Ján Lovíšek

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RELIABLE SOLUTION OF PARABOLIC OBSTACLE PROBLEMS WITH RESPECT TO UNCERTAIN DATA*

Ján Lovíšek, Bratislava

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Abstract. A class of parabolic initial-boundary value problems is considered, where admissible coefficients are given in certain intervals. We are looking for maximal values of the solution with respect to the set of admissible coefficients. We give the abstract general scheme, proposing how to solve such problems with uncertain data. We formulate a general maximization problem and prove its solvability, provided all fundamental assumptions are fulfilled. We apply the theory to certain Fourier obstacle type maximization problem.

Keywords: uncertain data, optimal design approach, parabolic obstacle problems, penalization method, Fourier problem

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0. Introduction

In engineering design problems, there are uncertainties associated with geometrical and material properties as well as with loads. Although by themselves these uncertainties may be negligible, their combination might result in unexpected behaviour which could lead to failure. Furthermore, most structures demand less weight or less cost, having at the same time a high performance and reliability measure. It thus becomes essential to deal with optimization problems considering probabilistic design aspects. An interesting objective of this is also to design optimal shape systems under a required reliability. It is a consequence of requirements of an industrial reliability that demands reduction of costs and fast and continuous evolution of the products.

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Many mathematical models involve data which are not easy to determine. This means that the coefficients of inequalities (or equations), right-hand sides or boundary values can often be prescribed only between certain lower and upper bounds, resulting from the accuracy of experimental measurements and the approximate identification problem. In the following, we assume that the main aim of the computations is to find the maximal value of a certain functional which depends on the solution of the mathematical model. Then we can formulate the corresponding maximization problem and employ methods of Optimal Design. In the present paper we apply a general approach which is called "the method of reliable solutions or worst scenario method', see ([8], [9]), to a class of nonlinear parabolic problems with uncertain coefficients. We give an abstract general scheme, proposing how to solve such problems with uncertain data. We formulate a general maximization problem and prove its solvability by using the method of penalization introduced by Lions ([11]) for the parabolic state problem. Roughly speaking, the solution of the state inequality is obtained as the limit of solutions of suitable approximate problems. We introduce a functional by means of which we can choose the "worst scenario", i.e., the worst admissible coefficients. This choice is then accomplished by formulating the corresponding maximization problem and we prove the existence of at least one maximizer. We concretize the abstract results by applying them to a certain Fourier obstacle type maximization problem (Fourier problem occurs in the modelling of several heat transfer phenomena), when the coefficients of the parabolic operator or the obstacles are given with some uncertainty.

1. Existence and uniqueness theorem for a parabolic variational inequality

1.1. Basic assumptions.

We describe some function spaces. More details can be found in the books [2], [5], [7] or [12], [16], [22]. Let E be a reflexive Banach space. If $1 \leqslant p \leqslant \infty$, we denote by $L_p(0,T,E)$ the space of all measurable functions $v\colon [0,T] \to E$ such that $\|v(\cdot)\|_E \in L_p(0,T)$, where $T \in (0,\infty)$ is fixed. The space $L_p(0,T,E)$ is the Banach space with the norm $\|v\|_{L_p(0,T,E)} = \big(\int_0^T \|v(t)\|_E^p \,\mathrm{d}t\big)^{1/p}$ if $1 \leqslant p < \infty$ and $\|v\|_{L_\infty(0,T,E)} = \operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|_E$. Let $C([0,T],E)$ stand for the usual Banach space of all continuous functions from $[0,T]$ to E. Further, $C^k([0,T],E)$ denotes the space of all k-times continuously differentiable functions $([0,T] \to E)$.

The spaces $L_p(0,T,E)$, $1 < p < \infty$, are reflexive and the dual spaces $[L_p(0,T,E)]^*$ can be identified with the spaces $L_q(0,T,E^*)$, $1/p + 1/q = 1$.

The space $L_\infty(0,T,E^*)$ can be identified with the dual space $[L_1(0,T,E)]^*$, i.e., for every $\mathcal{F} \in [L_1(0,T,E)]^*$ there exists a unique function $\theta \in L_\infty(0,T,E^*)$ satisfying the relations

$$\|\mathcal{F}\|_{[L_1(0,T,E)]^*} = \|\theta\|_{L_\infty(0,T,E^*)}$$

and

$$\mathcal{F}(v) = \int_0^T \langle \theta(t), v(t)\rangle_E \,\mathrm{d}t \quad \text{for every } \ v \in L_1(0,T,E).$$

On the other hand, if E is a Hilbert space with the inner product $(\cdot,\cdot)_E$, then $L_2(0,T,E)$ is a Hilbert space with the inner product

$$(v,z)_{L_2(0,T,E)} = \int_0^T (v(t), z(t))_E \,\mathrm{d}t, \quad v, z \in L_2(0,T,E).$$

Further, we introduce the Sobolev spaces of vector-valued functions. We denote by $W_p^m([0,T],E)$ the space of all functions $\nu \in L_p(0,T,E)$, $m \geqslant 1$, $1 \leqslant p \leqslant \infty$ such that there exist functions $\theta_i \in L_p(0,T,E)$, $i = 1,2,\dots,m$, satisfying the relations

$$\int_0^T \frac{\mathrm{d}^i\varphi(t)}{\mathrm{d}t^i}\nu(t)\,\mathrm{d}t = (-1)^i \int_0^T \varphi(t)\theta_i(t)\,\mathrm{d}t \quad \text{for every } \ \varphi \in C_0^\infty(0,T).$$

Functions θ_i are generalized derivatives of the i-th order and we set $\theta_i = \mathrm{d}^i\nu/\mathrm{d}t^i$, $i = 1,2,\dots,m$. It is clear that $W_p^m([0,T],E)$ is a Banach space with the norm

$$\|\nu\|_{W_p^m([0,T],E)} = (\|\nu\|^p_{L_p(0,T,E)} + \|\mathrm{d}\nu/\mathrm{d}t\|^p_{L_p(0,T,E)} + \dots + \|\mathrm{d}^m\nu/\mathrm{d}t^m\|^p_{L_p(0,T,E)})^{1/p},$$
$$1 \leqslant p < \infty,$$

and

$$\|\nu\|_{W_\infty^m([0,T],E)} = \|\nu\|_{L_\infty(0,T,E)} + \|\mathrm{d}\nu/\mathrm{d}t\|_{L_\infty(0,T,E)} + \dots + \|\mathrm{d}^m\nu/\mathrm{d}t^m\|_{L_\infty(0,T,E)}.$$

In particular, $v \in W_p^1([0,T],E)$ means that $v\colon\ [0,T] \to E$ is absolutely continuous, a.e. differentiable on $(0,T)$ and

$$v(t) = v(0) + \int_0^t (\mathrm{d}v(s)/\mathrm{d}s)\,\mathrm{d}s \quad \text{for } \ t \in [0,T], \quad \mathrm{d}v/\mathrm{d}s \in L_p(0,T,E).$$

Moreover, if E is a Hilbert space then $W_2^m([0,T],E)$ is a Hilbert space with the inner product

$$(\nu,\vartheta)_{W_2^m(0,T,E)} = (\nu,\vartheta)_{L_2(0,T,E)} + \Big(\frac{\mathrm{d}\nu}{\mathrm{d}t}, \frac{\mathrm{d}\vartheta}{\mathrm{d}t}\Big)_{L_2(0,T,E)} + \dots + \Big(\frac{\mathrm{d}^m\nu}{\mathrm{d}t^m}, \frac{\mathrm{d}^m\vartheta}{\mathrm{d}t^m}\Big)_{L_2(0,T,E)}.$$

Let V be a Hilbert space with the inner product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V$. Further, $L(V,V^*)$ is the space of all linear bounded operators from V into V^* with the norm $\|\cdot\|_{L(V,V^*)}$. We suppose that $V \subset H$, where H is a Hilbert space and V is dense in H. If we identify H with its dual we have $V \subset H \subset V^*$. We note that this inclusion holds both in the algebraic and the topological sense. The symbol $(\cdot,\cdot)_H$ denotes the scalar product in H. As a consequence of the previous identifications, the scalar product in H for $\mathcal{F} \in H$ and $v \in V$ is the same as the scalar product of $\mathcal{F}$ and v in the duality between V and V^*. We put

$$\langle \mathcal{F}, v\rangle_V = (\mathcal{F}, v)_H \quad \text{for any } \mathcal{F} \in H \text{ and for any } v \in V. \tag{1.1}$$

For a Banach space $\mathcal{X}$ and positive constants λ, Λ we denote by $\mathcal{E}_{[C^1,\mathcal{X}]}(\lambda,\Lambda)$ the class of the operator functions $\kappa(\cdot)\colon [0,T] \to L(\mathcal{X},\mathcal{X}^*)$ for which the assumptions

$$\text{(A0)} \qquad \begin{cases} 1.\ \lambda\|v\|_{\mathcal{X}}^2 \leqslant \langle \kappa(t)v, v\rangle_{\mathcal{X}} \leqslant \Lambda\|v\|_{\mathcal{X}}^2, \\ 2.\ \langle \kappa(t)v, z\rangle_{\mathcal{X}} = \langle \kappa(t)z, v\rangle_{\mathcal{X}} \quad \text{for all } v, z \in \mathcal{X} \text{ and } t \in [0,T], \\ 3.\ \kappa(\cdot) \in C^1([0,T], L(\mathcal{X},\mathcal{X}^*)) \end{cases}$$

hold.

We consider the initial value problem $(\mathcal{B})$

$$\begin{cases} u(t) \in \mathcal{K} \quad (t \text{ traversing the interval } [0,T]) \text{ such that} \\ \langle \mathrm{d}u/\mathrm{d}t, v - u(t)\rangle_V + \langle A(t)u(t), v - u(t)\rangle_V \geqslant \langle L(t), v - u(t)\rangle_V \\ \text{for all } v \in \mathcal{K}, \text{ for a.e. } t \in [0,T]; \ u(0) = u_0 \in \mathcal{K}, \end{cases} \tag{1.2}$$

where $\mathcal{K}$ is a closed convex subset of V, $\mathrm{d}u/\mathrm{d}t$ is the strong derivative of $u\colon [0,T] \to V^*$, and

$$\text{(A1)} \qquad \begin{cases} A(t) \in \mathcal{E}_{[C^1,V]}(\alpha, M) \quad \text{for a.e. } t \in [0,T], \\ L \in W_2^1([0,T], V^*) \cap C([0,T], H), \\ A(0)u_0 - L(0) \in H. \end{cases}$$

Let $I_{\mathcal{K}}$ be the indicator function of some closed convex subset $\mathcal{K}$ of V, i.e.

$$I_{\mathcal{K}}(v) = 0 \text{ if } v \in \mathcal{K}, \quad I_{\mathcal{K}}(v) = +\infty \text{ if } v \notin \mathcal{K}.$$

This is a convex, lower semicontinuous, proper mapping on V, and $v^* \in \partial I_{\mathcal{K}}(v) \subset V^*$ if $v \in \mathcal{K}$ and $\langle v - w, v^*\rangle_V \geqslant 0$ for any $w \in \mathcal{K}$. For every $v \in \mathcal{K}$, $\partial I_{\mathcal{K}}(v)$ is a closed convex cone in V^* with its vertex at zero, called the normal cone to $\mathcal{K}$ at v. If $v \notin \mathcal{K}$, then $\partial I_{\mathcal{K}}(v)$ is empty.

1.2. An approximation result for solutions to the initial value problem ($\mathcal{B}$).

Consider the approximating equations (a penalized parabolic initial-value problem) corresponding to the equality

$$(1.3)\qquad \begin{cases} \mathrm{d}u_\varepsilon(t)/\mathrm{d}t + A(t)u_\varepsilon(t) + (\partial I_{\mathcal{K}})_\varepsilon(u_\varepsilon(t)) = L(t), \quad \varepsilon > 0,\\ u_\varepsilon(0) = u_0. \end{cases}$$

We approximate

$$(1.4)\qquad \mathrm{d}u(t)/\mathrm{d}t + A(t)u(t) + \partial I_{\mathcal{K}}(u(t)) \ni L(t) \quad \text{for a.e. } t \in (0,T),$$

by replacing $I_{\mathcal{K}}$ by its Lipschitz-continuous Yosida approximation $(I_{\mathcal{K}})_\varepsilon$, $\varepsilon > 0$, where

$$(1.5)\qquad \begin{cases} (I_{\mathcal{K}})_\varepsilon(v) = (2\varepsilon)^{-1}\|v - P_{\mathcal{K}}(v)\|_V^2, \quad \varepsilon > 0, \quad v \in V,\\ (\partial I_{\mathcal{K}})_\varepsilon(v) \ \text{(the Fréchet derivative)} = \varepsilon^{-1}J(v - P_{\mathcal{K}}(v)), \ \text{and}\\ J \ \text{(duality mapping of } V\text{)}\colon\ V \to V^*,\\ \langle Jv, z\rangle_V = (v,z)_V, \ \ v, z \in V\\ \text{is the canonical isomorphism from } V \text{ into } V^*. \end{cases}$$

Here $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$, and it is monotone and Lipschitz continuous.

The projection operator is defined by ($P_{\mathcal{K}}\colon\ V \to \mathcal{K}$)

$$\|v - P_{\mathcal{K}}(v)\|_V = \min_{z\in\mathcal{K}} \|v - z\|_V, \quad v \in V,$$

and $P_{\mathcal{K}}$ has the following properties arising directly from its definition (see [21]):

$$(1.6)\qquad \begin{cases} 1^\circ \ \ P_{\mathcal{K}}(v) = v \Leftrightarrow v \in \mathcal{K};\\ 2^\circ \ \ (P_{\mathcal{K}}(v) - v, z - P_{\mathcal{K}}(v))_V \geqslant 0 \ \text{for all } v \in V, \ \ z \in \mathcal{K};\\ 3^\circ \ \ \|P_{\mathcal{K}}(v) - P_{\mathcal{K}}(z)\|_V \leqslant \|v - z\|_V \ \text{for all } v, z \in V. \end{cases}$$

On the other hand, the operator $(\partial I_{\mathcal{K}})_\varepsilon$ then fulfils the conditions

$$(1.7)\qquad \begin{cases} 1^\circ \ \ \varepsilon(\partial I_{\mathcal{K}})_\varepsilon(v) = 0 \Leftrightarrow v \in \mathcal{K};\\ 2^\circ \ \ \langle \varepsilon(\partial I_{\mathcal{K}})_\varepsilon(v) - \varepsilon(\partial I_{\mathcal{K}})_\varepsilon(z), v - z\rangle_V \geqslant 0;\\ 3^\circ \ \ \|\varepsilon(\partial I_{\mathcal{K}})_\varepsilon(v) - \varepsilon(\partial I_{\mathcal{K}})_\varepsilon(z)\|_{V^*} \leqslant 2\|v - z\|_V\\ \qquad \text{for all } v, z \in V. \end{cases}$$

This means that $\varepsilon(\partial I_{\mathcal{K}})_\varepsilon$ is monotone and Lipschitz continuous.

Theorem 1. *Let $T > 0$, $\varepsilon > 0$. Then there exists a unique solution $u_\varepsilon \in C^1([0,T],V)$ of the initial value problem* (1.3) *and the sequences $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ and $\{\mathrm{d}u_{\varepsilon_n}/\mathrm{d}t\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0_+$, are contained in a bounded subset of $L_2(0,T,V) \cap L_\infty(0,T,H)$.*

Proof. The initial problem can be rewritten in the form

$$\begin{cases} \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t + \mathcal{Z}_{\varepsilon_n}(t)u_{\varepsilon_n}(t) = L(t), \\ u_{\varepsilon_n}(0) = u_0, \end{cases} \tag{1.8}$$

with

$$\mathcal{Z}_{\varepsilon_n}(t)\colon\ V \to V^*,$$
$$\mathcal{Z}_{\varepsilon_n}(t) = A(t) + (\partial I_{\mathcal{K}})_{\varepsilon_n}.$$

Thus the operators $\mathcal{Z}_{\varepsilon_n}(t)$ are uniformly Lipschitz continuous and then due to [7] the initial value problem (1.8) has a unique solution which is also a unique solution the problem (1.3).

Let us set

$$z_{\varepsilon_n} = u_{\varepsilon_n} - u_0. \tag{1.9}$$

The function $z_{\varepsilon_n} \in C^1([0,T],V)$ is a solution of the initial value problem

$$\begin{cases} \mathrm{d}z_{\varepsilon_n}(t)/\mathrm{d}t + A(t)z_{\varepsilon_n}(t) + (\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(t)) = L(t) - A(t)u_0, \\ z_{\varepsilon_n}(0) = 0. \end{cases} \tag{1.10}$$

For any function v in $L_2(0,T,V)$ which satisfies $\mathrm{d}v/\mathrm{d}t \in L_2(0,T,V^*)$, the following equation holds (see [22]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_H^2 = 2\langle \mathrm{d}v(t)/\mathrm{d}t, v(t)\rangle_V. \tag{1.11}$$

This result will be used in the next step.

Performing duality pairing in (1.10), we obtain

$$\begin{aligned} \langle \mathrm{d}z_{\varepsilon_n}(t)/\mathrm{d}t, z_{\varepsilon_n}(t)\rangle_V &+ \langle A(t)z_{\varepsilon_n}(t), z_{\varepsilon_n}(t)\rangle_V \\ &+ \langle (\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(t)), z_{\varepsilon_n}(t)\rangle_V \\ &= \langle L(t) - A(t)u_0, z_{\varepsilon_n}(t)\rangle_V. \end{aligned} \tag{1.12}$$

By (1.2) one has $u_0 \in \mathcal{K}$, and hence $((\partial I_{\mathcal{K}})_{\varepsilon_n})(u_0) = 0$. This means that

$$\langle (\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(t)), z_{\varepsilon_n}(t)\rangle_V \geqslant 0, \tag{1.13}$$

due to the monotonicity of $(\partial I_{\mathcal{K}})_\varepsilon$.

Thus, by virtue of (1.13) and (1.11) we get the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_{\varepsilon_n}(t)\|_H^2 + 2\alpha\|z_{\varepsilon_n}(t)\|_V^2 \leqslant 2\langle L(t) - A(t)u_0, z_{\varepsilon_n}(t)\rangle_V. \tag{1.14}$$

The right-hand side of (1.14) is majorized by

$$2\|L(t) - A(t)u_0\|_{V^*}\|z_{\varepsilon_n}(t)\|_V \leqslant \alpha\|z_{\varepsilon_n}(t)\|_V^2 + \alpha^{-1}\|L(t) - A(t)u_0\|_{V^*}^2.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_{\varepsilon_n}(t)\|_H^2 + \alpha\|z_{\varepsilon_n}(t)\|_V^2 \leqslant \alpha^{-1}\|L(t) - A(t)u_0\|_{V^*}^2. \tag{1.15}$$

Integrating (1.15) from 0 to s, $0 < s < T$, we obtain in particular

$$\begin{aligned} \|z_{\varepsilon_n}(s)\|_H^2 &\leqslant \alpha^{-1}\int_0^s \|L(t) - A(t)u_0\|_{V^*}^2 \,\mathrm{d}t \\ &\leqslant \alpha^{-1}\int_0^T \|L(t) - A(t)u_0\|_{V^*}^2 \,\mathrm{d}t. \end{aligned} \tag{1.16}$$

Hence,

$$\sup_{s\in[0,T]} \|z_{\varepsilon_n}(s)\|_H^2 \leqslant \alpha^{-1}\int_0^T \|L(t) - A(t)u_0\|_{V^*}^2 \,\mathrm{d}t. \tag{1.17}$$

The right-hand side of (1.17) is finite and independent of ε_n, therefore

$(1.17)_1$ the sequence $\{z_{\varepsilon_n}\}_{n\in\mathbb{N}}$ remains in a bounded set of $L_\infty(0,T,H)$.

We then integrate (1.15) from 0 to T and get

$$\|z_{\varepsilon_n}(T)\|_H^2 + \alpha\int_0^T \|z_{\varepsilon_n}(t)\|_V^2 \,\mathrm{d}t \leqslant \alpha^{-1}\int_0^T \|L(t) - A(t)u_0\|_{V^*}^2 \,\mathrm{d}t. \tag{1.17_2}$$

This shows that the sequence $\{z_{\varepsilon_n}\}_{\varepsilon_n}$ remains in a bounded set of $L_2(0,T,V)$. This means that

$(1.17)_3$ the sequence $\{u_{\varepsilon_n}\}_{\varepsilon_n}$ is bounded in $L_2(0,T,V) \cap L_\infty(0,T,H)$ as $\varepsilon_n \to 0$.

On the other hand, in order to obtain an estimate for the sequence $\{\mathrm{d}u_{\varepsilon_n}/\mathrm{d}t\}_{\varepsilon_n}$ we formally differentiate equation (1.10) and arrive at

$$\mathrm{d}[\mathrm{d}z_{\varepsilon_n}(t)/\mathrm{d}t]/\mathrm{d}t + \mathrm{d}[A(t)z_{\varepsilon_n}(t)]/\mathrm{d}t + \mathrm{d}[(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(t))]/\mathrm{d}t = \mathrm{d}L(t)/\mathrm{d}t - (\mathrm{d}A(t)/\mathrm{d}t)u_0. \tag{1.18}$$

Next, we observe that the functions $(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(\cdot))\colon [0,T] \to V^*$ are Lipschitz continuous by virtue of $((1.7), 3°)$. As the space V^* is reflexive the functions $(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0 + z_{\varepsilon_n}(\cdot))$ belong to the space $W^1_\infty([0,T],V^*)$ (see [5]). Moreover, the functions $\mathcal{Z}_{\varepsilon_n}(\cdot)u_{\varepsilon_n}(\cdot)$, $L(\cdot)$ from equation (1.8) belong to the spaces $W^1_\infty([0,T],V^*)$ and $W^1_2([0,T],V^*)$, respectively. This means that $u_{\varepsilon_n} \in W^2_2([0,T],V^*)$ and by virtue of (1.9) and (1.18) we can write

$$\begin{aligned}
&\langle \mathrm{d}^2 u_{\varepsilon_n}(t)/\mathrm{d}t^2, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&\quad + \langle A(t)\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&\quad + \langle \mathrm{d}[(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_{\varepsilon_n}(t))]/\mathrm{d}t, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&= \langle \mathrm{d}L(t)/\mathrm{d}t - (\mathrm{d}A(t)/\mathrm{d}t)u_{\varepsilon_n}(t), \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&\quad \text{for a.e. } t \in [0,T].
\end{aligned} \tag{1.19}$$

Further, due to (1.11) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\|^2_H + 2\langle A(t)\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&\quad + 2\langle \mathrm{d}[(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_{\varepsilon_n}(t))]/\mathrm{d}t, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \\
&= 2\langle \mathrm{d}L(t)/\mathrm{d}t - (\mathrm{d}A(t)/\mathrm{d}t)u_{\varepsilon_n}(t), \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V.
\end{aligned} \tag{1.20}$$

However (due to the monotonicity of $(\partial I_{\mathcal{K}})_{\varepsilon_n}$), we can write

$$\langle \mathrm{d}[(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_{\varepsilon_n}(t))]/\mathrm{d}t, \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\rangle_V \geqslant 0 \quad \text{for a.e. } t \in [0,T]. \tag{1.21}$$

On the basis of (1.20), (A1), and (1.21) we obtain the inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\|^2_H + \alpha\|\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\|^2_V \\
&\leqslant 2\alpha^{-1}[(\|\mathrm{d}L(t)/\mathrm{d}t\|_{V^*})^2 + (\|(\mathrm{d}A(t)/\mathrm{d}t)u_{\varepsilon_n}(t)\|_{V^*})^2] \\
&\leqslant 2\alpha^{-1}[(\|\mathrm{d}L(t)/\mathrm{d}t\|_{V^*})^2 + \|(\mathrm{d}A(t)/\mathrm{d}t)\|^2_{L(V,V^*)}\|u_{\varepsilon_n}(t)\|^2_V],
\end{aligned} \tag{1.22}$$

and therefore, integrating (1.22) from 0 to s, $0 < s < T$, and using $(1.17)_3$, one has

$$\begin{aligned}
&\|\mathrm{d}u_{\varepsilon_n}(s)/\mathrm{d}t\|^2_H + \alpha\int_0^s \|\mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t\|^2_V \mathrm{d}t \\
&\qquad \leqslant \|\mathrm{d}u_{\varepsilon_n}(0)/\mathrm{d}t\|^2_H + 2\alpha^{-1}\Big[\int_0^T (\|\mathrm{d}L(t)/\mathrm{d}t\|_{V^*})^2 \mathrm{d}t + \mathrm{const}\Big].
\end{aligned} \tag{1.23}$$

On the other hand, putting $t = 0$ in the equality

$$\langle \mathrm{d}u_{\varepsilon_n}(t)/\mathrm{d}t, v\rangle_V + \langle A(t)u_{\varepsilon_n}(t), v\rangle_V + \langle (\partial I_{\mathcal{K}})_{\varepsilon_n}(u_{\varepsilon_n}(t)), v\rangle_V = \langle L(t), v\rangle_V$$

for any $v \in V$, $\varepsilon_n > 0$, we get (due to the previous estimates, $(\partial I_{\mathcal{K}})_{\varepsilon_n}(u_0) = 0$)

$$\langle \mathrm{d}u_{\varepsilon_n}(0)/\mathrm{d}t, v\rangle_V = \langle L(0) - A(0)u_0, v\rangle_V.$$

This yields

$$\mathrm{d}u_{\varepsilon_n}(0)/\mathrm{d}t = L(0) - A(0)u_0 \quad (\in H \ \text{ by (A1)}). \tag{1.24}$$

By (1.23) and (1.24) we see that

(1.25) the sequences $\{\mathrm{d}u_{\varepsilon_n}/\mathrm{d}t\}_{n\in\mathbb{N}}$ are bounded in the space $L_2(0,T,V) \cap L_\infty(0,T,H)$.

1.3. Solution of a parabolic variational inequality.

Due to the a priori estimates obtained above, we obtain existence and uniqueness for a solution of the unilateral problem $(\mathcal{B})$ introduced in (1.2).

Theorem 2. *There exists a unique solution $u \in W^1_\infty([0,T],H) \cap W^1_2([0,T],V)$ of the initial value problem $(\mathcal{B})$.*

Proof. Let $\varepsilon_n \to 0$, $\varepsilon_n > 0$. Then, due to the a priori estimates $(1.17)_3$ and (1.25), the sequence $\{u_{\varepsilon_n}\}_{\varepsilon_n}$ is bounded in the space $W^1_2([0,T],V)$ and in all spaces $W^1_p([0,T],H)$, $1 \leqslant p < \infty$. Hence, there exists a sequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, $\varepsilon_{n_k} > 0$, and a function $u_* \in W^1_2([0,T],V)$ such that

$$\begin{cases} \lim\limits_{k\to\infty} \varepsilon_{n_k} = 0, \\ u_{\varepsilon_{n_k}} \to u_* \ \text{ weakly in } \ W^1_2([0,T],V). \end{cases} \tag{1.26}$$

Further, due to ((1.3), 2°) we have the relation

$$(u_{\varepsilon_{n_k}}(t), v)_V = \left(\int_0^t (\mathrm{d}u_{\varepsilon_{n_k}}(s)/\mathrm{d}s)\,\mathrm{d}s, v\right)_V + (u_0, v)_V \tag{1.27}$$

for each $k \in \mathbb{N}$ and $v \in V$.

The expression $\left(\int_0^t (\mathrm{d}z(s)/\mathrm{d}s)\,\mathrm{d}s, v\right)_V$ for $z \in W^1_2([0,T],V)$ represents (for each fixed $t \in [0,T]$ and $v \in V$) a linear continuous functional over $W^1_2([0,T],V)$. This shows that the sequence $\{(u_{\varepsilon_{n_k}}(t), v)_V\}_{k\in\mathbb{N}}$ is (due to ((1.26), 2°)) convergent for every $t \in [0,T]$ and $v \in V$. Consequently, there exists a function $u\colon\ [0,T] \to V$ such that

$$u_{\varepsilon_{n_k}}(t) \to u(t) \quad \text{weakly in } V \text{ for each } t \in [0,T]. \tag{1.28}$$

By virtue of the Fatou Lemma and the Lebesgue Theorem ([5], App. 1), we see that

$$\begin{cases} u \in L_1(0,T,V), \\ u_{\varepsilon_{n_k}} \to u \quad \text{weakly in } L_1(0,T,V). \end{cases} \tag{1.29}$$

On the other hand, by comparing ((1.26), 2°) with (1.29) we conclude that $u(t) = u_*(t)$ for a.e. $t \in [0,T]$ and

$$u_{\varepsilon_{n_k}} \to u \quad \text{weakly in } W_2^1([0,T],V). \tag{1.30}$$

We observe that the a priori estimates $(1.17)_2$ or (1.23) imply that the sequences $\{u_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ and $\{\mathrm{d}u_{\varepsilon_{n_k}}/\mathrm{d}t\}_{k\in\mathbb{N}}$ are bounded in the space $L_\infty(0,T,H)$, which is the adjoint space to $L_1(0,T,H)$. Thus by virtue of (1.30) and due to a theorem of Banach-Alaoglu-Bourbaki ([6], Th. III 15) one has

$$\begin{cases} u_{\varepsilon_{n_k}} \to u & \text{weakly star in } L_{\infty(0,T,H)}, \\ \mathrm{d}u_{\varepsilon_{n_k}}/\mathrm{d}t \to \mathrm{d}u/\mathrm{d}t & \text{weakly star in } L_{\infty(0,T,H)}. \end{cases} \tag{1.31}$$

Then according to Proposition III.12 from [6] and by virtue of (1.31), we have the estimates

$$\begin{aligned} \|u - u_0\|_{L_\infty(0,T,H)} &\leqslant \liminf_{k\to\infty} \|u_{\varepsilon_{n_k}} - u_0\|_{L_\infty(0,T,H)}, \\ \|\mathrm{d}u/\mathrm{d}t\|_{L_\infty(0,T,H)} &\leqslant \liminf_{k\to\infty} \|\mathrm{d}u_{\varepsilon_{n_k}}/\mathrm{d}t\|_{L_\infty(0,T,H)}, \end{aligned}$$

which imply the estimates (using (1.23), (1.24))

$$\begin{cases} \|u - u_0\|_{L_\infty(0,T,H)} \leqslant \left[\alpha^{-1}\displaystyle\int_0^T \|L(t) - A(t)u_0\|_{V^*}^2\,\mathrm{d}t\right]^{\frac{1}{2}}, \\ \|\mathrm{d}u/\mathrm{d}t\|_{L_\infty(0,T,H)} \leqslant \Bigg[\|L(0)\|_{V^*}^2 + \|A(0)u_0\|_{V^*}^2 \\ \quad +2\alpha^{-1}\displaystyle\int_0^T \|\mathrm{d}L(t)/\mathrm{d}t\|_{V^*}^2\,\mathrm{d}t \\ \quad +\mathrm{const}(\displaystyle\max_{t\in[0,T]} \|\mathrm{d}A(t)/\mathrm{d}t\|_{L(V,V^*)}^2)\Bigg]^{\frac{1}{2}} \quad \text{for all } t \in [0,T]. \end{cases} \tag{1.32}$$

In virtue of the inequality ((1.3), 1°) we can write

$$\varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u_{\varepsilon_{n_k}}(t)) = \varepsilon_{n_k}[L(t) - \mathrm{d}u_{\varepsilon_{n_k}}(t)/\mathrm{d}t - A(t)u_{\varepsilon_{n_k}}(t)]$$

for every $t \in [0, T]$.

The sequences $\{u_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ and $\{\mathrm{d}u_{\varepsilon_{n_k}}/\mathrm{d}t\}_{k\in\mathbb{N}}$ are bounded in V and in H, respectively, for a.a. $t \in [0, T]$. Then one has (by virtue of ((1.26), 1°))

$$\lim_{k\to\infty} \varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u_{\varepsilon_{n_k}}(t)) = \lim_{k\to\infty} \varepsilon_{n_k}[L(t) - \mathrm{d}u_{\varepsilon_{n_k}}(t)/\mathrm{d}t - A(t)u_{\varepsilon_{n_k}}(t)] = 0$$

strongly in V^* for a.a. $t \in [0, T]$.

Moreover, using then the monotonocity of $\varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}$ and the relation (1.28), we obtain

$$\langle \varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(v), u(t) - v\rangle_V \leqslant 0 \quad \text{for a.a. } t \in [0, T],\ v \in V. \tag{1.33}$$

Then inserting $v = u(t) + \theta z$, $\theta > 0$, $z \in V$, into (1.33) we obtain

$$\langle \varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u(t) + \theta z), z\rangle_V \geqslant 0 \quad \text{for all } z \in V,$$

whence (due to the Lipschitz continuity of $\varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}$) the limiting process $\theta \to 0$ yields

$$\langle \varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u(t)), z\rangle_V \geqslant 0 \quad \text{for all } z \in V.$$

This means that

$$\varepsilon_{n_k}(\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u(t)) = 0 \quad \text{for almost all } t \in [0, T], \tag{1.34}$$

which due to ((1.7), 1°) gives the relation $u(t) \in \mathcal{K}$.

We have (after changing u on a set of zero measure)

$$u \in W^1_\infty([0, T], H) \cap C([0, T], H) \tag{1.35}$$

and thus,

$$u(t) = u(0) + \int_0^t (\mathrm{d}u(\xi)/\mathrm{d}\xi)\,\mathrm{d}\xi \quad \text{for every } t \in [0, T],\ 0 < \xi < t. \tag{1.36}$$

Simultaneously we obtain the relation

$$u_{\varepsilon_{n_k}}(t) = u_0 + \int_0^t (\mathrm{d}u_{\varepsilon_{n_k}}(\xi)/\mathrm{d}\xi)\,\mathrm{d}\xi \quad \text{for every } t \in [0, T],\ k \in \mathbb{N}. \tag{1.37}$$

Hence, due to the convergences (1.28) and (1.30) we obtain the initial condition $u(0) = u_0$.

Let us suppose again that z is given in $L_2(0, T, V)$ (being an arbitrary function), where

$$z(t) \in \mathcal{K} \quad \text{for a.e. } t \in [0, T]. \tag{1.38}$$

We then have the inequalities

$$\langle \varepsilon_{n_k} (\partial I_{\mathcal{K}})_{\varepsilon_{n_k}} (u_{\varepsilon_n}(t)), z(t) - u_{\varepsilon_{n_k}}(t) \rangle_V \leqslant 0, \tag{1.39}$$
$$\text{for a.e. } t \in [0, T] \text{ and every } n \in \mathbb{N}.$$

We then come back to the equations

$$\mathrm{d}u_{\varepsilon_{n_k}}(t)/\mathrm{d}t + A(t) u_{\varepsilon_{n_k}}(t) + (\partial I_{\mathcal{K}})_{\varepsilon_{n_k}}(u_{\varepsilon_{n_k}}(t)) = L(t), \tag{1.40}$$

and forming the scalar product of (1.40) and $[z(t) - u_{\varepsilon_{n_k}}(t)]$ and integrating from 0 to T, we arrive at the inequalities

$$\begin{aligned} \|u_{\varepsilon_{n_k}}(T)\|_H^2 &+ 2 \int_0^T \langle A(t) u_{\varepsilon_{n_k}}(t), u_{\varepsilon_{n_k}}(t) \rangle_V \, \mathrm{d}t \\ &\leqslant \|u_{\varepsilon_{n_k}}(0)\|_H^2 + 2 \int_0^T \langle A(t) u_{\varepsilon_{n_k}}(t), z(t) \rangle_V \, \mathrm{d}t \\ &\quad + 2 \int_0^T \langle \mathrm{d}u_{\varepsilon_{n_k}}(t)/\mathrm{d}t, z(t) \rangle_V \, \mathrm{d}t \\ &\quad + 2 \int_0^T \langle L(t), u_{\varepsilon_{n_k}}(t) - z(t) \rangle_V \, \mathrm{d}t \quad \text{for all } k \in \mathbb{N}. \end{aligned} \tag{1.41}$$

However, using the assumptions (A1), we can easily see that the functionals on the left-hand side of (1.41) are weakly lower semicontinuous on the spaces H and $L_2(0, T, V)$, respectively. The passage to the limit for $k \to \infty$ in the integrals of the inequalities (1.41) is easy, using the relations (1.28), (1.30) and the initial conditions in (1.2) and in (1.3). Hence, we find

$$\begin{aligned} \|u(T)\|_H^2 &+ 2 \int_0^T \langle A(t) u(t), u(t) \rangle_V \, \mathrm{d}t \\ &\leqslant \liminf_{k \to \infty} \left[\|u_{\varepsilon_{n_k}}(T)\|_H^2 + 2 \int_0^T \langle A(t) u_{\varepsilon_{n_k}}(t), u_{\varepsilon_{n_k}}(t) \rangle_V \, \mathrm{d}t \right] \\ &\leqslant \|u(0)\|_H^2 + 2 \int_0^T \langle \mathrm{d}u(t)/\mathrm{d}t, z(t) \rangle_V \, \mathrm{d}t \\ &\quad + 2 \int_0^T \langle A(t) u(t), z(t) \rangle_V \, \mathrm{d}t + 2 \int_0^T \langle L(t), u(t) - z(t) \rangle_V \, \mathrm{d}t, \end{aligned}$$

and this gives

$$\int_0^T \langle \mathrm{d}u(t)/\mathrm{d}t + A(t)u(t) - L(t), z(t) - u(t)\rangle_V \,\mathrm{d}t \geqslant 0 \tag{1.42}$$

for all $z \in L_2(0,T,V)$ such that $z(t) \in \mathcal{K}$ for a.e. $t \in [0,T]$.

We deduce from (1.42) that

$$\langle \mathrm{d}u(t)/\mathrm{d}t, v - u(t)\rangle_V + \langle A(t)u(t), v - u(t)\rangle_V \geqslant \langle L(t), v - u(t)\rangle_V \tag{1.43}$$

for all $v \in \mathcal{K}$ and for a.e. $t \in [0,T]$, which proves the inequality in (1.2).

Indeed, let $s \in [0,T]$ and $v \in V$ be arbitrary. We consider a family $\mathcal{O}_k$ of neighborhoods of the point s

$$\mathcal{O}_k = (s - 1/k, s + 1/k), \quad k \to \infty,$$

and define $z(t) = u(t)$ if $t \notin \mathcal{O}_k$ and $z(t) = v$ if $t \in \mathcal{O}_k$.

Then (1.42) yields

$$\frac{2}{k}\int_{\mathcal{O}_k} \Big\langle \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A(t)u(t) - L(t), v - u(t)\Big\rangle_V \,\mathrm{d}t \geqslant 0. \tag{1.44}$$

Passing to the limit with $k \to \infty$ and using the Lebesgue Theorem, we obtain

$$\langle \mathrm{d}u(s)/\mathrm{d}s + A(s)u(s) - L(s), v - u(s)\rangle_V \geqslant 0$$

for almost all $s \in [0,T]$. Thus (1.43) follows for a.a. $t \in [0,T]$. This inequality implies that u is a solution of the initial value problem $(\mathcal{B})$.

Let u_* and u_0 be two solutions of the problem $(\mathcal{B})$. We take successively

$$u = u_*, \quad v = u_0,$$
$$u = u_0, \quad v = u_*$$

in $(\mathcal{B})$. Then adding up these inequalities, we get (integrating from 0 to t)

$$\int_0^t \langle (\mathrm{d}u_*(\xi)/\mathrm{d}\xi - \mathrm{d}u_0(\xi)/\mathrm{d}\xi) + A(\xi)(u_*(\xi) - u_0(\xi)), u_*(\xi) - u_0(\xi)\rangle_V \,\mathrm{d}\xi \leqslant 0 \tag{1.45}$$

for every $t \in [0,T]$.

Let us denote $z = u_* - u_0$. The function z fulfils the initial condition

$$z(0) = 0. \tag{1.46}$$

The inequality (1.45) then implies (by the relation (1.11))

$$\|z(t)\|_H^2 + 2\int_0^t \langle A(\xi)z(\xi), z(\xi)\rangle_V \,\mathrm{d}\xi \leqslant 0 \quad \text{for all } t \in [0,T]. \tag{1.47}$$

This estimation together with (A1) gives

$$z(t) = u_*(t) - u_0(t) = 0$$

and

$$u_*(t) = u_0(t) \quad \text{for all } t \in [0,T],$$

which proves uniqueness of the solution of the initial value problem $(\mathcal{B})$. □

Remark 1. The a priori estimate $(1.17)_1$ shows the existence of an element u in $L_\infty(0,T,H)$ and a subsequence $\varepsilon_n \to 0$ (for $n \to \infty$) such that

$$u_{\varepsilon_n} \text{ converges to } u \text{ in the weak star topology of } L_\infty(0,T,H). \tag{1.48}$$

Then (1.48) means that for each $v \in L_1(0,T,H)$

$$\int_0^T \langle u_{\varepsilon_n}(t) - u(t), v(t)\rangle_H \,\mathrm{d}t \to 0, \quad \varepsilon_n \to 0. \tag{1.49}$$

By $(1.17)_3$ the subsequence $\{u_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ belongs to a bounded subset of $L_2(0,T,V)$, therefore another passage to a subsequence shows the existence of some $u_* \in L_2(0,T,V)$ and a subsequence $\{u_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ such that

$$u_{\varepsilon_{n_k}} \text{ converges to } u_* \text{ in the weak topology of } L_2(0,T,V). \tag{1.50}$$

The convergence (1.50) means

$$\int_0^T \langle u_{\varepsilon_{n_k}}(t) - u_*(t), v(t)\rangle_V \,\mathrm{d}t \to 0 \text{ for any } v \in L_2(0,T,V^*), \ \varepsilon_{n_k} \to 0.$$

In particular, by (1.1) one has

$$\int_0^T \langle u_{\varepsilon_{n_k}}(t), v(t)\rangle_H \,\mathrm{d}t \to \int_0^T \langle u_*(t), v(t)\rangle_H \,\mathrm{d}t \tag{1.51}$$

for each v in $L_2(0,T,H)$, $\varepsilon_{n_k} \to 0$. Thus, comparing (1.51) with (1.49), we see that

$$\int_0^T \langle u(t) - u_*(t), v(t)\rangle_H \,\mathrm{d}t = 0$$

for each v in $L_2(0,T,H)$, hence,

$$u = u_* \in L_2(0,T,V) \cap L_\infty(0,T,H).$$

2. A maximization problem. The worst scenario approach

2.1. Formulation of the problem.

We assume that the data in the problem $(\mathcal{B})$ depend on the input data $\mathcal{O}$ belonging to a compact subset $\mathcal{U}_{\rm ad}$ of a Banach space $\mathcal{U}$. We assume that the convex set of admissible states depends also on an input data parameter $\mathcal{O}$.

We consider the state problem

$$\left\{\begin{array}{l} u(t,\mathcal{O}) \in \mathcal{K}(\mathcal{O}) \ \text{ for a.e. } t \in [0,T], \\ \langle {\rm d}u(t,\mathcal{O})/{\rm d}t, v-u(t,\mathcal{O})\rangle_V + \langle A(t,\mathcal{O})u(t,\mathcal{O}), v-u(t,\mathcal{O})\rangle_V \\ \quad \geqslant \langle L(t,\mathcal{O}), v-u(t,\mathcal{O})\rangle_V \ \text{ for a.e. } t\in[0,T] \\ \quad \text{and for all } v \in \mathcal{K}(\mathcal{O}), \end{array}\right. \tag{2.1}$$

$$u(0,\mathcal{O}) = u_0(\mathcal{O}) \in \mathcal{K}(\mathcal{O}), \tag{2.2}$$

where $\mathcal{K}(\mathcal{O})$ is a closed convex subset of a Hilbert space V.

The maximization problem we consider here is (see the state problem (2.1))

$$\left\{\begin{array}{l} \text{Maximize } \Phi(\mathcal{O}, u(\mathcal{O})) \text{ with respect to } \mathcal{O} \in \mathcal{U}_{\rm ad}, \\ \text{where } u(\mathcal{O}) \in W_2^1([0,T],V) \text{ is the solution} \\ \text{of the state inequality in (2.1).} \end{array}\right. \tag{$\mathscr{P}$}$$

Here $\mathcal{U}_{\rm ad} \subset \mathcal{U}$ is compact and the criterion functional $\Phi(\mathcal{O}, u(\mathcal{O}))\colon \mathcal{U} \times W_2^1([0,T],V) \to \mathbb{R}$ is lower bounded and fulfils the assumption

$$\left\{\begin{array}{l} \text{If } v_n \to v \text{ weakly in } W_2^1([0,T],V) \text{ and } \mathcal{O}_n \to \mathcal{O} \text{ strongly in } \mathcal{U}, \\ \mathcal{O}_n \in \mathcal{U}_{\rm ad}, \text{ then one has } \limsup\limits_{n\to\infty} \Phi(\mathcal{O}_n, v_n) \leqslant \Phi(\mathcal{O}, v). \end{array}\right. \tag{E0}$$

In order to characterize the dependence $\mathcal{O} \to \mathcal{K}(\mathcal{O})$ we recall a special type of convergence of set sequences introduced in ([14]).

Definition 1. A sequence $\{K_n\}_{n\in\mathbb{N}}$ of subsets of a normed space W converges to a set $K \subset W$ if

$$\left\{\begin{array}{l} 1^\circ \ K \text{ contains all weak limits of the sequences } \{v_{n_k}\}_{k\in\mathbb{N}},\ v_{n_k} \in K_{n_k}, \\ \quad \text{where} \{K_{n_k}\}_{k\in\mathbb{N}} \text{ is an arbitrary subsequence of } \{K_n\}_{n\in\mathbb{N}}, \\ 2^\circ \text{ every element } v \in K \text{ is the strong limit of a sequence } \{v_n\}_{n\in\mathbb{N}}, \\ \quad v_n \in K_n. \end{array}\right. \tag{2.3}$$

N o t a t i o n. $K = \operatorname*{Lim}\limits_{n\to\infty} K_n$.

We introduce the system $\{\mathcal{K}(\mathcal{O})\}$ of convex closed subsets $\mathcal{K}(\mathcal{O}) \subset V$ and the family $\{A(t,\mathcal{O})\}_{\mathcal{O}\in\mathcal{U}_{\rm ad}}$ of linear bounded operators $A(\cdot,\mathcal{O}) \in C^1([0,T], L(V, V^*))$, $\mathcal{O} \in \mathcal{U}_{\rm ad}$, $t \in [0,T]$, the initial condition and functionals satisfying the following assumptions:

$$
\text{(H0)}\quad
\begin{cases}
1^\circ & \bigcap\limits_{\mathcal{O}\in\mathcal{U}_{\rm ad}} \mathcal{K}(\mathcal{O}) \neq \emptyset;\\
2^\circ & \mathcal{O}_n \to \mathcal{O} \text{ strongly in } \mathcal{U} \Rightarrow \mathcal{K}(\mathcal{O}) = \operatorname*{Lim}\limits_{n\to\infty} \mathcal{K}(\mathcal{O}_n);\\
3^\circ & \langle A(t,\mathcal{O})v, z\rangle_V = \langle A(t,\mathcal{O})z, v\rangle_V \text{ for all } v, z \in V,\\
 & t \in [0,T],\ \ \mathcal{O} \in \mathcal{U}_{\rm ad};\\
4^\circ & \|A(t,\mathcal{O})\|_{L(V,V^*)} \leqslant M_A \text{ for all } \mathcal{O} \in \mathcal{U}_{\rm ad} \text{ and } t \in [0,T];\\
5^\circ & \|{\rm d}A(t,\mathcal{O})/{\rm d}t\|_{L(V,V^*)} \leqslant \hat{M}_A \text{ for all } \mathcal{O} \in \mathcal{U}_{\rm ad} \text{ and } t \in [0,T];\\
6^\circ & \langle A(t,\mathcal{O})v, v\rangle_V \geqslant \alpha_A \|v\|_V^2,\ \ \alpha_A > 0 \text{ for all } v \in V,\\
 & t \in [0,T],\ \ \mathcal{O} \in \mathcal{U}_{\rm ad}\\
 & \text{(the real number } \alpha_A \text{ does not depend on } [\mathcal{O}, t] \text{ and } v;\\
 & A(t,\cdot) \text{ is said to be uniformly coercive with respect to } \mathcal{U}_{\rm ad});\\
7^\circ & \mathcal{O}_n \to \mathcal{O} \text{ strongly in } \mathcal{U} \Rightarrow A(\cdot,\mathcal{O}_n) \to A(\cdot,\mathcal{O})\\
 & \text{in } C^1([0,T], L(V,V^*));\\
8^\circ & u_0(\mathcal{O}_n) \to u_0(\mathcal{O}) \text{ strongly in } V \text{ if } \mathcal{O}_n \to \mathcal{O} \text{ strongly in } \mathcal{U},\\
 & \mathcal{O}_n \in \mathcal{U}_{\rm ad};\\
9^\circ & \|L(\cdot,\mathcal{O})\|_{W_2^1([0,T],V^*)} \leqslant M_L \text{ for all } \mathcal{O} \in \mathcal{U}_{\rm ad};\\
10^\circ & \{L(\cdot,\mathcal{O}_n)\}_{n\in\mathbb{N}} \text{ is a sequence in } C^1([0,T],V^*) \text{ such that}\\
 & L(\cdot,\mathcal{O}_n) \to L(\cdot,\mathcal{O}) \text{ in } C^1([0,T],V^*) \text{ as } \mathcal{O}_n \to \mathcal{O} \text{ strongly in } \mathcal{U};\\
11^\circ & A(0,\mathcal{O})u(\mathcal{O}) - L(0,\mathcal{O}) \in H \text{ for all } \mathcal{O} \in \mathcal{U}_{\rm ad};
\end{cases}
$$

where M_A, $\hat{M}_A$, M_L are constants independent of $\mathcal{O}$.

Theorem 3. *Let the assumptions* (H0) *and* (E0) *be satisfied. Then there exists at least one maximizer* $\mathcal{O}_* \in \mathcal{U}_{\rm ad}$ *of the optimal control problem* $(\mathcal{P})$.

Proof. According to Theorem 2, for every $\mathcal{O} \in \mathcal{U}_{\rm ad}$ there exists a unique solution $u(\mathcal{O}) \in W_\infty^1([0,T],H) \cap W_2^1([0,T],V)$ of the state initial value problem (2.1).

Let $\{\mathcal{O}_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\rm ad}$ be a maximizing sequence for the criterion functional $\Phi(\mathcal{O}, u(\mathcal{O}))$:

$$
\lim_{n\to\infty} \Phi(\mathcal{O}_n, u(\mathcal{O}_n)) = \sup_{\mathcal{O}\in\mathcal{U}_{\rm ad}} \Phi(\mathcal{O}, u(\mathcal{O})). \tag{2.4}
$$

Due to the compactness of the set $\mathscr{U}_{\rm ad}$ there exists an element $\mathcal{O}_* \in \mathscr{U}_{\rm ad}$ and a sequence $\{\mathcal{O}_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathcal{O}_{n_k} = \mathcal{O}_* \text{ in } \mathscr{U}. \tag{2.5}$$

Hence, the state problem (2.1) may be rewritten in the form

$$\begin{cases} u(t,\mathcal{O}_{n_k}) \in \mathcal{K}(\mathcal{O}_{n_k}), \\ \langle \mathrm{d}u(t,\mathcal{O}_{n_k})/\mathrm{d}t + A(t,\mathcal{O}_{n_k})u(t,\mathcal{O}_{n_k}), v - u(t,\mathcal{O}_{n_k})\rangle_V \\ \quad -\langle L(t,\mathcal{O}_{n_k}), v - u(t,\mathcal{O}_{n_k})\rangle_V \geqslant 0 \\ \quad \text{for all } v \in \mathcal{K}(\mathcal{O}_{n_k}) \text{ and for a.e. } t \in [0,T], \\ u(0,\mathcal{O}_{n_k}) = u_0(\mathcal{O}_{n_k}) \in \mathcal{K}(\mathcal{O}_{n_k}). \end{cases} \tag{2.6}$$

Due to the estimates $(1.17)_2$, (1.23), and (1.32) (Theorem 2), taking into account assumptions (H0), we see that

$$\begin{cases} \|u(\mathcal{O}_{n_k})\|_{W_2^1([0,T],V)} \leqslant M_{1,2}, \\ \|u(\mathcal{O}_{n_k})\|_{W_\infty^1([0,T],H)} \leqslant M_{1,\infty}, \end{cases} \tag{2.7}$$

where the constants $[M_{1,2}, M_{1,\infty}]$ involve only the constants $[\alpha_A, M_A, M_L, \hat{M}_A]$ from (H0) and the upper bound for the sequence $u_0(\mathcal{O}_{n_k})$. On the other hand, if we compare estimates $(1.17)_2$, (1.23), and (1.32) we can see that the constants $[M_{1,2}, M_{1,\infty}]$ do not depend on the sequence $\{\mathcal{K}(\mathcal{O}_{n_k})\}_{k\in\mathbb{N}}$. It follows by estimates (2.7) that there exists a function $u_* \in W_\infty^1([0,T],H) \cap W_2^1([0,T],V)$ and a subsequence of $\{\mathcal{O}_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that

$$\begin{cases} u(\mathcal{O}_{n_{k_j}}) \to u_* & \text{weakly in } W_2^1([0,T],V), \\ u(t,\mathcal{O}_{n_{k_j}}) \to u_*(t) & \text{weakly in } V \text{ for all } t \in [0,T], \end{cases} \tag{2.8}$$

and

$$\begin{cases} u(\mathcal{O}_{n_{k_j}}) \to u_* & \text{weakly star in } L_\infty(0,T,H), \\ \mathrm{d}u(\mathcal{O}_{n_{k_j}})/\mathrm{d}t \to \mathrm{d}u_*/\mathrm{d}t & \text{weakly star in } L_\infty(0,T,H). \end{cases} \tag{2.9}$$

On the other hand, we infer from relations (2.6), ((2.8), 2°), and assumption ((H0), 2°) that

$$u_*(t) \in \mathscr{K}(\mathcal{O}_*) \text{ for all } t \in [0,T]. \tag{2.10}$$

Next by virtue of the relations

$$\tag{2.11} \begin{cases} u(t,\mathcal{O}_{n_{k_j}}) = u_0(\mathcal{O}_{n_{k_j}}) + \int_0^t (\mathrm{d}u(\xi,\mathcal{O}_{n_{k_j}})/\mathrm{d}\xi)\,\mathrm{d}\xi, \\ u_*(t) = u_*(0) + \int_0^t (\mathrm{d}u_*(\xi)/\mathrm{d}\xi)\,\mathrm{d}\xi, \quad t\in[0,T], \end{cases}$$

we obtain, due to (2.8) and the assumption ((H0), 8°), that the initial condition satisfies

$$\tag{2.12} u_*(0) = u_0(\mathcal{O}_*) \in \mathcal{K}(\mathcal{O}_*).$$

Let $\kappa \in L_1([0,T],V)$ be an arbitrary function such that

$$\kappa(t) \in \mathcal{K}(\mathcal{O}_*) \text{ for a.e. } t\in[0,T].$$

The assumption ((H0), 2°) and Definition 1 imply the existence of a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that

$$v_k(t)\in\mathcal{K}(\mathcal{O}_{n_k}) \text{ for all } t\in[0,T],\ k\in\mathbb{N}, \text{ and } v_k(t)\to\kappa(t) \text{ strongly in } V$$

for a.e. $t\in[0,T]$.

On the other hand, since the sets $\mathcal{K}(\mathcal{O}_{n_k})$ are closed in the space V, we can use Lemma A.0 from ([5], App.), according to which for every (ε/k) $(\varepsilon>0)$ there exists a measurable function $v_k\colon [0,T]\to\mathcal{K}(\mathcal{O}_{n_k})$ with only a finite number of values and such that

$$\tag{2.13} \int_0^T \|\kappa(t)-v_k(t)\|_V\,\mathrm{d}t = (\varepsilon/k).$$

Then passing to the limit in (2.13), we obtain

$$\tag*{(2.13)$_1$} \lim_{k\to\infty}\|v_k-\kappa\|_{L_1(0,T,V)} = \lim_{k\to\infty}\int_0^T \|v_k(t)-\kappa(t)\|_V\,\mathrm{d}t = 0.$$

Furthermore, one has, for $t\in[0,T]$,

$$\tag{2.14} \begin{cases} A(t,\mathcal{O}_{n_{k_j}})v_j(t)\to A(t,\mathcal{O}_*)\omega(t) \quad \text{weakly in } V^*, \\ \langle A(t,\mathcal{O}_*)\omega(t),\omega(t)\rangle_V \leqslant \liminf\limits_{j\to\infty}\langle A(t,\mathcal{O}_{n_{k_j}})v_j(t),v_j(t)\rangle_V, \end{cases}$$

as $v_j(t)\to\omega(t)$ weakly in V and $\mathcal{O}_{n_{k_j}}\to\mathcal{O}_*$ strongly in $\mathscr{U}$.

Indeed, for any $\theta(t) \in V$ we can write

$$\begin{aligned}\lim_{j\to\infty} \langle A(t, \mathcal{O}_{n_{k_j}})v_j(t), \theta(t)\rangle_V &= \lim_{j\to\infty} \langle A(t, \mathcal{O}_{n_{k_j}})\theta(t), v_j(t)\rangle_V \\ &= \langle A(t, \mathcal{O}_*)\theta(t), \omega(t)\rangle_V \\ &= \langle A(t, \mathcal{O}_*)\omega(t), \theta(t)\rangle_V\end{aligned}$$

due to the assumptions ((H0) 3°, 7°).

Moreover, we have (in view of ((H0), 6°)

$$\langle A(t, \mathcal{O}_{n_{k_j}})(v_j(t) - \omega(t)), (v_j(t) - \omega(t))\rangle_V \geqslant 0.$$

Hence, we may write

$$\begin{aligned}&\lim_{j\to\infty} 2\langle A(t, \mathcal{O}_{n_{k_j}})\omega(t), v_j(t)\rangle_V \\ &\quad\leqslant \liminf_{j\to\infty} \langle A(t, \mathcal{O}_{n_{k_j}})v_j(t), v_j(t)\rangle_V + \lim_{j\to\infty} \langle A(t, \mathcal{O}_{n_{k_j}})\omega(t), \omega(t)\rangle_V.\end{aligned}$$

This yields ((2.14), 2°). By virtue of the inequality in (2.6) we get

$$\begin{aligned}\int_0^T \langle \mathrm{d}u(t, \mathcal{O}_{n_{k_j}})/\mathrm{d}t + A(t, \mathcal{O}_{n_{k_j}})u(t, \mathcal{O}_{n_{k_j}}) - L(t, \mathcal{O}_{n_{k_j}}),& \\ v_j(t) - u(t, \mathcal{O}_{n_{k_j}})\rangle_V \,\mathrm{d}t \geqslant 0.&\end{aligned} \tag{2.15}$$

The last inequality can be rewritten in the form

$$\begin{aligned}\|u(T, \mathcal{O}_{n_{k_j}})\|_H^2 &+ 2\int_0^T \langle A(t, \mathcal{O}_{n_{k_j}})u(t, \mathcal{O}_{n_{k_j}}), u(t, \mathcal{O}_{n_{k_j}})\rangle_V \,\mathrm{d}t \\ &\leqslant \|u(0, \mathcal{O}_{n_{k_j}})\|_H^2 + 2\int_0^T \langle A(t, \mathcal{O}_{n_{k_j}})u(t, \mathcal{O}_{n_{k_j}}), v_j(t)\rangle_V \,\mathrm{d}t \\ &\quad + 2\int_0^T \langle \mathrm{d}u(t, \mathcal{O}_{n_{k_j}})/\mathrm{d}t, v_j(t)\rangle_V \,\mathrm{d}t \\ &\quad + 2\int_0^T \langle L(t, \mathcal{O}_{n_{k_j}}), u(t, \mathcal{O}_{n_{k_j}}) - v_j(t)\rangle_V \,\mathrm{d}t.\end{aligned} \tag{2.16}$$

Thus by passing to the limit in (2.16), we have

$$\begin{aligned}\liminf_{j\to\infty} &\|u(T, \mathcal{O}_{n_{k_j}})\|_H^2 + 2\liminf_{j\to\infty} \int_0^T \langle A(t, \mathcal{O}_{n_{k_j}})u(t, \mathcal{O}_{n_{k_j}}), u(t, \mathcal{O}_{n_{k_j}})\rangle_V \,\mathrm{d}t \\ &\leqslant \lim_{j\to\infty} \|u_0(\mathcal{O}_{n_{k_j}})\|_H^2 + 2\lim_{j\to\infty} \int_0^T \langle A(t, \mathcal{O}_{n_{k_j}})u(t, \mathcal{O}_{n_{k_j}}), v_j(t)\rangle_V \,\mathrm{d}t \\ &\quad + 2\lim_{j\to\infty} \int_0^T \langle \mathrm{d}u(t, \mathcal{O}_{n_{k_j}})/\mathrm{d}t, v_j(t)\rangle_V \,\mathrm{d}t \\ &\quad + 2\lim_{j\to\infty} \int_0^T \langle L(t, \mathcal{O}_{n_{k_j}}), u(t, \mathcal{O}_{n_{k_j}}) - v_j(t)\rangle_V \,\mathrm{d}t,\end{aligned}$$

and hence (due to Definition 1, ((H0), 8°), (2.8), (2.9), Fatou lemma, (2.13), and (2.14)) we can write

$$\begin{aligned}(2.17)\qquad \|u_*(T)\|_H^2 &+ 2\int_0^T \langle A(t,\mathcal{O}_*)u_*(t), u_*(t)\rangle_V\,\mathrm{d}t \\ &\leqslant \|u_0(\mathcal{O}_*)\|_H^2 + 2\int_0^T \langle A(t,\mathcal{O}_*)u_*(t), \omega(t)\rangle_V\,\mathrm{d}t \\ &+ 2\int_0^T \langle \mathrm{d}u_*(t)/\mathrm{d}t, \omega(t)\rangle_V\,\mathrm{d}t \\ &+ 2\int_0^T \langle L(t,\mathcal{O}_*), u_*(t)-\omega(t)\rangle_V\,\mathrm{d}t.\end{aligned}$$

On the other hand, we infer from (2.17) that (using the initial condition in (2.12))

$$(2.18)\qquad \int_0^T \langle \mathrm{d}u_*(t)/\mathrm{d}t + A(t,\mathcal{O}_*)u_*(t) - L(t,\mathcal{O}_*), \omega(t)-u_*(t)\rangle_V\,\mathrm{d}t \geqslant 0$$

for all $\omega \in L_1(0,T,V)$ such that $\omega(t)\in\mathcal{K}(\mathcal{O}_*)$ for a.e. $t\in[0,T]$.

Then, inequality (2.18) takes the form

$$\begin{aligned}\langle \mathrm{d}u_*(t)/\mathrm{d}t, v(t)-u_*(t)\rangle_V &+ \langle A(t,\mathcal{O}_*)u_*(t), v(t)-u_*(t)\rangle_V \\ &\geqslant \langle L(t,\mathcal{O}_*), v(t)-u_*(t)\rangle_V\end{aligned}$$

for a.e. $t\in[0,T]$ and for all $v(t)\in\mathcal{K}(\mathcal{O}_*)$.

This inequality (together with (2.8), (2.9), and (2.12)) implies that (by the uniqueness of a solution of (2.1))

$$(2.19)\qquad \begin{cases} u_* = u(\mathcal{O}_*), \\ u(\mathcal{O}_n)\to u(\mathcal{O}_*) \text{ weakly in } W_2^1([0,T],V), \text{ and} \\ u(\mathcal{O}_n)\to u(\mathcal{O}_*) \text{ weakly star in } W_\infty^1(0,T,H).\end{cases}$$

The Aubin compactness criterion (see [1]) gives the result that for a subsequence $\{u(\mathcal{O}_{n_k})\}_{k\in\mathbb{N}}$ one has

$$u(\mathcal{O}_{n_k})\to u(\mathcal{O}_*) \text{ strongly in } L_2(0,T,H).$$

Finally, in view of the assumption (E0) and (2.4) we may write

$$\sup_{\mathcal{O}\in\mathscr{U}_{\mathrm{ad}}} \Phi(\mathcal{O},u(\mathcal{O})) = \lim_{n\to\infty}\Phi(\mathcal{O}_n,u(\mathcal{O}_n)) \leqslant \Phi(\mathcal{O}_*,u(\mathcal{O}_*)),$$

so that $\mathcal{O}_*$ is a maximizing element (maximizer $\mathcal{O}_*$ in $\mathscr{U}_{\mathrm{ad}}$).

3. Application to reliable solution of a Fourier obstacle problem

The parabolic obstacle problem occurs in the modelling of several heat transfer phenomena. Moreover, problems of this kind (optimal control problems in coefficients for systems described by parabolic equations) can be met in the technology of semiconductor devices and arise in mechanics and the theory of free boundary problems.

We start with notations. Let Ω denote an open bounded connected subset belonging to the three dimensional real space $\mathbb{R}^3$ ($\mathbf{x} = (x_1, x_2, x_3)$ is the generic point in $\mathbb{R}^3$) with the boundary $\partial\Omega = \partial\Omega_U \cup \partial\Omega_G$ (is the union of $\partial\Omega_U$ and $\partial\Omega_G$ such that $\operatorname{meas} \partial\Omega_U > 0$, $\operatorname{meas} \partial\Omega_G > 0$, $\operatorname{meas}(\partial\Omega_U \cap \partial\Omega_G) = 0$). Next, $(\mathbf{a}, \mathbf{b})_{\mathbb{R}^3}$ stands for the usual scalar product of $\mathbb{R}^3$, i.e., $(\mathbf{a}, \mathbf{b})_{\mathbb{R}^3} = \sum_{i=1}^{3} a_i b_i$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Moreover, we suppose that $\partial\Omega$ is sufficiently smooth (Lipschitz continuous, for example). By $H^1(\Omega)$ we denote the usual Sobolev space. For $v \in H^1(\Omega)$ the trace $\mathscr{M}_0 v (:= v|_{\partial\Omega})$ is well defined (the trace operator $\mathscr{M}_0$ is a linear continuous operator from $H_1(\Omega)$ to $L_2(\partial\Omega)$), and $H^m_\infty(\Omega)$ is the class of functions of $C^{m-1}(\overline{\Omega})$ whose derivatives of order $(m-1)$ satisfy a Lipschitz condition on $\overline{\Omega}$.

Assume that the coefficients of the differential operator of the second order (depending on the control e) and an obstacle $\mathscr{S}(\mathbf{x})$ are given with some uncertainty. To simplify notation they are denoted as a vector $\mathcal{O} \equiv [e, \mathscr{S}]^T \in \mathscr{U}(\Omega)$, where $\mathscr{U}(\Omega) = C(\overline{\Omega}) \times C(\overline{\Omega})$.

Moreover, we have

$$\begin{cases} \mathscr{U}_{\text{ad}}(\Omega) := \mathscr{U}^e_{\text{ad}}(\Omega) \times \mathscr{U}^{\mathscr{S}}_{\text{ad}}(\Omega), \quad \text{where} \\ \mathscr{U}^e_{\text{ad}}(\Omega) = \{e \in H^1_\infty(\Omega) \colon\ 0 < e_{\min} \leqslant e \leqslant e_{\max}, |\partial e_{\min}/\partial x_i| \leqslant \text{const}_{\langle i \rangle}, \\ \qquad\qquad i = 1, 2, 3\}. \end{cases}$$

We note that $\mathscr{U}^e_{\text{ad}}(\Omega)$ is clearly compact in the topology of $C(\overline{\Omega})$. Set

$$\begin{aligned} \mathscr{U}^{\mathscr{S}}_{\text{ad}}(\Omega) = \{\mathscr{S} \in H^1_\infty(\Omega) \colon\ & \text{const}_{1\mathscr{S}} \leqslant \mathscr{S}(\mathbf{x}) \leqslant \text{const}_{2\mathscr{S}} \\ & \text{for all } [\mathbf{x}] \in \Omega,\ |\partial\mathscr{S}/\partial x_i| \leqslant \text{const}_{\langle i \rangle},\ i = 1, 2, 3,\ \mathscr{S} \leqslant -c_p \text{ on } \partial\Omega_U \\ & \text{where } c_p = \text{const} > 0\}. \end{aligned}$$

We note that the constants involved are positive so that $\mathscr{U}^e_{\text{ad}}(\Omega)$ and $\mathscr{U}^{\mathscr{S}}_{\text{ad}}(\Omega)$ are nonempty.

For an arbitrary fixed $\mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega)$, let the control system be given by the solution of a nonlinear parabolic value problem (in the general form)

$$
(3.1)\qquad \begin{cases} {\rm d}u(\mathcal{O})/{\rm d}t + \mathscr{R}(e)u(\mathcal{O}) \geqslant L & \text{in } (0,T)\times\Omega,\\ u(\mathcal{O}) = 0 & \text{on } [0,T]\times\partial\Omega_U,\\ ([\mathcal{A}(e)]\,\mathbf{grad}\,u(\mathcal{O}),\mathbf{n})_{\mathbb{R}^3} + \omega u(\mathcal{O}) = G & \text{on } [0,T]\times\partial\Omega_G,\\ ({\rm d}u(\mathcal{O})/{\rm d}t + \mathscr{R}(e)u(\mathcal{O}) - L)(u(\mathcal{O}) - \mathscr{S}) = 0 & \text{in } (0,T)\times\Omega,\\ u(\mathcal{O}) \geqslant \mathscr{S} & \text{in } [0,T]\times\Omega,\\ u(0,\mathcal{O}) = u_0(\mathcal{O}) \geqslant \mathscr{S} & \text{in } \Omega. \end{cases}
$$

Here

$$
\mathscr{R}(e)u(\mathcal{O}) := -\,{\rm div}([\mathcal{A}(e)]\,\mathbf{grad}\,u(\mathcal{O})) + a_0(e)u(\mathcal{O})
$$

where $G(\mathbf{x})$ and $\omega(\mathbf{x})$ are given functions defined on $\partial\Omega$ with $G \in L_2(\partial\Omega_G)$, $\omega(\mathbf{x}) \in C(\partial\overline{\Omega})$, $\omega(\mathbf{x}) \geqslant 0$ on $\partial\Omega_G$, while the function $\mathscr{S}$ (the control variable) represents the obstacle; $\mathbf{n}$ is the outward unit vector normal at $\partial\Omega$, $\mathbf{grad}$ denotes the vector $\{\partial/\partial x_i\}_{i=1}^3$, $[\mathcal{A}(t,e)] = [\mathcal{A}([t,\cdot],e(\cdot))] = [a_{ij}([t,\cdot],e(\cdot))]$ denotes a $[3\times 3]$-matrix (the system of linear operators from $\mathbb{R}^3$ to $\mathbb{R}^3$ for any $t \in [0,T]$ depending on $\mathbf{x}$ over Ω, $a_0(t,e) = a_0([t,\cdot],e(\cdot))$, $t \in [0,T]$, is a scalar function and $e \in \mathscr{U}^{\rm e}_{\rm ad}(\Omega)$. The set of relations (3.1) will be referred to as a differential inequality.

In the following we assume that $[\mathcal{A}([t,\mathbf{x}],e)]$, $a_0([t,\mathbf{x}],e)$ are defined on $([0,T], \Omega)\times[e_{\min},e_{\max}]$ and satisfy the conditions

$$
({\rm A2})\qquad \begin{cases} 1^\circ & a_{ij}([\cdot,\mathbf{x}],h), a_0([\cdot,\mathbf{x}],h) \in C^1([0,T]) \text{ for a.e. } \mathbf{x}\in\Omega\\ & \text{and for any } h \in [e_{\min},e_{\max}],\\ & \text{where } a_0([t,\cdot],h) \geqslant \alpha_0 \quad \text{a.e on } \Omega,\ t\in[0,T],\\ & \alpha_0 = \text{const} \geqslant 0;\\ 2^\circ & a_{ij}([t,\cdot],h),\ {\rm d}a_{ij}([t,\cdot],h)/{\rm d}t,\ a_0([t,\cdot],h),\ {\rm d}a_0([t,\cdot],h)/{\rm d}t\\ & \text{are continuous functions on } \overline{\Omega} \text{ for every } h\in[e_{\min},e_{\max}],\ t\in[0,T],\\ & \text{and } a_{ij}([t,\mathbf{x}],\cdot),\ {\rm d}a_{ij}([t,\mathbf{x}],\cdot)/{\rm d}t,\ a_0([t,\mathbf{x}],\cdot),\\ & {\rm d}a_0([t,\mathbf{x}],\cdot)/{\rm d}t \text{ are continuous on } [e_{\min},e_{\max}]\\ & \text{for every } [t,\mathbf{x}]\in[0,T]\times\Omega;\\ 3^\circ & \text{the ellipticity condition: } ([\mathcal{A}([t,\mathbf{x}],e)]\boldsymbol{\xi},\boldsymbol{\xi})_{\mathbb{R}^3} \geqslant \alpha_*|\boldsymbol{\xi}|^2_{\mathbb{R}^3}\\ & \text{for any } \boldsymbol{\xi}\in\mathbb{R}^3,\ \text{for any } e\in\mathscr{U}^{\rm e}_{\rm ad}(\Omega) \text{ and for every}\\ & [t,\mathbf{x}]\in[0,T]\times\Omega,\ \text{where } \alpha_* = \text{const} > 0;\\ 4^\circ & a_{ij}([t,\mathbf{x}],h) = a_{ji}([t,\mathbf{x}],h). \end{cases}
$$

On the right-hand side of (3.1), $L(t)$ denotes a fixed functional defined below in (3.6) for any $t \in [0,T]$.

We shall employ the method of reliable solutions alias the worst scenario method (see [8], [9]), which consists of the following main steps:

$$(3.2)\qquad \begin{cases} 1^\circ & \text{choose a functional criterion } [\mathcal{O},u] \to \Phi(\mathcal{O},u), \\ 2^\circ & \text{solve the maximization problem: } \mathcal{O}_* = \operatorname*{Arg\,Max}_{\mathcal{O}\in\mathscr{U}_{\rm ad}(\Omega)} \Phi(\mathcal{O},u(\mathcal{O})), \end{cases}$$

where $u(\mathcal{O})$ denotes the (unique) solution of the state problem (3.1) for the input data $\mathcal{O}$.

Consider the following criterion Φ. Let us choose $0 < t < T$, subdomains $G_j \subset \Omega$, $j = 1,2,\ldots \mathcal{R}$, and define

$$\begin{cases} \mathscr{N}_j(v) = (\operatorname{meas} G_j)^{-1} \int_{G_j} v(t)\,\mathrm{d}\Omega, \\ \Phi(v) = \operatorname{Max}_{j\leqslant\mathcal{R}} \mathscr{N}_j(v), \end{cases}$$

where $\mathscr{N}_j(v)$ is the mean value of v over a given subdomain $G_j \subset \Omega$ or $G_j \subset \partial\Omega$, $\mathcal{R}$ is a positive integer.

We shall refer to ((3.2), 2°) as to the maximization problem $(\mathscr{P})$. Preparing our treatment we deal with the state inequality (3.1) for an arbitrary fixed $\mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega)$. Because of the above space assumptions we have to work in the framework of the space $W_2^1([0,T],V)$, where $V \subset H$, $V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \partial\Omega_U\}$, and $H = L_2(\Omega)$. This means that $u(\mathcal{O}) \in W_2^1([0,T],V)$ is a solution of (3.1) if and only if $u(\mathcal{O})$ is a solution of the symmetric operator equation

$$(3.3)\qquad \begin{cases} \mathrm{d}u(\mathcal{O})/\mathrm{d}t + A(e)u(\mathcal{O}) \geqslant L \text{ in } (0,T)\times\Omega \text{ (in the sense of distribution)} \\ \qquad \text{and} \\ u(\mathcal{O}) = 0 \text{ on } [0,T]\times\partial\Omega_U, \\ ([\mathcal{A}(e)]\,\mathbf{grad}\,u(\mathcal{O}),\mathbf{n})_{\mathbb{R}^3} + \omega u(\mathcal{O}) = G \text{ on } [0,T]\times\partial\Omega_G, \\ (\mathrm{d}u(\mathcal{O})/\mathrm{d}t + A(e)u(\mathcal{O}) - L)(u(\mathcal{O}) - \mathscr{S}) = 0 \text{ in } (0,T)\times\Omega, \\ u(\mathcal{O}) \geqslant \mathscr{S} \text{ in } [0,T]\times\Omega, \\ u(0,\mathcal{O}) = u_0(\mathcal{O}) \geqslant \mathscr{S} \text{ in } \Omega, \end{cases}$$

where $A(t,e)$ (the symmetric linear operator) is a bounded operator: $A(\cdot,e)\colon [0,T] \to L(V,V^*)$. It is defined by the identity

$$(3.4)\qquad \langle A(t,e)v,z\rangle_V := a([t,e]v,z) \text{ for any } v,z \in V \text{ and for any } t \in [0,T],\ e \in \mathscr{U}^{\rm e}_{\rm ad}(\Omega),$$

where we define the symmetric bilinear form $a([t,e]\cdot,\cdot)\colon\ V\times V\to\mathbb{R}$ for all $e\in\mathscr{U}^e_{\rm ad}(\Omega)$ and for any $t\in[0,T]$ by the relation

$$(3.5)\qquad a([t,e]v,z):=\int_\Omega\{([\mathcal{A}([t,\mathbf{x}],e)]\,\mathbf{grad}\,v,\mathbf{grad}\,z)_{\mathbb{R}^3}+a_0([t,\mathbf{x}],e)vz\}\,\mathrm{d}\Omega + \int_{\partial\Omega_G}\omega(\mathscr{M}_0 v)(\mathscr{M}_0 z)\,\mathrm{d}S.$$

Let us define a subset of V

$$(\mathrm{A3})\qquad D(A(t,e))=\{v\in V\colon\ z\to a([t,e]v,z)\ \text{is continuous on}\ V\ \text{in the topology of}\ H\}.$$

The right-hand side is an element belonging to V^* (for any $t\in[0,T]$), given by

$$(3.6)\qquad \langle L(t),v\rangle_{V(\Omega)}=\int_\Omega f(t)v\,\mathrm{d}\Omega+\int_{\partial\Omega_G}G\mathscr{M}_0 v\,\mathrm{d}S$$

for any $v\in V,\ f(t)\in L_2(\Omega),\ G\in L_2(\partial\Omega_G)$.

Moreover, we introduce a set of admissible state functions by

$$\mathcal{K}(\mathscr{S},\Omega):=\{v\in V\colon\ v(\mathbf{x})\geqslant\mathscr{S}(\mathbf{x})\ \text{for a.e.}\ \mathbf{x}\in\Omega\}.$$

Now we define

$$|[\mathscr{A}(t,e)](\mathbf{x})|=\sup_{\xi\in\mathbb{R}^3\setminus\{0\}}|[\mathcal{A}([t,\mathbf{x}],e)]\boldsymbol{\xi}|/|\boldsymbol{\xi}|_{\mathbb{R}^3}\ \text{for any}\ t\in[0,T].$$

Then by virtue of $((\mathrm{A2}),1^\circ,\,2^\circ)$ we easily find that the function: $\mathbf{x}\to|[\mathscr{A}(t,e)](\mathbf{x})|$ belongs to $L_\infty(\Omega)$ for all $t\in[0,T]$ and $e\in\mathscr{U}^e_{\rm ad}(\Omega)$.

Lemma 1. *The family* $\{A(t,e)\}$, $t\in[0,T]$, $e\in\mathscr{U}^e_{\rm ad}(\Omega)$ *of operators defined by* (3.4) *and* (3.5) *satisfies the assumptions* $((\mathrm{H0}),3^\circ$ *to* $7^\circ)$.

P r o o f. From the above hypotheses, using the continuity of the trace operator $\mathscr{M}_0\colon\ H^1(\Omega)\to L_2(\partial\Omega_G)$, we deduce that

$$(3.7)\qquad |\langle A(t,e)v,z)\rangle_V|\leqslant\max\big[\|[\mathcal{A}(t,e)]\|_{L_\infty(\Omega)},\|a_0(t,e)\|_{L_\infty(\Omega)},\ \mathrm{const}(\Omega)\|\omega\|_{L_\infty(\partial\Omega_G)}\big]\,\|v\|_V\,\|z\|_V$$

with some positive const(Ω) for all $t\in[0,T]$ and all $v,z\in V$, $e\in\mathscr{U}^e_{\rm ad}(\Omega)$ (it is a simple application of the Schwarz inequality; since $[\mathcal{A}(t,e)]\in L_\infty(\Omega)$ and $\omega\in L_\infty(\partial\Omega_G)$

for all $t \in [0,T]$, $e \in \mathscr{U}^e_{\rm ad}(\Omega)$). The same estimate (due to ((A2), 2°)) can be obtained for the operator $\mathrm{d}A/\mathrm{d}t$. Consequently, assumptions ((H0), 4°, 5°) are fulfilled. On the other hand, for the bilinear form (3.5) we have (due to ((A2), 1° to 3°))

(3.8) $\langle A(t,e)v, v\rangle_V \geqslant \min[\alpha_*, \alpha_0]\,\|v\|_V^2$ for all $t \in [0,T]$ and $v \in V, e \in \mathscr{U}^e_{\rm ad}(\Omega)$.

The condition ((H0), 6°) is verified.

Let $e_n \to e$ strongly in $\mathscr{U}^e(\Omega)$ for $n \to \infty$, $e_n \in \mathscr{U}^e_{\rm ad}(\Omega)$. Then one has

$$
\begin{aligned}
(3.9) \qquad & \big|\langle A(t,e_n)v,z\rangle_V - \langle A(t,e)v,z\rangle_V\big| \\
&\leqslant \int_\Omega \big|(([\mathcal{A}(t,e_n)] - [\mathcal{A}(t,e)])\,\mathbf{grad}\,v, \mathbf{grad}\,z)_{\mathbb{R}^3}\big|\,\mathrm{d}\Omega \\
&\quad + \int_\Omega |(a_0(t,e_n) - a_0(t,e))vz|\,\mathrm{d}\Omega \\
&\leqslant \max_{i,j}\max_{\mathbf{x}\in\Omega}\max_{t\in[0,T]} |a_{ij}([t,\mathbf{x}],e_n) - a_{ij}([t,\mathbf{x}],e)| \\
&\quad \times \int_\Omega 3|\,\mathbf{grad}\,v|_{\mathbb{R}^3}|\,\mathbf{grad}\,z|_{\mathbb{R}^3}\,\mathrm{d}\Omega \\
&\quad + \max_{\mathbf{x}\in\overline{\Omega}}\max_{t\in[0,T]} |a_0([t,\mathbf{x}],e_n) - a_0([t,\mathbf{x}],e)| \int_\Omega |vz|\,\mathrm{d}\Omega.
\end{aligned}
$$

Furthermore, by the hypotheses ((A2), 1°, 2°) and due to (3.9), an application of Theorem 3.10 ([12]) yields that

$$\lim_{n\to\infty}\max_{i,j}\max_{\mathbf{x}\in\Omega}\max_{t\in[0,T]} |a_{ij}([t,\mathbf{x}],e_n) - a_{ij}([t,\mathbf{x}],e)| = 0$$

and

$$\lim_{n\to\infty}\max_{\mathbf{x}\in\Omega}\max_{t\in[0,T]} |a_0([t,\mathbf{x}],e_n) - a_0([t,\mathbf{x}],e)| = 0.$$

This completes the proof of Lemma 1. □

Lemma 2. *For any $\mathscr{S} \in \mathscr{U}^{\mathscr{S}}_{\rm ad}(\Omega)$ the set $\mathcal{K}(\mathscr{S},\Omega)$ is a closed and convex subset of V and*

$$\{\mathscr{S}_n \to \mathscr{S} \text{ strongly in } C(\overline{\Omega});\ \ \mathscr{S}_n \in \mathscr{U}^{\mathscr{S}}_{\rm ad}(\Omega)\} \Rightarrow \mathcal{K}(\mathscr{S},\Omega) = \operatorname*{Lim}_{n\to\infty} \mathcal{K}(\mathscr{S}_n,\Omega).$$

P r o o f. Let us define for a fixed $v \in V$

$$\Omega_1 = \{\mathbf{x} \in \Omega\colon\ v(\mathbf{x}) - \mathscr{S}(\mathbf{x}) \leqslant c_p/2\}.$$

Since $v - \mathscr{S} \geqslant c_p$ holds on $\partial\Omega_U$, there exists a function $\varrho \in C^\infty(\overline{\Omega}) \cap V$ such that $0 \leqslant \varrho$ in Ω and $\varrho \geqslant 1$ in Ω_1. For any $v \in \mathcal{K}(\mathscr{S}, \Omega)$ we construct a sequence $v_n = v + \varrho\|\mathscr{S}_n - \mathscr{S}\|_{C(\overline{\Omega})}$. Then $v_n \in V$ and $v_n \geqslant \mathscr{S}_n$ a.e. in Ω. In fact, in Ω_1 we have

$$v_n - \mathscr{S}_n \geqslant v - \mathscr{S} + (\mathscr{S} - \mathscr{S}_n) + \|\mathscr{S}_n - \mathscr{S}\|_{C(\overline{\Omega})} \geqslant 0, \quad \text{for all } n,$$

whereas in $\Omega \backslash \Omega_1$

$$v_n - \mathscr{S}_n \geqslant v - \mathscr{S} + (\mathscr{S} - \mathscr{S}_n) > c_p/2 - \|\mathscr{S} - \mathscr{S}_n\|_{C(\overline{\Omega})} \geqslant 0$$

holds for n sufficiently great. Moreover,

$$\|v_n - v\|_V = \|\mathscr{S}_n - \mathscr{S}\|_{C(\overline{\Omega})}\|v\|_V \to 0.$$

Next, let $v_n \in \mathcal{K}(\mathscr{S}_n, \Omega)$, $v_n \to v$ weakly in V. Then $v \in V$ as V is weakly closed in $H^1(\Omega)$ and $v_n \to v$ in $L_2(\Omega)$ strongly due to the Rellich Theorem, $v_n \geqslant \mathscr{S}_n$ a.e. in Ω. From the Lebesgue Theorem, $v \geqslant \mathscr{S}$ follows a.e., so that $v \in \mathcal{K}(\mathscr{S}, \Omega)$.

Let the initial function $u_0(\mathcal{O}) \in \mathcal{K}(\mathscr{S}, \Omega)$ for $\mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega)$ be a solution of the elliptic variational inequality

$$\langle A(0, e)u_0(\mathcal{O}), v - u_0(\mathcal{O})\rangle_V \geqslant \langle L(0), v - u_0(\mathcal{O})\rangle_V \tag{3.10}$$

for all $v \in \mathcal{K}(\mathscr{S}, \Omega)$.

Let $\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathcal{O}_n \in \mathscr{U}_{\rm ad}(\Omega)$, be a sequence such that

$$\mathcal{O}_n \to \mathcal{O} \quad \text{strongly in } \mathscr{U}(\Omega). \tag{3.11}$$

Consider the variational inequality

$$\begin{cases} u_0(\mathcal{O}_n) \in \mathcal{K}(\mathscr{S}_n, \Omega), \\ \langle A(0, e_n)u_0(\mathcal{O}_n), v - u_0(\mathcal{O}_n)\rangle_V \geqslant \langle L(0), v - u_0(\mathcal{O}_n)\rangle_V \end{cases} \tag{3.12}$$

for all $v \in \mathcal{K}(\mathscr{S}_n, \Omega)$. Then due to (3.6), (3.7) and (3.8) and since the sequence $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ is bounded in $C(\overline{\Omega})$ we obtain an estimate

$$\|u_0(\mathcal{O}_n)\|_V \leqslant \text{const} \tag{3.13}$$

with a constant independent of n.

The sequence $\{u_0(\mathcal{O}_n)\}_{n\in\mathbb{N}}$ is bounded in V, hence there exists an element $u_{0\langle * \rangle}$ and a subsequence $\{u_0(\mathcal{O}_{n_k})\}_{k\in\mathbb{N}}$ such that

$$u_0(\mathcal{O}_{n_k}) \to u_{0\langle * \rangle} \text{ weakly in } V. \tag{3.14}$$

Moreover, as $u_0(\mathcal{O}_{n_k}) \in \mathcal{K}(\mathscr{S}_{n_k}, \Omega)$ due to Lemma 2 and Definition 1 we have

$$u_{0\langle * \rangle} \in \mathcal{K}(\mathscr{S}, \Omega). \tag{3.15}$$

By virtue of (3.7) and (3.13) we obtain

$$\|A(0, e_{n_k}) u_0(\mathcal{O}_{n_k})\|_{V^*} \leqslant M_0 \quad \text{for all } k.$$

Consequently, there exists an element $\mathcal{X} \in V^*$ and a subsequence

$$\{A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}})\}_{m \in \mathbb{N}}$$

such that

$$A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}) \to \mathcal{X} \quad \text{weakly in } V^*. \tag{3.16}$$

By assumption ((H0), 2°) there exists a sequence $\{a_m\}_{m \in \mathbb{N}}$, $a_m \in \mathcal{K}(\mathcal{O}_m, \Omega)$, such that $a_m \to u_{0\langle * \rangle}$ strongly in V.

Henceforth, we will often use the following implication: $\omega_n \to \omega$ weakly in V^*, $z_n \to z$ strongly in $V \Rightarrow \langle \omega_n, z_n \rangle_V \to \langle \omega, z \rangle_V$.

Now, let us take $v := a_m$ in (3.12). We may write (by virtue of (3.16))

$$\begin{aligned} \limsup_{m \to \infty} \langle A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) \rangle_V &\leqslant \limsup_{m \to \infty} \langle A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}), a_m \rangle_V \\ &\quad + \limsup_{m \to \infty} \langle L(0), u_0(\mathcal{O}_{n_{k_m}}) - a_m \rangle_V \\ &= \langle \mathcal{X}, u_{0\langle * \rangle} \rangle_V. \end{aligned} \tag{3.17}$$

Next, due to the assumption((A2), 3°), we have

$$\langle A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}) - A(0, e_{n_{k_m}}) v, u_0(\mathcal{O}_{n_{k_m}}) - v \rangle_V \geqslant 0 \quad \text{for all } v \in V. \tag{3.18}$$

Taking into account (3.14), (3.16), and (3.17), we derive

$$\langle \mathcal{X} - A(0, e) v, u_{0\langle * \rangle} - v \rangle_V \geqslant 0 \quad \text{for all } v \in V. \tag{3.19}$$

In fact, on the basis of (3.18) we may write

$$\begin{aligned} \limsup_{m \to \infty} \langle A(0, e_{n_{k_m}}) v, u_0(\mathcal{O}_{n_{k_m}}) - v \rangle_V &\leqslant \limsup_{m \to \infty} \langle A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) \rangle_V \\ &\quad + \limsup_{m \to \infty} \langle A(0, e_{n_{k_m}}) u_0(\mathcal{O}_{n_{k_m}}), -v \rangle_V \\ &= \langle \mathcal{X}, u_{0\langle * \rangle} \rangle_V + \langle \mathcal{X}, -v \rangle_V. \end{aligned}$$

This means that (3.19) follows from Lemma 1, ((H0), 7°) and (3.18), (3.14).

In (3.19) we set $v = u_{0\langle * \rangle} + \vartheta(w - u_{0\langle * \rangle})$, $\vartheta \in (0,1)$, and $w \in V$, and we get

$$\langle \mathcal{X} - A(0,e)(u_{0\langle * \rangle} + \vartheta(w - u_{0\langle * \rangle})), u_{0\langle * \rangle} - w\rangle_V \geqslant 0 \quad \text{for all } w \in V, \\ 0 < \vartheta < 1.$$

However, due to (3.7), if we set again $w = v$, we arrive at

$$\langle A(0,e)u_{0\langle * \rangle}, u_{0\langle * \rangle} - v\rangle_V \leqslant \langle \mathcal{X}, u_{0\langle * \rangle} - v\rangle_V \quad \text{for all } v \in V. \tag{3.20}$$

Next, substituting $v := u_{0\langle * \rangle}$ in (3.18), we obtain

$$\langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V \\ \geqslant \langle A(0,e_{n_{k_m}})u_{0\langle * \rangle}, u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V.$$

On the other hand, due to the relations (3.9) and (3.14), we have

$$\lim_{m\to\infty} \langle A(0,e_{n_{k_m}})u_{0\langle * \rangle}, u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V = 0$$

so that

$$\liminf_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V \geqslant 0. \tag{3.21}$$

Hence, combining (3.21) with the inequality

$$\limsup_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V \\ \leqslant \limsup_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}})\rangle_V \\ + \lim_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), -u_{0\langle * \rangle}\rangle_V \leqslant 0$$

which is a consequence of (3.17) and (3.16), we are led to the equation

$$\lim_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) - u_{0\langle * \rangle}\rangle_V = 0. \tag{3.22}$$

Given a $v \in \mathcal{K}(\mathscr{S}, \Omega)$, by Lemma 2 there exists a sequence $\{v_m\}_{m\in\mathbb{N}}$, $v_m \in \mathcal{K}(\mathscr{S}_m, \Omega)$, $v_m \to v$ strongly in V. Then, setting $v = v_m$ in (3.12) we may write

$$\lim_{m\to\infty} \langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}), u_0(\mathcal{O}_{n_{k_m}}) - v_m\rangle_V \\ \leqslant \lim_{m\to\infty} \langle L(0), u_0(\mathcal{O}_{n_{k_m}}) - v_m\rangle_V = \langle L(0), u_{0\langle * \rangle} - v\rangle_V.$$

The limit on the left-hand side exists and can be bounded below, since we can write

$$\begin{aligned}\lim_{m\to\infty}&\langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}),u_0(\mathcal{O}_{n_{k_m}})-u_{0\langle *\rangle}\rangle_V\\&+\lim_{m\to\infty}\langle A(0,e_{n_{k_m}})u_0(\mathcal{O}_{n_{k_m}}),u_{0\langle *\rangle}-v_m\rangle_V\\&=\langle \mathcal{X},u_{0\langle *\rangle}-v\rangle_V\geqslant\langle A(0,e)u_{0\langle *\rangle},u_{0\langle *\rangle}-v\rangle_V,\end{aligned}$$

where (3.22), (3.16), and (3.20) have been employed.

This leads (due to the continuity of the trace operator) to the inequality

$$\tag{3.23}\begin{cases}u_{0\langle *\rangle}\in\mathcal{K}(\mathscr{S},\Omega),\\\langle A(0,e)u_{0\langle *\rangle},v-u_{0\langle *\rangle}\rangle_V\geqslant\langle L(0),v-u_{0\langle *\rangle}\rangle_V\\\quad\text{for any }\ v\in\mathcal{K}(\mathscr{S},\Omega).\end{cases}$$

Hence, we deduce that $u_{0\langle *\rangle}=u_0(\mathcal{O})$ (since the element $v\in\mathcal{K}(\mathscr{S},\Omega)$ is chosen arbitrarily and the solution of the state problem (3.10) is unique) and we may write

$$\tag{3.24}u_0(\mathcal{O}_n)\to u_0(\mathcal{O})\quad\text{weakly in }\ V.$$

By virtue of Lemma 2 there exists a sequence $\{\theta_n\}_{n\in\mathbb{N}}$, $\theta_n\in\mathcal{K}(\mathscr{S}_n,\Omega)$, such that $\theta_n\to u_0(\mathcal{O})$ strongly in V. Inserting $v:=\theta_n$ in (3.12) and adding $\langle A(0,e_n)(u_0(\mathcal{O}_n)-\theta_n),u_0(\mathcal{O}_n)-\theta_n\rangle_V$ to both its sides, we obtain

$$\tag{3.25}\begin{aligned}\limsup_{n\to\infty}&\langle A(0,e_n)u_0(\mathcal{O}_n)-A(0,e_n)\theta_n,u_0(\mathcal{O}_n)-\theta_n\rangle_V\\&\leqslant\limsup_{n\to\infty}|\langle A(0,e_n)\theta_n,\theta_n-u_0(\mathcal{O}_n)\rangle_V|\\&\quad+\limsup_{m\to\infty}|\langle L(0),\theta_n-u_0(\mathcal{O}_n)\rangle_V|=0.\end{aligned}$$

We note that the last inequality follows from the implication

$$\begin{gathered}e_n\in\mathscr{U}^{\mathrm{e}}_{\mathrm{ad}}(\Omega),\ \ e_n\to e\ \text{ strongly in }\ \mathscr{U}^{\mathrm{e}}(\Omega)\ \text{ and }\ v_n\to v\ \text{ strongly in }\ V\\\text{for }\ n\to\infty\Rightarrow\|\langle A(0,e_n)v_n-A(0,e_n)v\|_{V^*}\leqslant M_A\|v_n-v\|_V\to 0,\end{gathered}$$

which is a consequence of $((\mathrm{H0}),4^\circ)$. On the other hand, due to the uniform monotonicity of $[A(0,e_n)]$ by $((\mathrm{H0}),6^\circ)$ and due to (3.25) we obtain the strong convergence $u_0(\mathcal{O}_n)\to u_0(\mathcal{O})$ strongly in V for $n\to\infty$. This means that the assumption $((\mathrm{H0}),8^\circ)$ is verified. The operator $A(0,e)$ is defined by (3.4) at $t=0$. Then $u_0(\mathcal{O})\in D(A([0,\cdot]))$ is equivalent (due to the assumption (A3)) to $u_0(\mathcal{O})\in V$, $A([0,\cdot])u_0(\mathcal{O})\in H$ and $\int_\Omega A([0,\cdot])u_0(\mathcal{O})v\,\mathrm{d}\Omega=a([0,\cdot]u_0(\mathcal{O}),v)$ for any $v\in V$. On the other hand, in view of (3.6) (for $G=0$ on $\partial\Omega_G$) one has $L(0)\in H$. Hence, we have $A([0,\cdot])u_0(\mathcal{O}) - L(0) \in H$ for all $\mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega)$. This means that the condition $((\mathrm{H0}), 11^\circ)$ is verified. Now if we combine the above arguments we may conclude that all the assumptions (H0) are satisfied.

Let us consider the criterion Φ of the form (3.2). We shall verify the assumption (E0).

Let $v_n \to v$ strongly in $C([0,T],H)$, then $v_n(t) \to v(t)$ strongly in $L_2(\Omega)$ and we may write

$$
\begin{aligned}
|\mathscr{N}_j(v_n) - \mathscr{N}_j(v)| &= (\operatorname{meas} G_j)^{-1} \int_{G_j} (v_n(t) - v(t))\,\mathrm{d}\Omega \\
&\leqslant \text{const}\, \|v_n(t) - v(t)\|_{L_2(\Omega)} \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}
$$

Then one has

$$
\lim_{n\to\infty} \Phi(v_n) = \lim_{n\to\infty} \operatorname*{Max}_j \mathscr{N}_j(v_n) = \operatorname*{Max}_j \lim_{n\to\infty} \mathscr{N}_j(v_n) = \operatorname*{Max}_j \mathscr{N}_j(v) = \Phi(v).
$$

Now, we are able to define the main task for the maximization problem for a parabolic inequality. Find

$$
(\mathscr{P}_{\rm heat}) \qquad \mathcal{O}_* = \operatorname*{Arg\,Max}_{\mathcal{O}\in\mathscr{U}_{\rm ad}(\Omega)} \Phi(\mathcal{O}, u(\mathcal{O})),
$$

where $u(\mathcal{O}) \in W_2^1([0,T],V)$ denotes the solution of the State Problem

$$
(3.26) \qquad \begin{cases} u(t,\mathcal{O}) \in \mathcal{K}(\mathscr{S},\Omega), \ \ t \in [0,T], \ \ \mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega), \\ \langle \mathrm{d}u(t,\mathcal{O})/\mathrm{d}t + A(e)u(t,\mathcal{O}), v - u(t,\mathcal{O})\rangle_V \geqslant \langle L(t), v - u(t,\mathcal{O})\rangle_V. \end{cases}
$$

Finally, as a consequence of Lemmas 1, 2 and due to the assertions given above, we conclude that the assumptions of Theorem 3 are fulfilled. We thus obtain:

Theorem 4. *There exists a unique solution of the State Problem* (3.26) *for any* $\mathcal{O} \in \mathscr{U}_{\rm ad}(\Omega)$. *The Maximization Problem* ($\mathscr{P}_{\rm heat}$) *has at least one solution.*

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Author's address: *Ján Lovíšek*, Department of Mechanics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovak Republic, e-mail: `lovisek@svf.stuba.sk`.