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ON A DIFFERENTIAL-ALGEBRAIC PROBLEM

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Abstract. The method of quasilinearization is a procedure for obtaining approximate solutions of differential equations. In this paper, this technique is applied to a differential-algebraic problem. Under some natural assumptions, monotone sequences converge quadratically to a unique solution of our problem.

Keywords: differential-algebraic problem, monotone sequences, quadratic convergence

MSC 2000: 34A45, 34B99

Section 1

Consider the problem

$$(1)\qquad \begin{cases} x'(t) = f\big(t, x(t), y(t)\big), & t \in J = [0, b], \\ x(0) = k_0, \\ y(t) = g\big(t, x(t), y(t)\big), & t \in J, \end{cases}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $k_0 \in \mathbb{R}$ are given. Problem (1) is called a differential-algebraic problem. A solution of (1) is a pair $(x, y) \in C^1(J, \mathbb{R}) \times C(J, \mathbb{R})$.

There are some methods which can be applied to construct approximate solutions of such problems. Convergence of approximate iterations for (1) is proved, for example, in papers [3], [6] (see also [1]). In [5], an approximate solution is constructed by some numerical methods. Another useful method is based on monotone iterations. The aim of this paper is to apply this procedure. Using the monotone iterative technique we can construct monotone sequences of approximate solutions that converge uniformly to the unique solution of (1). As we shall see, the convergence is quadratic, which is useful in practice.

Section 2

Let $x_0, u_0 \in C^1(J,\mathbb{R})$, $y_0, v_0 \in C(J,\mathbb{R})$ be such that $x_0(t) \leqslant u_0(t)$, $y_0(t) \leqslant v_0(t)$ on J. Define the closed set

$$\Omega = \{(t,x,y)\colon\ t \in J, \quad x_0(t) \leqslant x \leqslant u_0(t), \quad y_0(t) \leqslant y \leqslant v_0(t)\}.$$

Theorem 1. *Let $f, g \in C(\Omega,\mathbb{R})$. Assume that f_x, f_y, g_x, g_y exist and $g_y(t,u,v) < 1$ on Ω. Then problem (1) has at most one solution.*

Proof. Assume that problem (1) has two distinct solutions $(\overline{x},\overline{y})$ and $(\overline{\overline{x}},\overline{\overline{y}})$. Put $p = \overline{x} - \overline{\overline{x}}$, $q = \overline{y} - \overline{\overline{y}}$. Note that $p(0) = 0$. Using the mean value theorem we obtain

$$\begin{aligned} p'(t) &= f\bigl(t,\overline{x}(t),\overline{y}(t)\bigr) - f\bigl(t,\overline{\overline{x}}(t),\overline{\overline{y}}(t)\bigr) \\ &= f\bigl(t,\overline{x}(t),\overline{y}(t)\bigr) - f\bigl(t,\overline{\overline{x}}(t),\overline{y}(t)\bigr) + f\bigl(t,\overline{\overline{x}}(t),\overline{y}(t)\bigr) - f\bigl(t,\overline{\overline{x}}(t),\overline{\overline{y}}(t)\bigr) \\ &= f_x\bigl(t,\xi,\overline{y}(t)\bigr)p(t) + f_y\bigl(t,\overline{\overline{x}}(t),\sigma\bigr)q(t) \end{aligned}$$

and

$$\begin{aligned} q(t) &= g\bigl(t,\overline{x}(t),\overline{y}(t)\bigr) - g\bigl(t,\overline{\overline{x}}(t),\overline{\overline{y}}(t)\bigr) \\ &= g\bigl(t,\overline{x}(t),\overline{y}(t)\bigr) - g\bigl(t,\overline{\overline{x}}(t),\overline{y}(t)\bigr) + g\bigl(t,\overline{\overline{x}}(t),\overline{y}(t)\bigr) - g\bigl(t,\overline{\overline{x}}(t),\overline{\overline{y}}(t)\bigr) \\ &= g_x\bigl(t,\bar{\xi},\overline{y}(t)\bigr)p(t) + g_y\bigl(t,\overline{\overline{x}}(t),\bar{\sigma}\bigr)q(t) \end{aligned}$$

with ξ, $\bar{\xi}$ between $\overline{x}$ and $\overline{\overline{x}}$, and σ, $\bar{\sigma}$ between $\overline{y}$ and $\overline{\overline{y}}$. Indeed,

$$q(t) = \frac{g_x\bigl(t,\bar{\xi},\overline{y}(t)\bigr)}{1 - g_y\bigl(t,\overline{\overline{x}}(t),\bar{\sigma}\bigr)}p(t),$$

and hence we have

$$p'(t) = p(t)k(t) \quad \text{for} \quad k(t) = f_x\bigl(t,\xi,\overline{y}(t)\bigr) + \frac{f_y\bigl(t,\overline{\overline{x}}(t),\sigma\bigr)g_x\bigl(t,\bar{\xi},\overline{y}(t)\bigr)}{1 - g_y\bigl(t,\overline{\overline{x}}(t),\bar{\sigma}\bigr)}.$$

Solving the linear differential equation we get

$$p(t) = p(0)\mathrm{e}^{\int_0^t k(\tau)\,\mathrm{d}\tau}, \quad t \in J,$$

so $p(t) = q(t) = 0$, $t \in J$. This proves Theorem 1. $\square$

Definition 1. A pair $(x_0, y_0) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a lower solution of problem (1) if

$$\begin{cases} x_0'(t) \leqslant f\big(t, x_0(t), y_0(t)\big), & t \in J, \\ x_0(0) \leqslant k_0, \\ y_0(t) \leqslant g\big(t, x_0(t), y_0(t)\big), & t \in J \end{cases} \tag{2}$$

and a pair (u_0, v_0) is an upper solution of (1) if the inequalities are reversed.

Theorem 2. *Assume that $f, g \in C(\Omega,\mathbb{R})$, and*

1. $(x_0, y_0) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ *and* $(u_0, v_0) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ *are lower and upper solutions of problem* (1), *respectively, such that* $x_0(t) \leqslant u_0(t)$, $y_0(t) \leqslant v_0(t)$, $t \in J$,
2. f_{xx}, f_{xy}, f_{yx}, f_{yy}, g_{xx}, g_{xy}, g_{yx}, g_{yy} *exist, are continuous and*
3. $f_{xx}(t,u,v) \geqslant 0$, $f_{xy}(t,u,v) \geqslant 0$, $f_{yy}(t,u,v) \geqslant 0$, $f_y(t,u,v) \geqslant 0$, $g_{xx}(t,u,v) \geqslant 0$, $g_{xy}(t,u,v) \geqslant 0$, $g_{yy}(t,u,v) \geqslant 0$, $g_x(t,u,v) \geqslant 0$, $g_y(t,x,y) \leqslant B < 1$ *on* Ω.

Then there exist monotone sequences (x_n, y_n), (u_n, v_n) which converge uniformly and monotonically on J to the unique solution of problem (1), *and this convergence is quadratic.*

P r o o f. For $n = 0, 1, \dots$ and $t \in J$, let us define the following sequences:

$$\begin{aligned} x_{n+1}'(t) &= f\big(t, x_n(t), y_n(t)\big) + f_x\big(t, x_n(t), y_n(t)\big)[x_{n+1}(t) - x_n(t)] \\ &\quad + f_y\big(t, x_n(t), y_n(t)\big)[y_{n+1}(t) - y_n(t)], \quad x_{n+1}(0) = k_0, \\ y_{n+1}(t) &= g\big(t, x_n(t), y_n(t)\big) + g_x\big(t, x_n(t), y_n(t)\big)[x_{n+1}(t) - x_n(t)] \\ &\quad + g_y\big(t, x_n(t), y_n(t)\big)[y_{n+1}(t) - y_n(t)], \end{aligned}$$

and

$$\begin{aligned} u_{n+1}'(t) &= f\big(t, u_n(t), v_n(t)\big) + f_x\big(t, x_n(t), y_n(t)\big)[u_{n+1}(t) - u_n(t)] \\ &\quad + f_y\big(t, x_n(t), y_n(t)\big)[v_{n+1}(t) - v_n(t)], \quad u_{n+1}(0) = k_0, \\ v_{n+1}(t) &= g\big(t, u_n(t), v_n(t)\big) + g_x\big(t, x_n(t), y_n(t)\big)[u_{n+1}(t) - u_n(t)] \\ &\quad + g_y\big(t, x_n(t), y_n(t)\big)[v_{n+1}(t) - v_n(t)]. \end{aligned}$$

First we shall show that

$$x_0(t) \leqslant x_1(t) \leqslant u_1(t) \leqslant u_0(t), \tag{3}$$

$$y_0(t) \leqslant y_1(t) \leqslant v_1(t) \leqslant v_0(t) \tag{4}$$

on J. Let $p(t) = x_0(t) - x_1(t)$, $q(t) = y_0(t) - y_1(t)$ on J. Note that

$$
\begin{aligned}
p'(t) &= x_0'(t) - x_1'(t) \\
&\leqslant f\big(t, x_0(t), y_0(t)\big) - f\big(t, x_0(t), y_0(t)\big) \\
&\quad - f_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] - f_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] \\
&= f_x\big(t, x_0(t), y_0(t)\big)p(t) + f_y\big(t, x_0(t), y_0(t)\big)q(t), \\
q(t) &= y_0(t) - y_1(t) \\
&\leqslant g\big(t, x_0(t), y_0(t)\big) - g\big(t, x_0(t), y_0(t)\big) \\
&\quad - g_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] - g_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] \\
&= g_x\big(t, x_0(t), y_0(t)\big)p(t) + g_y\big(t, x_0(t), y_0(t)\big)q(t)
\end{aligned}
$$

since (x_0, y_0) is a lower solution of problem (1). Hence

$$
q(t) \leqslant \frac{g_x\big(t, x_0(t), y_0(t)\big)}{1 - g_y\big(t, x_0(t), y_0(t)\big)}p(t), \quad t \in J,
$$

so

$$
p'(t) \leqslant K(t)p(t), \quad t \in J, \quad p(0) \leqslant 0,
$$

where

$$
K(t) = f_x\big(t, x_0(t), y_0(t)\big) + \frac{f_y\big(t, x_0(t), y_0(t)\big)g_x\big(t, x_0(t), y_0(t)\big)}{1 - g_y\big(t, x_0(t), y_0(t)\big)}.
$$

This yields the inequality

$$
p(t) \leqslant p(0)\mathrm{e}^{\int_0^t K(\tau)\,\mathrm{d}\tau} \leqslant 0, \quad t \in J,
$$

so $x_0(t) \leqslant x_1(t)$ and $y_0(t) \leqslant y_1(t)$ on J.

If we now put $p(t) = u_1(t) - u_0(t)$, $q(t) = v_1(t) - v_0(t)$, $t \in J$, then

$$
\begin{aligned}
p'(t) &= u_1'(t) - u_0'(t) \\
&\leqslant f\big(t, u_0(t), v_0(t)\big) + f_x\big(t, x_0(t), y_0(t)\big)[u_1(t) - u_0(t)] \\
&\quad + f_y\big(t, x_0(t), y_0(t)\big)[v_1(t) - v_0(t)] - f\big(t, u_0(t), v_0(t)\big) \\
&= f_x\big(t, x_0(t), y_0(t)\big)p(t) + f_y\big(t, x_0(t), y_0(t)\big)q(t), \quad p(0) \leqslant 0,
\end{aligned}
$$

and

$$
\begin{aligned}
q(t) &= v_1(t) - v_0(t) \\
&\leqslant g\big(t, u_0(t), v_0(t)\big) + g_x\big(t, x_0(t), y_0(t)\big)[u_1(t) - u_0(t)] \\
&\quad + g_y\big(t, x_0(t), y_0(t)\big)[v_1(t) - v_0(t)] - g\big(t, u_0(t), v_0(t)\big) \\
&= g_x\big(t, x_0(t), y_0(t)\big)p(t) + g_y\big(t, x_0(t), y_0(t)\big)q(t).
\end{aligned}
$$

Similarly as in the previous case, we immediately have $p(t) \leqslant 0$, $q(t) \leqslant 0$ on J, so $u_1(t) \leqslant u_0(t)$, $v_1(t) \leqslant v_0(t)$, $t \in J$.

To show that $x_1(t) \leqslant u_1(t)$, $y_1(t) \leqslant v_1(t)$ on J we need some relations on f and g. Observe that Taylor's formula yields

$$\begin{aligned} f(t,u,\alpha) &= f(t,u,\alpha) - f(t,v,\alpha) + f(t,v,\alpha) - f(t,v,\beta) + f(t,v,\beta) \\ &= f(t,v,\beta) + f_x(t,v,\alpha)(u-v) + \frac{1}{2} f_{xx}(t,\xi,\alpha)(u-v)^2 \\ &\quad + f_y(t,v,\beta)(\alpha-\beta) + \frac{1}{2} f_{yy}(t,v,\delta)(\alpha-\beta)^2 \end{aligned}$$

where ξ is between u and v, while δ is between α and β. Assume that $u \geqslant v$, $\alpha \geqslant \beta$. Since $f_{xx} \geqslant 0$, $f_{yy} \geqslant 0$, $f_{xy} \geqslant 0$, we have

$$f(t,u,\alpha) \geqslant f(t,v,\beta) + f_x(t,v,\beta)(u-v) + f_y(t,v,\beta)(\alpha-\beta). \tag{5}$$

In the same way, we can prove that

$$g(t,u,\alpha) \geqslant g(t,v,\beta) + g_x(t,v,\beta)(u-v) + g_y(t,v,\beta)(\alpha-\beta) \tag{6}$$

provided $u \geqslant v$ and $\alpha \geqslant \beta$.

Now, put $p(t) = x_1(t) - u_1(t)$, $q(t) = y_1(t) - v_1(t)$. Note that $p(0) = 0$. Basing on (5) and (6), we have

$$\begin{aligned} p'(t) &= f\big(t,x_0(t),y_0(t)\big) + f_x\big(t,x_0(t),y_0(t)\big)[x_1(t)-x_0(t)] \\ &\quad + f_y\big(t,x_0(t),y_0(t)\big)[y_1(t)-y_0(t)] - f\big(t,u_0(t),v_0(t)\big) \\ &\quad - f_x\big(t,x_0(t),y_0(t)\big)[u_1(t)-u_0(t)] - f_y\big(t,x_0(t),y_0(t)\big)[v_1(t)-v_0(t)] \\ &\leqslant - f_x\big(t,x_0(t),y_0(t)\big)[u_0(t)-x_0(t)] - f_y\big(t,x_0(t),y_0(t)\big)[v_0(t)-y_0(t)] \\ &\quad + f_x\big(t,x_0(t),y_0(t)\big)[x_1(t)-x_0(t)-u_1(t)+u_0(t)] \\ &\quad + f_y\big(t,x_0(t),y_0(t)\big)[y_1(t)-y_0(t)-v_1(t)+v_0(t)] \\ &= f_x\big(t,x_0(t),y_0(t)\big)p(t) + f_y\big(t,x_0(t),y_0(t)\big)q(t), \\ q(t) &= g\big(t,x_0(t),y_0(t)\big) + g_x\big(t,x_0(t),y_0(t)\big)[x_1(t)-x_0(t)] \\ &\quad + g_y\big(t,x_0(t),y_0(t)\big)[y_1(t)-y_0(t)] - g\big(t,u_0(t),v_0(t)\big) \\ &\quad - g_x\big(t,x_0(t),y_0(t)\big)[u_1(t)-u_0(t)] - g_y\big(t,x_0(t),y_0(t)\big)[v_1(t)-v_0(t)] \\ &\leqslant - g_x\big(t,x_0(t),y_0(t)\big)[u_0(t)-x_0(t)] - g_y\big(t,x_0(t),y_0(t)\big)[v_0(t)-y_0(t)] \\ &\quad + g_x\big(t,x_0(t),y_0(t)\big)[x_1(t)-x_0(t)-u_1(t)+u_0(t)] \\ &\quad + g_y\big(t,x_0(t),y_0(t)\big)[y_1(t)-y_0(t)-v_1(t)+v_0(t)] \\ &= g_x\big(t,x_0(t),y_0(t)\big)p(t) + g_y\big(t,x_0(t),y_0(t)\big)q(t). \end{aligned}$$

From the above, we have $p(t) \leqslant 0$, $q(t) \leqslant 0$ on J, so $x_1(t) \leqslant u_1(t)$, $y_1(t) \leqslant v_1(t)$, $t \in J$. This proves that (3) and (4) are satisfied.

In the next step we have to show that (x_1, y_1) and (u_1, v_1) are lower and upper solutions of problem (1), respectively. To show this we will use (5) and (6), obtaining

$$\begin{aligned}
x_1'(t) &= f\big(t, x_0(t), y_0(t)\big) + f_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] \\
&\quad + f_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] \\
&\leqslant f\big(t, x_1(t), y_1(t)\big) - f_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] \\
&\quad - f_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] + f_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] \\
&\quad + f_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] \\
&= f\big(t, x_1(t), y_1(t)\big), \\
y_1(t) &= g\big(t, x_0(t), y_0(t)\big) + g_x\big(t, x_0(t), y_0(t)\big)[x_1(t) - x_0(t)] \\
&\quad + g_y\big(t, x_0(t), y_0(t)\big)[y_1(t) - y_0(t)] \\
&\leqslant g(t, x_1(t), y_1(t)),
\end{aligned}$$

and

$$\begin{aligned}
u_1'(t) &= f\big(t, u_0(t), v_0(t)\big) + f_x\big(t, x_0(t), y_0(t)\big)[u_1(t) - u_0(t)] \\
&\quad + f_y\big(t, x_0(t), y_0(t)\big)[v_1(t) - v_0(t)] \\
&\geqslant f\big(t, u_1(t), v_1(t)\big) + f_x\big(t, u_1(t), v_1(t)\big)[u_0(t) - u_1(t)] \\
&\quad + f_y\big(t, u_1(t), v_1(t)\big)[v_0(t) - v_1(t)] + f_x\big(t, x_0(t), y_0(t)\big)[u_1(t) - u_0(t)] \\
&\quad + f_y\big(t, x_0(t), y_0(t)\big)[v_1(t) - v_0(t)] \\
&\geqslant f\big(t, u_1(t), v_1(t)\big), \\
v_1(t) &= g\big(t, u_0(t), v_0(t)\big) + g_x\big(t, x_0(t), y_0(t)\big)[u_1(t) - u_0(t)] \\
&\quad + g_y\big(t, x_0(t), y_0(t)\big)[v_1(t) - v_0(t)] \\
&\geqslant g\big(t, u_1(t), v_1(t)\big),
\end{aligned}$$

since f_x, f_y, g_x and g_y are nondecreasing with respect to the last two variables. This shows that (9) and (10) are satisfied.

Let us assume that

$$\begin{aligned}
x_0(t) \leqslant x_1(t) \leqslant \ldots \leqslant x_k(t) \leqslant u_k(t) \leqslant \ldots \leqslant u_1(t) \leqslant u_0(t), \\
y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_k(t) \leqslant v_k(t) \leqslant \ldots \leqslant v_1(t) \leqslant v_0(t),
\end{aligned}$$

$t \in J$, and let (x_k, y_k), (u_k, v_k) be lower and upper solutions of problem (1), respectively, for some $k > 1$.

We shall prove that

$$
\begin{cases} x_k(t) \leqslant x_{k+1}(t) \leqslant u_{k+1}(t) \leqslant u_k(t), & t \in J, \\ y_k(t) \leqslant y_{k+1}(t) \leqslant v_{k+1}(t) \leqslant v_k(t), & t \in J. \end{cases} \tag{7}
$$

As before, we set $p(t) = x_k(t) - x_{k+1}(t)$, $q(t) = y_k(t) - y_{k+1}(t)$, $t \in J$. We see that $p(0) = 0$ and

$$
\begin{aligned}
p'(t) &\leqslant f\big(t, x_k(t), y_k(t)\big) - f\big(t, x_k(t), y_k(t)\big) - f_x\big(t, x_k(t), y_k(t)\big)[x_{k+1}(t) - x_k(t)] \\
&\quad - f_y\big(t, x_k(t), y_k(t)\big)[y_{k+1}(t) - y_k(t)] \\
&= f_x(t, x_k(t), y_k(t))p(t) + f_y\big(t, x_0(t), y_0(t)\big)q(t), \\
q(t) &\leqslant g\big(t, x_k(t), y_k(t)\big) - g\big(t, x_k(t), y_k(t)\big) - g_x\big(t, x_k(t), y_k(t)\big)[x_{k+1}(t) - x_k(t)] \\
&\quad - g_y\big(t, x_k(t), y_k(t)\big)[y_{k+1}(t) - y_k(t)] \\
&= g_x\big(t, x_k(t), y_k(t)\big)p(t) + g_y\big(t, x_k(t), y_k(t)\big)q(t).
\end{aligned}
$$

Hence we have $p(t) \leqslant 0$ and $q(t) \leqslant 0$ on J, so $x_k(t) \leqslant x_{k+1}(t)$, $y_k(t) \leqslant y_{k+1}(t)$, $t \in J$. Similarly, we can show that $u_{k+1}(t) \leqslant u_k(t)$ and $v_{k+1}(t) \leqslant v_k(t)$, $t \in J$. Now let $p(t) = x_{k+1}(t) - u_{k+1}(t)$, $q(t) = y_{k+1}(t) - v_{k+1}(t)$, $t \in J$. Then in view of (5) and (6) we get

$$
\begin{aligned}
p'(t) &= f\big(t, x_k(t), y_k(t)\big) + f_x\big(t, x_k(t), y_k(t)\big)[x_{k+1}(t) - x_k(t)] \\
&\quad + f_y\big(t, x_k(t), y_k(t)\big)[y_{k+1}(t) - y_k(t)] - f\big(t, u_k(t), v_k(t)\big) \\
&\quad - f_x\big(t, x_k(t), y_k(t)\big)[u_{k+1}(t) - u_k(t)] - f_y\big(t, x_k(t), y_k(t)\big)[v_{k+1}(t) - v_k(t)] \\
&\leqslant f_x\big(t, x_k(t), y_k(t)\big)p(t) + f_y\big(t, x_k(t), y_k(t)\big)q(t),
\end{aligned}
$$

and

$$
\begin{aligned}
q(t) &= g\big(t, x_k(t), y_k(t)\big) + g_x\big(t, x_k(t), y_k(t)\big)[x_{k+1}(t) - x_k(t)] \\
&\quad + g_y\big(t, x_k(t), y_k(t)\big)[y_{k+1}(t) - y_k(t)] - g\big(t, u_k(t), v_k(t)\big) \\
&\quad - g_x\big(t, x_k(t), y_k(t)\big)[u_{k+1}(t) - u_k(t)] - g_y\big(t, x_k(t), y_k(t)\big)[v_{k+1}(t) - v_k(t)] \\
&\leqslant g_x\big(t, x_k(t), y_k(t)\big)p(t) + g_y\big(t, x_k(t), y_k(t)\big)q(t).
\end{aligned}
$$

As a result we have

$$
x_{k+1}(t) \leqslant u_{k+1}(t) \quad \text{and} \quad y_{k+1}(t) \leqslant v_{k+1}(t), \quad t \in J
$$

so (7) holds.

Basing on (5) and (6) we can show that (x_{k+1}, y_{k+1}), (u_{k+1}, v_{k+1}) are lower and upper solutions of problem (1), respectively.

Hence, by induction, we have

$$x_0(t) \leqslant x_1(t) \leqslant \ldots \leqslant x_n(t) \leqslant u_n(t) \leqslant \ldots \leqslant u_1(t) \leqslant u_0(t),$$
$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_n(t) \leqslant v_n(t) \leqslant \ldots \leqslant v_1(t) \leqslant v_0(t),$$

$t \in J$, for all $n > 1$, and $\big(x_n(t), y_n(t)\big)$, $\big(u_n(t), v_n(t)\big)$ are lower and upper solutions of problem (1), respectively.

Employing Dini's theorem we can show that the sequences (x_n, y_n), (u_n, v_n) converge uniformly and monotonically to the corresponding solutions of problem (1). Since problem (1) has at most one solution (x, y), so (x_n, y_n), (u_n, v_n) converge to the unique solution of (1).

We shall prove that the convergence of (x_n, y_n), (u_n, v_n) to (x, y) is quadratic. First, we put $p_{n+1}(t) = x(t) - x_{n+1}(t) \geqslant 0$, $q_{n+1}(t) = y(t) - y_{n+1}(t) \geqslant 0$, $t \in J$. Note that $p_{n+1}(0) = 0$. Then, by the mean value theorem and the monotonicity of f_x, f_y, g_x g_y we have

$$\begin{aligned}
p'_{n+1}(t) &= f\big(t, x(t), y(t)\big) - f\big(t, x_n(t), y_n(t)\big) - f_x\big(t, x_n(t), y_n(t)\big)[x_{n+1}(t) - x_n(t)] \\
&\quad - f_y\big(t, x_n(t), y_n(t)\big)[y_{n+1}(t) - y_n(t)] \\
&= f\big(t, x(t), y(t)\big) - f\big(t, x_n(t), y(t)\big) + f\big(t, x_n(t), y(t)\big) - f\big(t, x_n(t), y_n(t)\big) \\
&\quad - f_x\big(t, x_n(t), y_n(t)\big)[x_{n+1}(t) - x(t) + x(t) - x_n(t)] \\
&\quad - f_y\big(t, x_n(t), y_n(t)\big)[y_{n+1}(t) - y(t) + y(t) - y_n(t)] \\
&= f_x\big(t, \xi_1, y(t)\big)p_n(t) + f_y\big(t, x_n(t), \delta_1\big)q_n(t) \\
&\quad + f_x\big(t, x_n(t), y_n(t)\big)[p_{n+1}(t) - p_n(t)] + f_y\big(t, x_n(t), y_n(t)\big)[q_{n+1}(t) - q_n(t)] \\
&\leqslant \big[f_x\big(t, x(t), y(t)\big) - f_x\big(t, x_n(t), y(t)\big) \\
&\quad + f_x\big(t, x_n(t), y(t)\big) - f_x\big(t, x_n(t), y_n(t)\big)\big]p_n(t) \\
&\quad + f_x\big(t, x_n(t), y_n(t)\big)p_{n+1}(t) + f_y\big(t, x_n(t), y_n(t)\big)q_{n+1}(t) \\
&= \big[f_{xx}\big(t, \xi_2, y(t)\big)p_n(t) + f_{xy}\big(t, x_n(t), \delta_2\big)q_n(t)\big]p_n(t) \\
&\quad + f_x\big(t, x_n(t), y_n(t)\big)p_{n+1}(t) + f_y\big(t, x_n(t), y_n(t)\big)q_{n+1}(t),
\end{aligned}$$

and

$$\begin{aligned}
q_{n+1}(t) &= g\big(t, x(t), y(t)\big) - g\big(t, x_n(t), y(t)\big) + g\big(t, x_n(t), y(t)\big) - g\big(t, x_n(t), y_n(t)\big) \\
&\quad - g_x\big(t, x_n(t), y_n(t)\big)[x_{n+1}(t) - x(t) + x(t) - x_n(t)] \\
&\quad - g_y\big(t, x_n(t), y_n(t)\big)[y_{n+1}(t) - y(t) + y(t) - y_n(t)] \\
&= g_x\big(t, \xi_3, y(t)\big)p_n(t) + g_y\big(t, x_n(t), \delta_3\big)q_n(t) \\
&\quad + g_x\big(t, x_n(t), y_n(t)\big)[p_{n+1}(t) - p_n(t)] + g_y\big(t, x_n(t), y_n(t)\big)[q_{n+1}(t) - q_n(t)]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \big[g_x\big(t,x(t),y(t)\big)-g_x\big(t,x_n(t),y(t)\big)+g_x\big(t,x_n(t),y(t)\big)\\
&\quad -g_x\big(t,x_n(t),y_n(t)\big)\big]p_n(t)+g_x\big(t,x_n(t),y_n(t)\big)p_{n+1}(t)\\
&\quad +g_y\big(t,x_n(t),y_n(t)\big)q_{n+1}(t)+\big[g_y\big(t,x_n(t),y(t)\big)-g_y\big(t,x_n(t),y_n(t)\big)\big]q_n(t)\\
&=\big[g_{xx}\big(t,\xi_4,y(t)\big)p_n(t)+g_{xy}\big(t,x_n(t),\delta_4\big)q_n(t)\big]p_n(t)+g_{yy}\big(t,x_n(t),\delta_5\big)q_n^2(t)\\
&\quad +g_y\big(t,x_n(t),y_n(t)\big)q_{n+1}(t)+g_x(t,x_n(t),y(t))p_{n+1}(t)
\end{aligned}$$

where $x_n(t)<\xi_1,\xi_2,\xi_3,\xi_4<x(t)$, $y_n(t)<\delta_1,\delta_2,\delta_3,\delta_4,\delta_5<y(t)$. Hence

$$\begin{aligned}
q_{n+1}(t)&\leqslant B_1p_n^2(t)+B_2p_n(t)q_n(t)+B_3q_n^2(t)+Bq_{n+1}(t)+B_4p_{n+1}(t)\\
&\leqslant \Big(B_1+\frac{1}{2}B_2\Big)p_n^2(t)+\Big(B_3+\frac{1}{2}B_2\Big)q_n^2(t)+Bq_{n+1}(t)+B_4p_{n+1}(t),
\end{aligned}$$

so

$$q_{n+1}(t)\leqslant \frac{B_4}{1-B}p_{n+1}(t)+b_1p_n^2(t)+b_2q_n^2(t),\quad t\in J, \tag{8}$$

since $B<1$, where

$$|g_{xx}(t,u,v)|\leqslant B_1,\quad |g_{xy}(t,u,v)|\leqslant B_2,\quad |g_{yy}(t,u,v)|\leqslant B_3,$$
$$|g_x(t,u,v)|\leqslant B_4 \text{ on } \Omega,$$
$$b_1=\frac{B_1+\frac{1}{2}B_2}{1-B},\quad b_2=\frac{B_3+\frac{1}{2}B_2}{1-B}.$$

Moreover, we have

$$\begin{aligned}
p'_{n+1}(t)&\leqslant A_1p_n^2(t)+A_2p_n(t)q_n(t)+A_3p_{n+1}(t)+A_4q_{n+1}(t)\\
&\leqslant \Big(A_1+\frac{1}{2}A_2\Big)p_n^2(t)+\frac{1}{2}A_2q_n^2(t)+A_3p_{n+1}(t)+A_4q_{n+1}(t)
\end{aligned} \tag{9}$$

where $|f_{xx}(t,u,v)|\leqslant A_1$, $|f_{xy}(t,u,v)|\leqslant A_2$, $|f_x(t,u,v)|\leqslant A_3$, $|f_y(t,u,v)|\leqslant A_4$ on Ω. Combining (8) and (9) we finally get

$$p'_{n+1}(t)\leqslant a_1p_{n+1}(t)+a_2p_n^2(t)+a_3q_n^2(t),\quad t\in J,$$

where

$$a_1=A_3+\frac{B_4A_4}{1-B},\quad a_2=A_1+\frac{1}{2}A_2+A_4b_1,\quad a_3=\frac{1}{2}A_2+A_4b_2.$$

By Gronwall's inequality, we see that

$$\begin{aligned}
p_{n+1}(t)&\leqslant \int_0^t [a_2p_n^2(s)+a_3q_n^2(s)]\mathrm{e}^{a_1(t-s)}\,\mathrm{d}s\\
&\leqslant [a_2\max_s p_n^2(s)+a_3\max_s q_n^2(s)]c,\quad c=b\mathrm{e}^{a_1b},\quad t\in J.
\end{aligned}$$

Thus

$$\max_{t\in J}|x_{n+1}(t)-x(t)| \leqslant c_2\max_{t\in J}|x_n(t)-x(t)|^2 + c_3\max_{t\in J}|y_n(t)-y(t)|^2,$$

where $c_i = ca_i$ for $i=2,3$. Hence and by (8), we directly obtain

$$\max_{t\in J}|y_{n+1}(t)-y(t)| \leqslant \overline{a}_2\max_{t\in J}|x_n(t)-x(t)|^2 + \overline{a}_3\max_{t\in J}|y_n(t)-y(t)|^2,$$

where

$$\overline{a}_i = \frac{B_4c_i}{1-B} + b_{i-1}, \quad i=2,3.$$

In the same way we can show that the convergence of (u_n, v_n) to (x, y) is quadratic, so

$$\begin{aligned}
\max_{t\in J}|u_{n+1}(t)-x(t)| \leqslant{}& d_1\max_{t\in J}|u_n(t)-x(t)|^2 + d_2\max_{t\in J}|v_n(t)-y(t)|^2\\
&+ d_3\max_{t\in J}|x_n(t)-x(t)|^2 + d_4\max_{t\in J}|y_n(t)-y(t)|^2,\\
\max_{t\in J}|v_{n+1}(t)-y(t)| \leqslant{}& e_1\max_{t\in J}|u_n(t)-x(t)|^2 + e_2\max_{t\in J}|v_n(t)-y(t)|^2\\
&+ e_3\max_{t\in J}|x_n(t)-x(t)|^2 + e_4\max_{t\in J}|y_n(t)-y(t)|^2,
\end{aligned}$$

where $|f_{yy}(t,u,v)| \leqslant A_5$ on Ω,

$$\begin{aligned}
d_1 &= c\left[A_2 + \frac{3}{2}A_1 + A_4\frac{B_2+\frac{3}{2}B_1}{1-B}\right], \quad & d_2 &= c\left[A_2 + \frac{3}{2}A_5 + A_4\frac{B_2+\frac{3}{2}B_3}{1-B}\right],\\
d_3 &= \frac{1}{2}c\left[A_2 + A_1 + A_4\frac{B_2+B_1}{1-B}\right], & d_4 &= \frac{1}{2}c\left[A_2 + A_5 + A_4\frac{B_2+B_3}{1-B}\right],
\end{aligned}$$

and

$$\begin{aligned}
e_1 &= \frac{B_2+\frac{3}{2}B_1+d_1B_4}{1-B}, \quad & e_2 &= \frac{B_2+\frac{3}{2}B_3+d_2B_4}{1-B},\\
e_3 &= \frac{1}{2}\frac{B_2+B_1+d_3B_4}{1-B}, & e_4 &= \frac{1}{2}\frac{B_2+B_3+d_4B_4}{1-B}.
\end{aligned}$$

□

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