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NUMERICAL SOLUTIONS FOR SECOND-KIND VOLTERRA INTEGRAL EQUATIONS BY GALERKIN METHODS

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Abstract. In this paper, we study the global convergence for the numerical solutions of nonlinear Volterra integral equations of the second kind by means of Galerkin finite element methods. Global superconvergence properties are discussed by iterated finite element methods and interpolated finite element methods. Local superconvergence and iterative correction schemes are also considered by iterated finite element methods. We improve the corresponding results obtained by collocation methods in the recent papers [6] and [9] by H. Brunner, Q. Lin and N. Yan. Moreover, using an interpolation post-processing technique, we obtain a global superconvergence of the $O(h^{2r})$-convergence rate in the piecewise-polynomial space of degree not exceeding $(r-1)$. As a by-product of our results, all these higher order numerical methods can also provide an a posteriori error estimator, which gives critical and useful information in the code development.

Keywords: Volterra integral equations, Galerkin methods, convergence and superconvergence, interpolation post-processing, iterative correction, a posteriori error estimators

MSC 2000: 65R20, 65B05, 65N30

1. Introduction

In this note we are concerned with finite element methods for the Volterra integral equation of the second kind,

$$y(t) = g(t) + \int_0^t k(t, s, y(s))\,\mathrm{d}s, \quad t \in I := [0, 1], \tag{1.1}$$

where $g\colon I \to R$ and $k\colon D \times R \to R$ (with $D := \{(t, s)\colon 0 \leqslant s \leqslant t \leqslant 1\}$) denote given functions. In our analysis we sometimes employ the linear counterpart of (1.1),

$$y(t) = g(t) + \int_0^t K(t, s)y(s)\,\mathrm{d}s, \quad t \in I := [0, 1]. \tag{1.2}$$

It will always be assumed that both the problem (1.1) and its finite element numerical method possess a unique solution, namely, the given functions $g(t)$ and $k(t,s,y)$ will be subject to the conditions that $g \in C(I)$ and k, which is continuous for all $(t,s) \in D$ and all y, satisfies the (uniform) Lipschitz conditions (compare also [5] and [8]):

$$\text{(V1)} \qquad |k(t,s,y_1) - k(t,s,y_2)| \leqslant L_1|y_1 - y_2|,$$

$$\text{(V2)} \qquad |k_t(t,s,y_1) - k_t(t,s,y_2)| \leqslant L_2|y_1 - y_2|$$

for all $t \in I$, $(t,s) \in D$, and $y_1, y_2 \in R$, with Lipschitz constants L_1 and L_2 being independent of y_1 and y_2.

The study of convergence properties of collocation methods for the Volterra integral equation (1.1) (as well as for the second-kind Fredholm integral equations) and of methods for accelerating the convergence orders has received considerable attention since the early 1980s (compare, for example, [1], [2], [5], [12] and [16]), and the literature is now quite extensive. See, for example, the survey paper [4] and the references cited therein. The recent progress in this research area has been achieved in [6], [8] and [9] for collocation methods, and in [7] for the finite element methods.

However, to the authors' knowledge, finite element methods for the nonlinear Volterra integral equation (1.1) have few results, even for the general convergence to be proved in the coming Section 2 of the paper. The main motivation of this paper derives from [6] and [9]: by means of Galerkin methods we will improve the corresponding results given in [6] and [9] not only for the linear version (1.2), which has been analyzed in [6] and [9] by collocation methods, but also for the nonlinear case (1.1).

The paper is organized in the following way. In Section 2 we give some necessary preliminaries and study global convergence properties by finite element methods for the problem (1.1). Section 3 is devoted to obtaining global superconvergence by virtue of iterated finite element methods and interpolated finite element methods. In Section 4 we study local superconvergence by means of iterated finite element methods. Here, using an interpolation post-processing technique, we can also get higher approximations of the $O(h^{2r})$-convergence rate in the piecewise-polynomial space of degree at most $(r-1)$ for the problem (1.1). In addition, as an application of these superconvergence properties, some a posteriori error estimators, by which the finite element error bound can be determined, are obtained. In Section 5 we discuss iterative correction approximations and some a posteriori error estimators based on the iterative correction method.

2. Global convergence of finite element solutions

In this section we formulate the Galerkin finite element schemes and investigate the global convergence properties for the problem (1.1). For this purpose, we first define a nonlinear integral operator $G\colon\ C(I) \to C(I)$ by

$$(G\varphi)(t) := g(t) + \int_0^t k(t,s,\varphi(s))\,\mathrm{d}s.$$

Then, the problem (1.1) reads: Find $y = y(t)$ such that

$$y(t) = (Gy)(t), \quad t \in I, \tag{2.1}$$

and its weak form is to find $y \in L^2(I)$ such that

$$(y,v) = (Gy,v), \quad v \in L^2(I), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the usual inner product in the L^2-space.

Let $T_h\colon\ 0 = t_0 < t_1 < \ldots < t_M = 1$ be a given mesh for the interval I, and denote the finite element space by

$$S_{r-1}^{(-1)}(T_h) := \{u\colon\ u|_{\sigma_k} \in P_{r-1}(0 \leqslant k \leqslant M-1)\}.$$

Here P_m denotes the space of (real) polynomials of degree not exceeding m, and we have set

$$\sigma_k := \begin{cases} [t_0,t_1], & \text{if } k = 0,\\ (t_k,t_{k+1}], & \text{if } 1 \leqslant k \leqslant M-1, \end{cases}$$

$h_k := t_{k+1} - t_k, \quad h := \max\limits_{(k)}\{h_k\}$. Note that we use the superscript (-1) in the notation for the above finite element space to emphasize that it is not a subspace of $C(I)$.

Our Galerkin approximation of (2.2) is now defined as: Find $u^N \in S_{r-1}^{(-1)}(T_h)$ such that

$$(u^N,v) = (Gu^N,v), \quad v \in S_{r-1}^{(-1)}(T_h). \tag{2.3}$$

Let $P_h\colon\ L^2(I) \to S_{r-1}^{(-1)}(T_h)$ be the L^2-projection operator defined by

$$(y,v) = (P_h y, v), \quad \forall v \in S_{r-1}^{(-1)}(T_h).$$

Then, the problem (2.3) can be equivalently written as: Find $u^N \in S_{r-1}^{(-1)}(T_h)$ such that

$$u^N = P_h G u^N. \tag{2.4}$$

Since $S_{r-1}^{(-1)}(T_h)$ is a discontinuous piecewise-polynomial space, and P_h possesses localization, we have

$$\int_{\sigma_k} vP_h u \,\mathrm{d}t = \int_{\sigma_k} vu \,\mathrm{d}t, \quad \forall v \in P_{r-1} \tag{2.5}$$

with

$$\|P_h y - y\|_{0,\infty} \leqslant Ch^r \|y\|_{r,\infty},$$

where, for nonnegative integer m,

$$\|v\|_{m,\infty} := \max_{0\leqslant k\leqslant m} \{\|v^{(k)}\|_\infty\}.$$

In this case, P_h is defined on every element, and it can be regarded as an interpolation operator of degree r (it is a kind of interpolation in average which is different from the standard Lagrange interpolation) associated with the mesh T_h.

Here and below, C denotes a generic constant whose particular meaning will become clear by the context in which it arises.

Lemma 2.1. *If the conditions* $(V1)$ *and* $(V2)$ *are fulfilled, then the problem* (2.3) *(or* (2.4)*) is uniquely solvable whenever the mesh size* h *is sufficiently small.*

Proof. Define an operator $E\colon S_{r-1}^{(-1)}(T_h) \to S_{r-1}^{(-1)}(T_h)$ by $E := P_h G$. Then, in order to prove Lemma 2.1, it is sufficient to show that the operator E has a unique fixed point $u^N \in S_{r-1}^{(-1)}(T_h)$, which is the unique solution of (2.4). To this end, by the standard contraction mapping principle, we need only to prove that the operator $E^n\colon S_{r-1}^{(-1)}(T_h) \to S_{r-1}^{(-1)}(T_h)$ is a contraction as n is sufficiently large so that operators E and E^n have the identical fixed points.

Decompose the operator E into

$$E = P_h G = (P_h - I)G + G := E_1 + E_2,$$

where I is the identity operator. For the operator E_1, by the approximation property of the L^2-projection operator P_h, from the conditions $(V1)$ and $(V2)$ we find that for any $u_1, u_2 \in S_{r-1}^{(-1)}(T_h)$ we have

$$\begin{aligned} \|E_1 u_1 - E_1 u_2\|_{0,\infty} &= \|(P_h - I)(Gu_1 - Gu_2)\|_{0,\infty} \\ &\leqslant Ch\|Gu_1 - Gu_2\|_{1,\infty} \leqslant Ch\|u_1 - u_2\|_{0,\infty}. \end{aligned} \tag{2.6}$$

For the operator E_2, from the condition $(V1)$ we obtain that for any $u_1, u_2 \in S_{r-1}^{(-1)}(T_h)$ we have

$$\begin{aligned} |(E_2 u_1)(t) - (E_2 u_2)(t)| &\leqslant \int_0^t |k(t,s,u_1(s)) - k(t,s,u_2(s))| \,\mathrm{d}s \\ &\leqslant L_1 \int_0^t |u_1(s) - u_2(s)| \,\mathrm{d}s \leqslant L_1 t \|u_1 - u_2\|_{0,\infty}, \end{aligned}$$

and then

$$\begin{aligned}|(E_2^2u_1)(t)-(E_2^2u_2)(t)| &= |(E_2(E_2u_1))(t)-(E_2(E_2u_2))(t)| \\ &\leqslant L_1\int_0^t |(E_2u_1)(s)-(E_2u_2)(s)|\,\mathrm{d}s \\ &\leqslant L_1\int_0^t Ls\|u_1-u_2\|_{0,\infty}\,\mathrm{d}s = \frac{L_1^2}{2!}t^2\|u_1-u_2\|_{0,\infty}.\end{aligned}$$

This recurrently leads to

$$|(E_2^n u_1)(t)-(E_2^n u_2)(t)| \leqslant \frac{L_1^n}{n!}t^n\|u_1-u_2\|_{0,\infty} \leqslant \frac{L_1^n}{n!}\|u_1-u_2\|_{0,\infty}$$

or

$$\|E_2^n u_1 - E_2^n u_2\|_{0,\infty} \leqslant \frac{L_1^n}{n!}\|u_1-u_2\|_{0,\infty}, \tag{2.7}$$

which yields that there exists a positive integer N_0 such that

$$\|E_2^{N_0}u_1 - E_2^{N_0}u_2\|_{0,\infty} \leqslant \frac{1}{3}\|u_1-u_2\|_{0,\infty}. \tag{2.8}$$

Thus, it follows from (2.6) through (2.8) and the identity

$$E^{N_0} = (E_1+E_2)^{N_0} = \sum_{j=1}^{N_0} C_{N_0}^j E_1^j E_2^{N_0-j} + E_2^{N_0}$$

that when $0<h<\min\Big\{1,\Big(3\sum\limits_{j=1}^{N_0} C_{N_0}^j\frac{L_1^{N_0-j}}{(N_0-j)!}\Big)^{-1}\Big\}$,

$$\begin{aligned}\|E^{N_0}u_1 - E^{N_0}u_2\|_{0,\infty} &\leqslant \sum_{j=1}^{N_0} C_{N_0}^j\|(E_1^jE_2^{N_0-j})u_1-(E_1^jE_2^{N_0-j})u_2\|_{0,\infty} \\ &\quad + \|E_2^{N_0}u_1 - E_2^{N_0}u_2\|_{0,\infty} \\ &\leqslant \sum_{j=1}^{N_0} C_{N_0}^j h^j\frac{L_1^{N_0-j}}{(N_0-j)!}\|u_1-u_2\|_{0,\infty} + \frac{1}{3}\|u_1-u_2\|_{0,\infty} \\ &\leqslant \frac{2}{3}\|u_1-u_2\|_{0,\infty},\end{aligned}$$

where $C_{N_0}^j := \dfrac{N_0!}{j!(N_0-j)!}$; that is, $E^{N_0}\colon S_{r-1}^{(-1)}(T_h)\to S_{r-1}^{(-1)}(T_h)$ is a contraction whenever h is sufficiently small. Thus, we complete the proof of Lemma 2.1. $\square$

Let $e^N := y - u^N$ be the finite element error corresponding to the finite element solution u^N of (1.1). Then, for the kernel function $k(t,s,y)$ in (1.1), the Mean-Value Theorem implies that there exists a function ξ, whose value $\xi(t)$ at t is between $y(t)$ and $u^N(t)$, such that

$$k(t,s,y) - k(t,s,u^N) = k_y(t,s,\xi)e^N(t). \tag{2.9}$$

Set

$$\begin{aligned}(G'\varphi)(t) &:= \int_0^t k_y(t,s,y(s))\varphi(s)\,\mathrm{d}s,\\ (G'_h\varphi)(t) &:= \int_0^t k_y(t,s,\xi(s))\varphi(s)\,\mathrm{d}s.\end{aligned} \tag{2.10}$$

Thus, we have the following lemma [10]:

Lemma 2.2. *We have*

$$\lim_{h\to 0} \|G'_h - G'\|_{C(I)\to C(I)} = 0,$$

where

$$\|A\|_{C(I)\to C(I)} := \sup_{\varphi\in C(I)} \frac{\|A\varphi\|_{0,\infty}}{\|\varphi\|_{0,\infty}}.$$

Throughout this paper, we resort to the standard hypothesis that the problem (1.2) is well-posed, such that $(I-G')^{-1}$ always exists and is bounded on $C(I)$. Now we are prepared to get our global convergence result for the problem (1.1). In fact, we have

Theorem 2.1. *In* (1.1), *assume that* $g \in C^r(I)$ *and* $k \in C^r(D\times R)$ *such that the Volterra integral equation* (1.1) *possesses a unique solution* $y \in C^r(I)$. *Then the finite element error* e^N *satisfies*

$$\|e^N\|_{0,\infty} \leqslant Ch^r\|y\|_{r,\infty}.$$

Proof. From (2.1) one obtains that

$$P_h y = P_h G y,$$

which together with (2.4), (2.9) and (2.10) leads to

$$P_h y - u^N = P_h(Gy - Gu^N) = P_h G'_h e^N. \tag{2.11}$$

Thus, from (2.11) and the definition of e^N one finds the identity

$$e^N - P_h G_h' e^N = y - P_h y$$

or

$$(I - P_h G_h') e^N = (I - P_h) y. \tag{2.12}$$

It is easy to see that

$$\lim_{h \to 0} \|(I - P_h) G'\|_{C(I) \to C(I)} = 0. \tag{2.13}$$

Since the operator $(I - G')$ has a continuous inverse operator $(I - G')^{-1}$, we get the identity

$$\begin{aligned} I - P_h G_h' &= (I - P_h G') + P_h (G' - G_h') \\ &= (I - G') + (I - P_h) G' + P_h (G' - G_h') \\ &= (I - G')\{I + (I - G')^{-1}[(I - P_h) G' + P_h (G' - G_h')]\}, \end{aligned}$$

which together with (2.13), Lemma 2.2 and the uniform boundedness of the L^2-projection operator P_h,

$$\|P_h \varphi\|_{0,\infty} \leqslant C \|\varphi\|_{0,\infty}, \tag{2.14}$$

demonstrates that $(I - P_h G_h')^{-1}$ exists and is bounded uniformly on $C(I)$ for all $h \in (0, h_0)$, $h_0 > 0$ sufficiently small. Therefore, from (2.12) we finally obtain

$$\|e^N\|_{0,\infty} \leqslant \|(I - P_h G_h')^{-1}\|_{C(I) \to C(I)} \cdot \|(I - P_h) y\|_{0,\infty} \leqslant C h^r \|y\|_{r,\infty}.$$

Thus, the proof of Theorem 2.1 is completed. □

3.1. Global superconvergence of iterated finite element solutions. The iterated finite element solution u_{it}^N corresponding to the finite element solution u^N given by (2.3), is defined as follows:

$$u_{it}^N(t) := g(t) + \int_0^t k(t, s, u^N(s))\,\mathrm{d}s \tag{3.1}$$

with

$$P_h u_{it}^N = u^N. \tag{3.2}$$

Here we shall prove that the iterated finite element solution u_{it}^N has the global superconvergence properties. First of all, we need ([5])

Lemma 3.1. *Let the functions g and K characterizing the integral equation* (1.2) *be continuous on I and D, respectively. Then this equation has a unique solution $y \in C(I)$ given by*

$$y(t) = g(t) + \int_0^t R(t, s) g(s)\,\mathrm{d}s, \quad t \in I,$$

where $R \in C(D)$ is the resolvent kernel associated with the given kernel K and defined by $R(t, s) := \sum\limits_{m=1}^{\infty} K_m(t, s)$, $(t, s) \in D$ with $K_1(t, s) := K(t, s)$ and $K_n(t, s) := \int_s^t K_1(t, \tau) K_{n-1}(\tau, s)\,\mathrm{d}\tau$, $(t, s) \in D$ $(n \geqslant 2)$. Moreover, the resolvent kernel satisfies the identities (usually called the Fredholm identities)

$$R(t, s) = K(t, s) + \int_s^t K(t, \tau) R(\tau, s)\,\mathrm{d}\tau, \quad (t, s) \in D,$$

and

$$R(t, s) = K(t, s) + \int_s^t R(t, \tau) K(\tau, s)\,\mathrm{d}\tau, \quad (t, s) \in D.$$

Let δ^N be the residual (or: defect) function defined by

$$\delta^N(t) := -u^N(t) + g(t) + \int_0^t k(t, s, u^N(s))\,\mathrm{d}s. \tag{3.3}$$

Then we know from (3.1) and (3.2) that

$$\delta^N = (I - P_h) u_{it}^N, \quad u^N = G u^N - \delta^N. \tag{3.4}$$

Assume that $k_{yy}(t,s,y)$ is bounded uniformly on $D \times R$. Then it follows from (2.1), (2.10), (3.4), Theorem 2.1 and Taylor's formula that

$$\begin{aligned} e^N &= \delta^N + (Gy - Gu^N) \\ &= \delta^N + G'e^N + \frac{1}{2}\int_0^t k_{yy}(t,s,\eta(s))(e^N(s))^2\,\mathrm{d}s, \\ &= \delta^N + G'e^N + O(h^{2r})\|y\|_{r,\infty}^2, \end{aligned}$$

where η is a function whose value $\eta(s)$ at s is between $u^N(s)$ and $y(s)$. Thus, setting $F := \delta^N + O(h^{2r})\|y\|_{r,\infty}^2$, we derive from Lemma 3.1 that

$$\begin{aligned} e^N &= F(t) + \int_0^t R^*(t,s)F(s)\,\mathrm{d}s \\ &= \delta^N(t) + \int_0^t R^*(t,s)\delta^N(s)\,\mathrm{d}s + O(h^{2r})\|y\|_{r,\infty}^2, \end{aligned} \tag{3.5}$$

where $R^*(t,s)$ is the resolvent kernel associated with $K^*(t,s) := k_y(t,s,y(s))$, which inherits the same smoothness of $K^*(t,s)$ and satisfies the Fredholm identity

$$R^*(t,s) = K^*(t,s) + \int_s^t K^*(t,\tau)R^*(\tau,s)\,\mathrm{d}\tau, \quad (t,s) \in D.$$

Therefore, using the Fredholm identity we obtain by exchanging the order of integration with respect to s and τ, (3.4) and (3.5) that

$$\begin{aligned} e_{it}^N := y - u_{it}^N = Gy - Gu^N &= G'e^N + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= \int_0^t K^*(t,s)\delta^N(s)\,\mathrm{d}s + \int_0^t K^*(t,s)\left(\int_0^s R^*(s,\tau)\delta^N(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= \int_0^t R^*(t,s)\delta^N(s)\,\mathrm{d}s + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= \int_0^t R^*(t,s)(I - P_h)u_{it}^N(s)\,\mathrm{d}s + O(h^{2r})\|y\|_{r,\infty}^2. \end{aligned} \tag{3.6}$$

Setting

$$(R_h^*\varphi)(t) := \int_0^t R^*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s \tag{3.7}$$

one finds from (3.6) that

$$\begin{aligned} e_{it}^N &= -R_h^* u_{it}^N + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= -R_h^*(u_{it}^N - y + y) + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= R_h^* e_{it}^N - R_h^* y + O(h^{2r})\|y\|_{r,\infty}^2. \end{aligned} \tag{3.8}$$

Lemma 3.2. *The operator $(I - R_h^*)^{-1}$ exists and is bounded uniformly on $C(I)$, where the operator R_h^* is given by* (3.7).

P r o o f. For any $t \in \sigma_k$ $(0 \leqslant k \leqslant M-1)$, from (2.5) one derives that

$$\begin{aligned}
(3.9) \qquad |(R_h^*\varphi)(t)| &= \left|\int_0^{t_k} R^*(t,s)(I-P_h)\varphi(s)\,\mathrm{d}s + \int_{t_k}^t R^*(t,s)(I-P_h)\varphi(s)\,\mathrm{d}s\right| \\
&\leqslant \sum_{j=0}^{k-1}\left|\int_{\sigma_j}(I-P_h)R^*(t,s)(I-P_h)\varphi(s)\,\mathrm{d}s\right| \\
&\quad + \left|\int_{t_k}^t R^*(t,s)(I-P_h)\varphi(s)\,\mathrm{d}s\right| \\
&\leqslant Ch^r\|\varphi\|_{0,\infty} + C(t-t_k)\|\varphi\|_{0,\infty} \leqslant Ch\|\varphi\|_{0,\infty},
\end{aligned}$$

which implies

$$(3.10) \qquad \|R_h^*\|_{C(I)\to C(I)} \leqslant Ch.$$

Therefore, $(I - R_h^*)^{-1}$ exists and is bounded uniformly on $C(I)$ for all $h \in (0,\sigma)$, with $\sigma > 0$ sufficiently small. □

Theorem 3.1. *In* (1.1), *assume that $g \in C^r(I)$, $k \in C^r(D \times R)$ and $k_{yy}(t,s,y)$ is bounded uniformly on $D \times R$. Then the iterated finite element error $e_{it}^N := y - u_{it}^N$ satisfies*

$$(3.11) \qquad \|e_{it}^N\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}.$$

P r o o f. From the procedure of obtaining (3.9) we can also derive the estimate

$$(3.12) \qquad \|R_h^* y\|_{0,\infty} \leqslant Ch^{2r}\|y\|_{r,\infty} + Ch^{r+1}\|y\|_{r,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}$$

which, together with (3.8) and Lemma 3.2, leads to

$$\begin{aligned}
\|e_{it}^N\|_{0,\infty} &= \|-(I-R_h^*)^{-1}R_h^* y\|_{0,\infty} + O(h^{2r})\|y\|_{r,\infty}^2 \\
&\leqslant \|(I-R_h)^{-1}\|_{C(I)\to C(I)} \cdot \|R_h^* y\|_{0,\infty} + Ch^{2r}\|y\|_{r,\infty}^2 \leqslant Ch^{r+1}\|y\|_{r,\infty}.
\end{aligned}$$

Thus, Theorem 3.1 is proved. □

We would like to point out that a simple and direct proof of Theorem 3.1 for the linear version (1.2) is available. In fact, it is easy to see from (3.1) and (3.2) corresponding to (1.2) that

$$(I - KP_h)u_{it}^L = g,$$

where u^L_{it} is the iterated finite element solution corresponding to the finite element solution u^L of (1.2) and $K\colon L^2(I) \to C(I)$ is the linear Volterra integral operator associated with the known kernel $K(t,s)$ in (1.2):

$$(K\varphi)(t) := \int_0^t K(t,s)\varphi(s)\,\mathrm{d}s, \quad t \in I.$$

This together with (1.2) implies that

$$\begin{aligned}(I - KP_h)(y - u^L_{it}) &= (I - KP_h)y - g \\ &= (I - KP_h)y - (I - K)y \\ &= K(I - P_h)y.\end{aligned}$$

Assume that 1 is not an eigenvalue of the Volterra integral operator K so that $(I - K)^{-1}$ always exists. Then the inverse operator $(I - KP_h)^{-1}$ exists and is uniformly bounded on $C(I)$ for all $h \in (0, \sigma)$ with $\sigma > 0$ sufficiently small. And thus, from (3.12) we have

$$y - u^L_{it} = (I - KP_h)^{-1}K(I - P_h)y. \tag{3.13}$$

Also, it follows from the procedure of obtaining (3.9) that

$$\|K(I - P_h)y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty},$$

which together with (3.13) leads to

$$\|y - u^L_{it}\|_{0,\infty} \leqslant \|(I - KP_h)^{-1}\|_{C(I)\to C(I)} \cdot \|K(I - P_h)y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}. \tag{3.14}$$

Remark 3.1. In [9], where the global superconvergence for (1.2) was studied by collocation methods, it is assumed for the exact solution y of (1.2) that $y \in C^{r+1}(I)$ to get (3.14). However, here we need a weaker regularity, $y \in C^r(I)$, for the exact solution y of (1.2), which is a merit of the finite element method. In addition, in the following section, we will obtain a global superconvergence of the $O(h^{2r})$-convergence rate by means of an interpolation post-processing technique.

The following discussions will involve how to get a posteriori error estimators for the finite element solution of the problem (1.1). We know that it is very important for the finite element computations to have a computable a posteriori error bound by which we can determine the finite element error bound. As an application of Theorem 3.1, we will show that the resulting superconvergence property in the theorem can provide a useful a posteriori error estimator.

Theorem 3.2. *Under the conditions of Theorem* 3.1 *we have*

$$\|y - u^N\|_{0,\infty} = \|u^N - u^N_{it}\|_{0,\infty} + O(h^{r+1}). \tag{3.15}$$

In addition, if there exist positive constant C_0 and small $\varepsilon \in (0,1)$ such that

$$\|y - u^N\|_{0,\infty} \geqslant C_0 h^{r+1-\varepsilon}, \tag{3.16}$$

then we have

$$\lim_{h\to 0} \frac{\|y - u^N\|_{0,\infty}}{\|u^N - u^N_{it}\|_{0,\infty}} = 1. \tag{3.17}$$

P r o o f. It follows from Theorem 3.1 and

$$y - u^N = (u^N_{it} - u^N) + (y - u^N_{it})$$

that

$$\|y - u^N\|_{0,\infty} = \|u^N - u^N_{it}\|_{0,\infty} + O(h^{r+1}).$$

Thus, by (3.16) we obtain

$$\frac{\|u^N - u^N_{it}\|_{0,\infty}}{\|y - u^N\|_{0,\infty}} + Ch^{\varepsilon} \geqslant 1$$

or

$$\underline{\lim}_{h\to 0} \frac{\|u^N - u^N_{it}\|_{0,\infty}}{\|y - u^N\|_{0,\infty}} \geqslant 1. \tag{3.18}$$

Similarly, it follows from (3.16) and

$$\|u^N_{it} - u^N\|_{0,\infty} = \|y - u^N\|_{0,\infty} + O(h^{r+1})$$

that

$$\overline{\lim}_{h\to 0} \frac{\|u^N - u^N_{it}\|_{0,\infty}}{\|y - u^N\|_{0,\infty}} \leqslant 1,$$

which, together with (3.18), leads to (3.17). Hence, we complete the proof of Theorem 3.2. □

We know from (3.15) that the computable estimate bound $\|u^N - u_{it}^N\|_{0,\infty}$ is the principal part of the finite element error $\|y - u^N\|_{0,\infty}$, and can be used as an a posteriori error estimator to obtain the bound of the finite element error. From (3.17) we further see that under the condition (3.16), which is a reasonable assumption since h^r is the optimal convergence rate of the finite element solution $u^N \in S_{r-1}^{(-1)}(T_h)$ from the view point of the approximation theory, $\|u^N - u_{it}^N\|_{0,\infty}$ is a reliable a posteriori error estimator.

3.2. Global superconvergence of interpolated finite element solutions. In the previous subsection, the iteration post-processing method has been used to accelerate the approximation procedure. This method is efficacious, and prevalent. However, we will find that another acceleration method, the interpolation post-processing method which has been utilized many times for various partial differential equations and integro-differential equations in our previous work (see, for example, [7], [13], [14], [15]), can also be used to attain the same goal. Such an interpolation post-processing method is simpler than the iteration post-processing.

Theorem 3.3. *In* (1.1), *assume that* $g \in C^r(I)$, $k \in C^r(D \times R)$ *and* $k_{yy}(t,s,y)$ *is bounded uniformly on* $D \times R$. *Then we have*

$$\|P_h y - u^N\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}.$$

Proof. From (2.1) we get that

$$P_h y = P_h G y \tag{3.19}$$

which, together with (2.4), (2.14) and Theorem 2.1, leads to

$$P_h y - u^N = P_h(Gy - Gu^N) = P_h G' e^N + O(h^{2r})\|y\|_{r,\infty}^2. \tag{3.20}$$

Therefore, we find from (3.20) that

$$(P_h y - u^N) - P_h G'(P_h y - u^N) = P_h G'(I - P_h)y + O(h^{2r})\|y\|_{r,\infty}^2. \tag{3.21}$$

And thus, it follows from the standard assumption that $I - G'$ has a continuous inverse operator such that $(I - P_h G')^{-1}$ exists and is uniformly bounded on $C(I)$ for all $h \in (0, \sigma)$, with $\sigma > 0$ sufficiently small so that

$$P_h y - u^N = (I - P_h G')^{-1} P_h G'(I - P_h)y + O(h^{2r})\|y\|_{r,\infty}^2. \tag{3.22}$$

Following the steps to obtain (3.9), we can also get that

$$\|G'(I-P_h)y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}$$

which together with (3.22) yields

$$\begin{aligned}\|P_h y - u^N\|_{0,\infty} &\leqslant \|(I-P_hG)^{-1}P_h\|_{C(I)\to C(I)} \cdot \|G'(I-P_h)y\|_{0,\infty} + Ch^{2r}\|y\|^2_{r,\infty} \\ &\leqslant Ch^{r+1}\|y\|_{r,\infty}.\end{aligned}$$

□

R e m a r k 3.2. The above proof of Theorem 3.3 has nothing to do with Theorem 3.1 in order to emphasize that the interpolation post-processing and the iteration post-processing are two independent post-processing methods. However, if we utilize the result of Theorem 3.1, the proof of Theorem 3.3 is much simpler. In fact, we find from (3.2), (2.14) and Theorem 3.1 that

$$\|P_h y - u^N\|_{0,\infty} = \|P_h(y-u^N_{it})\|_{0,\infty} \leqslant C\|y-u^N_{it}\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}.$$

By virtue of Theorem 3.3, we can obtain global superconvergence of order $r+1$ by an interpolation post-processing method instead of the iteration post-processing method. To this end, we assume that T_h has been obtained from T_{2h} with mesh size $2h$ by subdividing each element of T_{2h} into two elements (i.e., each element of T_{2h} is obtained by a combination of each 2-element in T_h), so that the number of elements M for T_h is even. Then, we can define a higher interpolation operator I^r_{2h} of degree r associated with the mesh T_{2h} according to the following conditions:

$$I^r_{2h}u\,\big|_{\sigma_i\cup\sigma_{i+1}} \in P_r, \qquad i=0,2,4,\dots,M-2, \quad \text{such that}$$

$$\int_{\sigma_i} I^r_{2h}u\,\mathrm{d}s = \int_{\sigma_i} u\,\mathrm{d}s, \qquad \int_{\sigma_{i+1}} I^r_{2h}u\,\mathrm{d}s = \int_{\sigma_{i+1}} u\,\mathrm{d}s$$

and

$$\int_{\sigma_i\cup\sigma_{i+1}} vI^r_{2h}u\,\mathrm{d}s = \int_{\sigma_i\cup\sigma_{i+1}} vu\,\mathrm{d}s, \quad \forall v \in P_{r-1}(\sigma_i\cup\sigma_{i+1}).$$

It is easy to check that

$$\begin{aligned} I^r_{2h}P_h &= I^r_{2h}, \\ \|I^r_{2h}v\|_{0,\infty} &\leqslant C\|v\|_{0,\infty}, \quad \forall v \in S^{(-1)}_{r-1}(T_h), \\ \|I^r_{2h}v - v\|_{0,\infty} &\leqslant Ch^{r+1}\|v\|_{r+1,\infty}.\end{aligned}$$

Theorem 3.4. *In* (1.1), *assume that* $g \in C^{r+1}(I)$ *and* $k \in C^{r+1}(D \times R)$ *such that the Volterra integral equation* (1.1) *possesses a unique solution* $y \in C^{r+1}(I)$. *Then, we have the superconvergence property*

$$\|I_{2h}^r u^N - y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r+1,\infty}.$$

P r o o f. Due to the properties of the operator I_{2h}^r, we have

$$I_{2h}^r u^N - y = I_{2h}^r(u^N - P_h y) + (I_{2h}^r y - y).$$

Therefore, it follows from Theorem 3.3 and the interpolation theorem that

$$\|I_{2h}^r u^N - y\|_{0,\infty} \leqslant C\|u^N - P_h y\|_{0,\infty} + \|I_{2h}^r y - y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r+1,\infty}.$$

□

As a by-product of Theorem 3.4 we have

$$\|y - u^N\|_{0,\infty} = \|I_{2h}^r u^N - u^N\|_{0,\infty} + O(h^{r+1}),$$

in which the estimator $\|I_{2h}^r u^N - u^N\|_{0,\infty}$ is easier to compute than that in Theorem 3.2.

4. Local superconvergence of iterated finite element solutions

In this section we are concerned with the study of the local superconvergence property for the iterated finite element solution of the problem (1.1).

Theorem 4.1. *In* (1.1) *assume that* $g \in C^r(I)$, $k \in C^r(D \times R)$ *and* k_{yy} *is bounded uniformly on* $D \times R$. *Then there is the following superconvergence at the points of the mesh* T_h:

$$\max_{1 \leqslant k \leqslant M} |y(t_k) - u_{it}^N(t_k)| \leqslant Ch^{2r}\|y\|_{r,\infty}. \tag{4.1}$$

P r o o f. From (2.5) and (3.7) one obtains that

$$\begin{aligned}
|(R_h^* y)(t_k)| &= \left|\int_0^{t_k} R^*(t_k, s)(I - P_h)y(s)\,\mathrm{d}s\right| \\
&= \left|\sum_{j=0}^{k-1}\int_{\sigma_j} R^*(t_k, s)(I - P_h)y(s)\,\mathrm{d}s\right| \\
&= \left|\sum_{j=0}^{k-1}\int_{\sigma_j} (I - P_h)R^*(t_k, s)(I - P_h)y(s)\right| \leqslant Ch^{2r}\|y\|_{r,\infty}.
\end{aligned}$$

Now, it is easy to find from (2.5), (3.7) and (3.11) that

$$\begin{aligned}\left|(R_h^* e_{it}^N)(t_k)\right| &= \left|\int_0^{t_k} R^*(t_k,s)(I-P_h)e_{it}^N(s)\,\mathrm{d}s\right| \\ &= \left|\int_0^{t_k} (I-P_h)R^*(t_k,s)(I-P_h)e_{it}^N(s)\,\mathrm{d}s\right| \\ &\leqslant Ch^r\|e_{it}^N\|_{0,\infty} \leqslant Ch^{2r+1}\|y\|_{r,\infty},\end{aligned}$$

which together with (3.8) leads to

$$|e_{it}^N(t_k)| \leqslant \left|(R_h^* e_{it}^N)(t_k)\right| + |(R_h^* y)(t_k)| + Ch^{2r}\|y\|_{r,\infty}^2 \leqslant Ch^{2r}\|y\|_{r,\infty}.$$

Thus, Theorem 4.1 follows. □

Remark 4.1. In Theorem 4.1 we only need that the exact solution of the problem (1.1) satisfies $y \in C^r(I)$ to obtain (4.1). However, for the collocation method, the case is quite different in that $y \in C^{2r}(I)$ is assumed. This is the main difference between the two numerical methods.

Notice that the superconvergence property (4.1) holds only at the points of the mesh T_h. However, it will be shown that by virtue of (4.1), one can obtain the global superconvergence approximation of order $2r$ by using the interpolation post-processing technique. For this reason, we need to define a higher interpolation operator.

For easy exposition, we demonstrate our idea mainly for the case of $r = 2$. Let the number M of elements for T_h be a multiple of 3 so that we can define an interpolation operator I_{3h}^3 of degree 3 associated with T_{3h} as follows:

$$I_{3h}^3 u|_{\sigma_{k-1}\cup\sigma_k\cup\sigma_{k+1}} \in P_3, \quad k = 3l+1, \quad l = 0,1,\ldots,\frac{M}{3}-1$$

and

$$I_{3h}^3 u(t_i) = u(t_i), \quad i = k-1, k, k+1, k+2 \quad (1 \leqslant k \leqslant M-2).$$

Similarly, we can also define an interpolation operator $I_{(2r-1)h}^{2r-1}$ of degree $(2r-1)$ associated with the mesh $T_{(2r-1)h}$.

Theorem 4.2. *In (1.1), assume that $g \in C^{2r}(I), k \in C^{2r}(D \times R)$. Then we have*

$$\|I_{(2r-1)h}^{2r-1} u_{it}^N - y\|_{0,\infty} \leqslant Ch^{2r}\|y\|_{2r,\infty}.$$

P r o o f. Denoting the basis function corresponding to $\{t_j\}$ by $\{\varphi_j\}(1 \leqslant j \leqslant M)$, we have

$$I^3_{3h}(u^N_{it} - y)(t) = \sum_{j=1}^{M} (u^N_{it} - y)(t_j)\varphi_j(t),$$

which together with (4.1) and the uniform boundedness of $\{\varphi_j\}_1^M$ leads to

$$\|I^3_{3h}(u^N_{it} - y)\|_{0,\infty} \leqslant \sum_{j=1}^{M} Ch^4 \|y\|_{2,\infty} \|\varphi_j\|_{0,\infty} \leqslant Ch^4 \|y\|_{2,\infty}.$$

Thus, using the interpolation property $\|I^3_{3h}y - y\|_{0,\infty} \leqslant Ch^4\|y\|_{4,\infty}$ we obtain that

$$\|I^3_{3h}u^N_{it} - y\|_{0,\infty} \leqslant \|I^3_{3h}(u^N_{it} - y)\|_{0,\infty} + \|I^3_{3h}y - y\|_{0,\infty} \leqslant Ch^4\|y\|_{4,\infty}.$$

For the general case $r \geqslant 3$ the proof is similar to that given above, so we omit it. □

By the way, we point out that for a posteriori estimators we can also obtain from Theorems 4.1 and 4.2 by means of arguments analogous to those in Theorem 3.2 that

$$\max_{1\leqslant j\leqslant M} |y(t_j) - u^N(t_j)| = \max_{1\leqslant j\leqslant M} |u^N(t_j) - u^N_{it}(t_j)| + O(h^{2r}) \tag{4.2}$$

under the conditions of Theorem 4.1, and

$$\|y - u^N\|_{0,\infty} = \left\|u^N - I^{2r-1}_{(2r-1)h}u^N_{it}\right\|_{0,\infty} + O(h^{2r}) \tag{4.3}$$

under the conditions of Theorem 4.2.

Since h^r is the optimal convergence rate of the finite element solution u^N in the finite element space $S^{(-1)}_{r-1}(T_h)$, the a posteriori error estimators provided by (4.2) and (4.3) are more practical than that obtained in Theorem 3.2.

5. The iterative correction of finite element solutions

In this section, we will study an iterative correction method, of which the scheme proposed in [6] is a special case. In addition, by virtue of the superconvergence analysis technique used before, we can improve the iterative correction approximation obtained in [6].

From (3.8) one derives a recurrence formula of the form

$$e^N_{it} = -\sum_{k=1}^{r} (R^*_h)^k y + (R^*_h)^r e^N_{it} + O(h^{2r})\|y\|^2_{r,\infty}. \tag{5.1}$$

Lemma 5.1. *Under the conditions of Theorem* 3.1 *we have*

$$\|(R_h^*)^r e_{it}^N\|_{0,\infty} \leqslant Ch^{2r+1}\|y\|_{r,\infty}.$$

P r o o f. We know from (3.10) and Theorem 3.1 that

$$\|(R_h^*)^r e_{it}^N\|_{0,\infty} \leqslant \|(R_h^*)^r\|_{C(I)\to C(I)} \cdot \|e_{it}^N\|_{0,\infty} \leqslant Ch^{2r+1}\|y\|_{r,\infty},$$

and hence Lemma 5.1 is proved. □

According to Lemma 5.1 we can write (5.1) as

$$e_{it}^N = -\sum_{k=1}^{r}(R_h^*)^k y + O(h^{2r})\|y\|_{r,\infty}^2. \tag{5.2}$$

Theorem 5.1. *In* (1.1), *assume that the conditions of Theorem* 3.1 *hold. Then, the* $(n-1)$*st iterative correction* $\tilde{u}_n^N$ *of the iterated finite element solution* u_{it}^N *corresponding to the finite element solution* $u^N \in S_{r-1}^{(-1)}(T_h)$ *satisfies*

$$\|y - \tilde{u}_n^N\|_{0,\infty} \leqslant Ch^{r+n}\|y\|_{r,\infty}, \quad 1 \leqslant n \leqslant r,$$

where $\tilde{u}_n^N := \sum_{k=1}^{n}(-1)^{k-1}C_n^k(A_1^N)^k y$, *and* $A_1^N\colon C(I) \to C(I)$ *is the iterated finite element operator corresponding to the problem* (1.1), *defined by* $A_1^N y := u_{it}^N$.

P r o o f. From (5.2) we derive that

$$(I - A_1^N)y = -\sum_{k=1}^{r}(R_h^*)^k y + O(h^{2r})\|y\|_{r,\infty}^2.$$

Therefore, we obtain from the boundedness of the operator $(I - A_1^N)$ that

$$\begin{aligned}(I - A_1^N)^2 y &= -(I - A_1^N)\left(\sum_{k=1}^{r}(R_h^*)^k y\right) + O(h^{2r})\|y\|_{r,\infty}^2 \\ &= \sum_{k=1}^{r}\sum_{j=1}^{r}(R_h^*)^{k+j} y + O(h^{2r})\|y\|_{r,\infty}^2.\end{aligned} \tag{5.3}$$

By virtue of (3.10) and (3.12) we know that

$$\|R_h^*\|_{C(I)\to C(I)} \leqslant Ch \quad \text{and} \quad \|R_h^* y\|_{0,\infty} \leqslant Ch^{r+1}\|y\|_{r,\infty}.$$

Therefore, we derive

$$\|(R_h^*)^2 y\|_{0,\infty} \leqslant \|R_h^*\|_{C(I)\to C(I)} \cdot \|R_h^* y\|_{0,\infty} \leqslant Ch^{r+2}\|y\|_{r,\infty}.$$

Now, this implies by (5.3) that

$$(I - A_1^N)^2 y = O(h^{r+2})\|y\|_{r,\infty}.$$

Inductively, we can eventually conclude

$$(I - A_1^N)^n y = O(h^{r+n})\|y\|_{r,\infty}, \quad 1 \leqslant n \leqslant r,$$

and the left-hand side is exactly

$$(I - A_1^N)^n y = y - \tilde{u}_n^N.$$

□

From Theorem 5.1 we know that for the nonlinear problem (1.1) the iterative correction method is valid subject to the iterative number n not exceeding r. However, for the linear problem (1.2), the case is quite different in that this iterative process can be continued to generate approximations of higher and higher orders. In fact, following the procedure for obtaining (3.8) one can get

$$e_{it}^L := y - u_{it}^L = R_h e_{it}^L - R_h y, \tag{5.4}$$

where R_h is a linear Volterra integral operator defined by

$$(R_h\varphi)(t) := \int_0^t R(t,s)(I - P_h)\varphi(s)\,\mathrm{d}s.$$

This implies a recurrence formula

$$e_{it}^L = -\sum_{k=1}^{n} R_h^k y + R_h^n e_{it}.$$

Consequently, we obtain

$$\|y - \tilde{u}_n^L\|_{0,\infty} \leqslant Ch^{n+r}\|y\|_{r,\infty}, \quad n \geqslant 1, \tag{5.5}$$

where $\tilde{u}_n^L := \sum_{k=0}^{n-1}(-1)^k C_n^{k+1}(A_1^L)^{k+1} y$ with $A_1^L\colon C(I) \to C(I)$ being the iterated finite element operator corresponding to the problem (1.2), defined by $A_1^L y := u_{it}^L$.

R e m a r k 5.1. From (5.5) we observe that when $u^L \in S_0^{(-1)}(T_h)$, that is $r = 1$, the convergence rate of the $(n-1)$st iterative correction $\tilde{u}_n^L$ is h^{n+1}, which improves the corresponding result in [6] where the convergence rate of $\tilde{u}_n^L$ is h^n.

Like the superconvergence properties in the previous section, the convergence results in Theorem 5.1 and (5.5) can also provide us with some a posteriori error estimators:

$$\|y - \tilde{u}_n^L\|_{0,\infty} = \|\tilde{u}_{n+1}^L - \tilde{u}_n^L\|_{0,\infty} + O(h^{r+n+1}), \quad n \geqslant 1,$$

and

$$\left\|y - \tilde{u}_n^N\right\|_{0,\infty} = \left\|\tilde{u}_{n+1}^N - \tilde{u}_n^N\right\|_{0,\infty} + O(h^{r+n+1}), \quad 1 \leqslant n \leqslant r.$$

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