{(b_2 + \widehat{\beta}_2)K^2 + 2p_y[(b_1 + \widehat{\beta}_1)p_x - (b_2 + \widehat{\beta}_2)p_y]\} \end{aligned} \tag{19}$$

on the caustic as Yang obtains away from the caustic. We note that c_{gx} and c_{gy} also may be determined from Hamilton's equations if we replace the raypath parameter

γ with t. Then Hamilton's equations become

(20)

$$
\begin{aligned}
\frac{dx}{\mathrm{d}t} &= \frac{\partial H}{\partial p_x} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = c_{gx} \\
\frac{dy}{\mathrm{d}t} &= \frac{\partial H}{\partial p_y} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = c_{gy} \\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= -\frac{\partial H}{\partial x} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = -\Big\{p_y\frac{\partial v}{\partial x} - \frac{p_x}{K^2}\frac{\partial \widehat{\beta}_1}{\partial x} + \frac{p_y}{K^2}\Big(\frac{\partial b_2}{\partial x} + \frac{\partial \widehat{\beta}_2}{\partial x}\Big)\Big\} \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= -\frac{\partial H}{\partial y} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = -\Big\{p_x\frac{\partial u}{\partial y} - \frac{p_x}{K^2}\Big(\frac{\partial \widehat{\beta}_1}{\partial y} + \frac{\partial b_1}{\partial y}\Big) + \frac{p_y}{K^2}\frac{\partial \widehat{\beta}_2}{\partial y}\Big\} \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\partial H}{\partial \gamma} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = p_x\frac{\partial u}{\partial t} + p_y\frac{\partial v}{\partial t} - \frac{p_x}{K^2}\frac{\partial b_1}{\partial t} + \frac{p_y}{K^2}\frac{\partial b_2}{\partial t} \\
\frac{\mathrm{d}t}{\mathrm{d}t} &= -\frac{\partial H}{\partial \omega} \Big/ \Big(-\frac{\partial H}{\partial \omega}\Big) = 1.
\end{aligned}
$$

Further, we note the equations that model the large-scale structural evolution of the wave packet are also identical on and off the caustic, namely,

(21)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(p_x^2 + p_y^2) = &- 2p_xp_y\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) \\
&\frac{2}{K^2}\Big[p_x^2\frac{\partial \widehat{\beta}_1}{\partial x} - p_y^2\frac{\partial \widehat{\beta}_2}{\partial y} + p_xp_y\Big(\frac{\partial b_1}{\partial y} + \frac{\partial \widehat{\beta}_1}{\partial y} + \frac{\partial b_2}{\partial x} + \frac{\partial \widehat{\beta}_2}{\partial x}\Big)\Big] \\
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{p_y}{p_x}\Big) = &\frac{\partial u}{\partial y} - \Big(\frac{p_y}{p_x}\Big)^2\frac{\partial v}{\partial x} \\
&- \frac{1}{K^2}\Big[\Big(\frac{\partial b_1}{\partial y} + \frac{\partial \widehat{\beta}_1}{\partial y}\Big) - \Big(\frac{p_y}{p_x}\Big)\Big(\frac{\partial \widehat{\beta}_1}{\partial x} + \frac{\partial \widehat{\beta}_2}{\partial y}\Big) + \Big(\frac{p_y}{p_x}\Big)^2\Big(\frac{\partial b_2}{\partial x} + \frac{\partial \widehat{\beta}_2}{\partial x}\Big)\Big].
\end{aligned}
$$

The first of these equations describes the time evolution of the square of the wavevector (momentum). The second equation provides a measure of the time evolution of the relative directionality.

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