Zdeněk Skalák; Petr Kučera

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REGULARITY OF PRESSURE IN THE NEIGHBOURHOOD OF REGULAR POINTS OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS*

Zdeněk Skalák, Petr Kučera, Praha

Abstract. In the context of the weak solutions of the Navier-Stokes equations we study the regularity of the pressure and its derivatives in the space-time neighbourhood of regular points. We present some global and local conditions under which the regularity is further improved.

Keywords: Navier-Stokes equations, regularity of weak solutions, regular and singular points

MSC 2000: 35Q35, 35Q30

Introduction

Let Ω be either $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$ with $\mathcal{C}^{2+\mu}$ boundary $\partial\Omega$ ($\mu > 0$), $T > 0$ and $Q_T = \Omega \times (0, T)$. In Q_T we deal with the Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \mathcal{P} = 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u(x, 0) = u_0 \tag{3}$$

and (if Ω is a bounded domain)

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{4}$$

where $u = (u_1, u_2, u_3)$ and $\mathcal{P}$ denote the velocity and pressure and $\nu > 0$ is the viscosity coefficient.

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As is usual in the standard theory of the Navier-Stokes equations, define $D(\Omega) = \{\psi \in C_0^\infty(\Omega)^3; \nabla \cdot \psi = 0 \text{ in } \Omega\}$ and let $H(\Omega)$ and $V(\Omega)$ be the completion of $D(\Omega)$ in $L^2(\Omega)^3$ and $W^{1,2}(\Omega)^3$, respectively. Define also $D_T = \{\eta \in C_0^\infty(\Omega \times [0,T))^3; \nabla \cdot \eta = 0 \text{ in } \Omega \times [0,T)\}$.

Definition 1. Let $u_0 \in H(\Omega)$. A measurable function $u\colon Q_T \to \mathbb{R}^3$ is called a weak solution of the problem (1)–(4) if $u \in L^2(0,T,V(\Omega)) \cap L^\infty(0,T,H(\Omega))$ and

$$\int_0^T \int_\Omega \left[u \cdot \frac{\partial \eta}{\partial t} - \nu \nabla u \cdot \nabla \eta - u \cdot \nabla u \cdot \eta \right] \mathrm{d}x \, \mathrm{d}t = - \int_\Omega u_0 \cdot \eta(\cdot, 0) \, \mathrm{d}x$$

for all $\eta \in D_T$.

The existence of a weak solution of (1)–(4) was proved in many articles (see e.g. [1], [3] or [9]). However, the problem of regularity and uniqueness of weak solutions has not yet been solved. It is not known whether weak solutions obtained by different methods coincide or not. Let us point out that if not stated explicitly we do not consider any concrete weak solution (constructed, for example, by the Faedo-Galerkin method—see [9]) in this chapter but all the conclusions hold for any weak solution from Definition 1.

We say that a point $(x_0, t_0) \in Q_T$ is a regular point of u if there exists a neighbourhood $\mathcal{U}$ of (x_0, t_0) in Q_T such that $u \in L^\infty(\mathcal{U})^3$.

Let $(x_0, t_0) \in Q_T$ be a regular point. In this paper we are mainly interested in the smoothness of the time derivative $\partial u/\partial t$ in a neighbourhood of (x_0, t_0). More precisely, we ask whether $\partial u/\partial t \in C(Q')^3$, where $Q' = B_\delta(x_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and δ, ε are some positive numbers. The problem was already studied in [5] where it was concluded (among other) that

$$\frac{\partial u}{\partial t} \in C(Q')^3. \tag{5}$$

To prove this result (see Theorem 7 in [5]) the author used the second part of the Main Theorem from [7]. He verified all the necessary assumptions, particularly $\partial u/\partial t \in L^{4/3}(t_0 - \varepsilon, t_0 + \varepsilon, L^2(B_\delta(x_0))^3)$ for some $\varepsilon, \delta > 0$ and concluded finally that (5) is an immediate consequence of the Main Theorem. Unfortunately, in our opinion, it is not the case. In fact, Serrin's theorem does not yield such a strong regularity of $\partial u/\partial t$ in Q' and therefore, we still consider the question mentioned above an open problem. Its solution would be useful in the study of the local regularity of solutions of the Navier-Stokes equations.

We begin the next chapter with the presentation of a few observations concerning the regularity of $\partial u/\partial t$ and $\mathcal{P}$ near regular points—Theorem 2 and Theorem 4. We stress that due to the non-existence of the boundary the results for $\Omega = \mathbb{R}^3$ are stronger than those for a bounded Ω. Next, we present some simple (global and local) additional conditions on u and $\mathcal{P}$ under which the regularity of $\partial u/\partial t$ and $\mathcal{P}$ in the neighbourhood of a regular point of u can be improved.

1. Discussion of known results

Let us start with some obvious properties of weak solutions of (1)–(4) in a space-time neighbourhood of regular points which follow from [7] and [5].

Thus, let $(x_0, t_0) \in Q_T$ be a regular point of u. Then there exist $\varepsilon, \delta > 0$, $Q' = B_\delta(x_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that

(i) $u(\cdot, t) \in C^\infty(B_\delta(x_0))^3$ for almost every $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and every space derivative of u is bounded in compact subregions of Q' (see [7]);

(ii) $\partial u/\partial t \in L^{4/3}(t_0 - \varepsilon, t_0 + \varepsilon, L^2(B_\delta(x_0))^3)$ (see [5]);

(iii) $D_x^\gamma u(x, \cdot)$ are absolutely continuous functions of time in $(t_0 - \varepsilon, t_0 + \varepsilon)$ for almost every $x \in B_\delta(x_0)$ and every multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, where $D_x^\gamma = \partial^{|\gamma|}/(\partial x_1^{\gamma_1} \ldots \partial x_3^{\gamma_3})$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$ (see [7]).

It follows from (i) that the functions $D_x^\gamma u(\cdot, t)$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \setminus \mathcal{A}$, where $\mathcal{A}$ is a subset of $\mathbb{R}$ of Lebesgue measure 0, are uniformly bounded and uniformly continuous in $(\overline{B_{\delta_1}(x_0)})^3$ for every $\delta_1 \in (0, \delta)$. Using the Arzelà-Ascoli theorem and the weak continuity of u as a function from $(t_0 - \varepsilon, t_0 + \varepsilon)$ into $L^2(B_\delta(x_0))$, we get that

$$u(\cdot, t) \in C^\infty(B_\delta(x_0))^3 \ \text{ for } \textit{every } t \in (t_0 - \varepsilon, t_0 + \varepsilon) \tag{6}$$

and $D_x^\gamma u$ are continuous functions in $B_\delta(x_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ for every γ.

In the following two theorems we present further results on regularity of space derivatives of pressure and space derivatives of the time derivative of velocity. Since we have no information concerning the behavior of the pressure derivatives on $\partial\Omega$, we use the cut-off function technique (see e.g. [5]). Therefore, the results are determined by the initial global regularity of pressure as presented in [8] or [6]. The second theorem is stated exclusively for $\Omega = \mathbb{R}^3$. In this case we use the integral representation of $\mathcal{P}$ (see [1]) which holds for any weak solution. It is interesting that due to the non-existence of the boundary $\partial\Omega$, Theorem 4 gives stronger results than Theorem 2.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary or $\Omega = R^3$, let u be a weak solution of* (1)–(4) *and $\mathcal{P}$ the associated pressure. Let further $(x_0, t_0) \in Q_T$ be a regular point of u and ε, δ the numbers from* (i), (ii) *and* (iii). *Then $D_x^\gamma \partial u/\partial t, D_x^\gamma \mathcal{P} \in L^\alpha(t_0 - \varepsilon, t_0 + \varepsilon, L^\infty(B_{\delta_1}(x_0))^3)$ for every multi-index γ, $|\gamma| \geqslant 0$, $\delta_1 \in (0, \delta)$ and $\alpha \in (1, 2)$.*

P r o o f. Let $\delta_1 \in (0,\delta)$ and $\alpha \in (1,2)$. As follows from [8] or [6] we can suppose that $\mathcal{P} \in L^r(\xi, T, L^s(\Omega_1))$ for every $\xi > 0$ provided $\Omega_1 \subset \Omega$ is a bounded domain, $2/r + 3/s = 3$, $r \in (1,2)$ and $s \in (3/2, 3)$. Thus, we have

$$\mathcal{P} \in L^{\alpha}(t_0-\varepsilon, t_0+\varepsilon; L^1(B_{\delta}(x_0))). \tag{7}$$

Let $\varphi \in C^{\infty}(\mathbb{R}^3)$,

$$\varphi \in \langle 0,1\rangle \text{ in } \mathbb{R}^3,\quad \varphi \equiv 1 \text{ in } B_{(2\delta_1+\delta)/3}(x_0) \text{ and } \varphi \equiv 0 \text{ in } \mathbb{R}^3 \setminus B_{(2\delta+\delta_1)/3}(x_0). \tag{8}$$

For almost every $t \in (t_0-\varepsilon, t_0+\varepsilon$ and for every $x \in B_{\delta_1}(x_0)$ it is possible to write

$$\begin{aligned} \mathcal{P}(x,t) = (\varphi\mathcal{P})(x,t) &= \frac{1}{4\pi}\int_{\Omega} \frac{1}{|x-y|}\varphi(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y \\ &+ \frac{1}{4\pi}\int_{\Omega}\frac{1}{|x-y|}\mathcal{P}(y,t)\Delta\varphi(y)\,\mathrm{d}y + \frac{1}{2\pi}\int_{\Omega}\frac{x-y}{|x-y|^3}\nabla\varphi(y)\mathcal{P}(y,t)\,\mathrm{d}y \\ &= p_1(x,t) + p_2(x,t) + p_3(x,t). \end{aligned} \tag{9}$$

Using (i), (8) and the equality

$$\frac{\partial}{\partial x_k}\frac{1}{|x-y|} = -\frac{\partial}{\partial y_k}\frac{1}{|x-y|},$$

we see that

$$\frac{\partial p_1}{\partial x_k}(x,t) = \frac{1}{4\pi}\int_{\Omega}\frac{1}{|x-y|}\frac{\partial}{\partial y_k}\Big(\varphi(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\Big)\,\mathrm{d}y, \quad k=1,2,3 \tag{10}$$

and the following conclusion can be derived from (10) and (i):

$$\nabla p_1 \in L^{\infty}(t_0-\varepsilon, t_0+\varepsilon; L^{\infty}(B_{\delta_1}(x_0))^3). \tag{11}$$

Differentiating further (10), we obtain

$$D_x^{\gamma} p_1 \in L^{\infty}(t_0-\varepsilon, t_0+\varepsilon; L^{\infty}(B_{\delta_1}(x_0))^3) \tag{12}$$

for every γ. Using (7), (8) and the facts that

$$\nabla\varphi \equiv 0 \text{ on } B_{(2\delta_1+\delta)/3}(x_0) \cup (\mathbb{R}^3 \setminus B_{(2\delta+\delta_1)/3}(x_0))$$

and

$$\frac{1}{|x-y|} \in C^{\infty}(\mathbb{R}^3 \setminus \{x\}),$$

we get by differentiating the last two integrals in (9) that

$$D_x^\gamma p_2, D_x^\gamma p_3 \in L^\alpha(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3). \tag{13}$$

It follows from (12) and (13) that

$$D_x^\gamma \mathcal{P} \in L^\alpha(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3). \tag{14}$$

Finally, using (i), (1) and (14), we conclude that

$$D_x^\gamma \frac{\partial u}{\partial t} \in L^\alpha(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$$

and the proof is complete. □

R e m a r k 3. Suppose that the assumptions of Theorem 2 are satisfied and $\mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^1(B_\delta(x_0)))$, $\beta \in \langle 2, \infty\rangle$. Then the proof of Theorem 2 gives that $D_x^\gamma p_2, D_x^\gamma p_3 \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$ and consequently $D_x^\gamma \partial u/\partial t, D_x^\gamma \mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$.

Since $D_x^\gamma \partial u/\partial t \in L^\alpha(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$, we have

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0+\varepsilon} \operatorname*{ess\,sup}_{x\in B_{\delta_1}(x_0)} \left| D_x^\gamma \frac{\partial u}{\partial t}(x,t)\right|^\alpha \mathrm{d}t \\ = \left\| D_x^\gamma \frac{\partial u}{\partial t}\right\|^\alpha_{L^\alpha(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)} = c^\alpha < \infty.\end{aligned} \tag{15}$$

It follows from (15), (i) and (iii) that

$$\operatorname*{ess\,sup}_{x\in B_{\delta_1}(x_0)} \{\|D_x^\gamma u(x,\cdot)\|_{W^{1,\alpha}(t_0-\varepsilon, t_0+\varepsilon)}\} \leqslant c < \infty.$$

Therefore, using also the continuity of $D_x^\gamma u$ in Q' and the reflexivity of the space $W^{1,\alpha}(t_0-\varepsilon, t_0+\varepsilon)$, we get that

$$D_x^\gamma u(x,\cdot) \in W^{1,\alpha}(t_0-\varepsilon, t_0+\varepsilon) \text{ for } \textit{every } x \in B_{\delta_1}(x_0) \tag{16}$$

and

$$\sup_{x\in B_{\delta_1}(x_0)} \{\|D_x^\gamma u(x,\cdot)\|_{W^{1,\alpha}(t_0-\varepsilon, t_0+\varepsilon)}\} \leqslant c < \infty.$$

Using (6) and (16), we obtain that $D_x^\gamma u \in C^{0,1-1/\alpha}(B_{\delta_1}(x_0) \times (t_0-\varepsilon, t_0+\varepsilon))^3$, $\alpha \in (1,2)$, where the distance in the space-time is defined as $d((x,t),(x',t')) = |x-x'| + |t-t'|$. Note that Ladyzhenskaya and Seregin proved a similar result in [6].

Theorem 4. *Let $\Omega = R^3$ and let the assumptions of Theorem 2 be satisfied. Then $D_x^\gamma \partial u/\partial t, D_x^\gamma \mathcal{P} \in L^\infty(t_0-\varepsilon, t_0+\varepsilon, L^\infty(B_{\delta_1}(x_0))^3)$ for every multi-index γ, $|\gamma| \geqslant 0$ and $\delta_1 \in (0,\delta)$.*

Proof. Let $\delta_1 < \delta_2 < \delta$. It is known (see e.g. [1]) that for almost every $t \in (t_0-\varepsilon, t_0+\varepsilon)$ and every $x \in B_{\delta_1}(x_0)$,

$$\begin{aligned}
\mathcal{P}(x,t) &= \frac{1}{4\pi}\int_{R^3} \frac{1}{|x-y|}\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y \\
&= \frac{1}{4\pi}\int_{B_{\delta_2}(x_0)} \frac{1}{|x-y|}\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_{\delta_2}(x_0)} \frac{1}{|x-y|}\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y \\
&= \hat{p}_1(x,t) + \hat{p}_2(x,t).
\end{aligned} \tag{17}$$

It follows from (i) that $\hat{p}_1 \in L^\infty(t_0-\varepsilon, t_0+\varepsilon, L^\infty(B_{\delta_1}(x_0))^3)$. If we use twice integration by parts and take into consideration that $u\nabla u$, $|u|^2 \in L^1(R^3)$ we get

$$\begin{aligned}
\hat{p}_2(x,t) &= \frac{1}{4\pi}\int_{\partial B_{\delta_2}(x_0)} \frac{1}{|x-y|}\frac{\partial u_i}{\partial y_j}(y,t)u_j(y,t)n_i(y)\,\mathrm{d}_yS \\
&\quad - \frac{1}{4\pi}\int_{\partial B_{\delta_2}(x_0)} \frac{\partial}{\partial y_i}\Big(\frac{1}{|x-y|}\Big)u_i(y,t)u_j(y,t)n_j(y)\,\mathrm{d}_yS \\
&\quad + \frac{1}{4\pi}\int_{R^3\setminus B_{\delta_2}(x_0)} \frac{\partial^2}{\partial y_i\partial y_j}\Big(\frac{1}{|x-y|}\Big)u_i(y,t)u_j(y,t)\,\mathrm{d}y,
\end{aligned} \tag{18}$$

where $n = (n_1, n_2, n_3)$ is the outer normal vector. Using (18), (i) and the facts that $u_iu_j \in L^\infty(t_0-\varepsilon, t_0+\varepsilon, L^1(\mathbb{R}^3))$ and the functions $\psi(y) = 1/|x-y|$, $x \in B_{\delta_1}(x_0)$ and all their first and second space derivatives in y are uniformly bounded in $\mathbb{R}^3 \setminus B_{\delta_2}(x_0)$, we see that $\hat{p}_2 \in L^\infty(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$. From this and (17) one can conclude

$$\mathcal{P} \in L^\infty(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3). \tag{19}$$

Now the proof follows immediately from Theorem 2 and Remark 3. □

Let us remark that since $D_x^\gamma \partial u/\partial t \in L^\infty(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$, we have

$$\operatorname*{ess\,sup}_{x\in B_{\delta_1}(x_0)} \{\|D_x^\gamma u(x,\cdot)\|_{W^{1,\beta}(t_0-\varepsilon,t_0+\varepsilon)}\} \leqslant c < \infty$$

for every $\beta \in \langle 1, \infty\rangle$ and

$$D_x^\gamma u(x,\cdot) \in W^{1,\beta}(t_0-\varepsilon, t_0+\varepsilon) \text{ holds for } \textit{every } x \in B_{\delta_1}(x_0) \tag{20}$$

provided $\beta \in \langle 1, \infty)$. Moreover, we also have $u \in C^{0,\lambda}(B_{\delta_1}(x_0) \times (t_0-\varepsilon, t_0+\varepsilon))^3$ for every $\lambda \in (0,1)$.

R e m a r k 5. Let u be a weak solution of (1)–(4) and let $\mathcal{P}$ be the associated pressure. Then $\mathcal{P} \in L^{5/3}(Q_T)$.

P r o o f. We use the integral representation of $\mathcal{P}$ (the first equation in (17)) and apply twice integration by parts. We get for almost every $t \in (0,T)$ that

$$\mathcal{P}(x,t) = -\frac{1}{3}|u(x,t)|^2 + \lim_{\varepsilon\to 0_+} \frac{1}{4\pi} \int_{R^3_\varepsilon(x)} \frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{|x-y|}\right) u_i(y,t) u_j(y,t)\,\mathrm{d}y$$

for almost every $x \in \mathbb{R}^3$, where $\mathbb{R}^3_\varepsilon(x) = \mathbb{R}^3 \setminus \overline{B_\varepsilon(x)}$. The conclusion of Remark 5 is obtained by using the Calderón-Zygmund theorem (see [2] or [4]) and the fact that $u \in L^{10/3}(Q_T)$. $\square$

2. Application of additional conditions

From now on till the end of the chapter suppose that Ω is a bounded domain. Our intention is to prove Theorem 2 for $\alpha \geqslant 2$. Unfortunately, we are not able to prove it generally. At first, we present a few global additional conditions on u and p, under which Theorem 2 holds for $\alpha \geqslant 2$. Thus, let the assumptions of Theorem 2 be satisfied and $\delta_1 \in (0,\delta)$. Moreover, we suppose that it is possible to write

$$(t_0-\varepsilon, t_0+\varepsilon) = \bigcup_{\alpha\in\Gamma} I_\alpha \cup G, \tag{21}$$

where Γ is at most countable, I_α are open disjoint intervals, one-dimensional Lebesgue measure of G is zero and u, $\mathcal{P}$ are smooth functions in $\Omega \times I_\alpha$. These assumptions hold, for example, for the weak solution of (1)–(4) constructed by the Faedo-Galerkin method (see [9]) or for the weak solutions of (1)–(4) satisfying the strong energy inequality (see [4]).

There exists an orthogonal decomposition of $L^2(\Omega)^3$ (see [9]),

$$L^2(\Omega)^3 = H_1 \oplus H_2 \oplus H_3, \tag{22}$$

where

$$H_1 = \{v \in L^2(\Omega)^3,\ \operatorname{div} v = 0,\ \gamma_\nu v = 0\},$$
$$H_2 = \{v = \nabla p,\ p \in W^{1,2}_0(\Omega)\}$$

and

$$H_3 = \{v = \nabla q,\ q \in W^{1.2}(\Omega),\ \Delta q = 0\},$$

where $\gamma_\nu v$ is the restriction of $v \cdot \nu$ to $\partial\Omega$ for every $v \in C^\infty(\overline{\Omega})$ and ν is the normal vector to $\partial\Omega$ (see [9]). Coming back to the equation (1), we can decompose all terms in (1) on the intervals I_α. Denote by P_1, P_2, P_3, P_{12}, P_{13}, P_{23}, respectively, the projections from $(L^2(\Omega))^3$ onto the spaces H_1, H_2, H_3, $H_1 \oplus H_2$, $H_1 \oplus H_3$, $H_2 \oplus H_3$. Then

$$\frac{\partial u}{\partial t} = P_1\left(\frac{\partial u}{\partial t}\right) \in H_1, \tag{23}$$

$$\nu \cdot \Delta u = P_1(\nu \cdot \Delta u) + P_3(\nu \cdot \Delta u) \in H_1 \oplus H_3, \tag{24}$$

$$\nabla \mathcal{P} = P_2(\nabla \mathcal{P}) + P_3(\nabla \mathcal{P}) \in H_2 \oplus H_3 \tag{25}$$

and

$$u\nabla u = P_1(u\nabla u) + P_2(u\nabla u) + P_3(u\nabla u) \in H_1 \oplus H_2 \oplus H_3. \tag{26}$$

Lemma 6. *If u is a weak solution of the system* (1)–(4) *and $\mathcal{P}$ is an associated pressure then*

$$(u \cdot \nabla)u \in L^\infty(t_0 - \varepsilon, t_0 + \varepsilon; (W^{3,2}(\Omega)^3)^*), \tag{27}$$

$$\Delta \mathcal{P} = -\frac{\partial u_i}{\partial y_j} \cdot \frac{\partial u_j}{\partial y_i} \in L^2(t_0 - \varepsilon, t_0 + \varepsilon; (W^{3,2}(\Omega))^*). \tag{28}$$

Proof. Let $\psi \in W^{3,2}(\Omega)^3$. Then

$$\int_\Omega (u \cdot \nabla)u \cdot \psi \,\mathrm{d}y = -\int_\Omega u_j u_i \frac{\partial \psi_i}{\partial y_j} \,\mathrm{d}y.$$

We estimate

$$\left| \int_\Omega u_j u_i \frac{\partial \psi_i}{\partial y_j} \,\mathrm{d}y \right| \leqslant c\|u\|^2_{L^2(\Omega)^3} \|\psi\|_{W^{3,2}(\Omega)^3}$$

and (27) follows. Similarly, if $\psi \in W^{3,2}(\Omega)$, then

$$\int_\Omega \frac{\partial u_i}{\partial y_j} \frac{\partial u_j}{\partial y_i} \psi \,\mathrm{d}y = -\int_\Omega u_j \frac{\partial u_j}{\partial y_j} \frac{\partial \psi}{\partial y_i} \,\mathrm{d}y$$

and

$$\left| \int_\Omega u_j \frac{\partial u_i}{\partial y_j} \frac{\partial \psi}{\partial y_i} \,\mathrm{d}y \right| \leqslant c\|u\|_{L^2(\Omega)^3} \|\nabla u\|_{L^2(\Omega)^9} \|\psi\|_{W^{3,2}(\Omega)}.$$

Therefore (28) holds. Lemma 6 is proved. $\square$

Let us show now that there exists a function $\eta_x = \eta_x(y)$ from $W^{3,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap \mathcal{C}^\infty(\overline{\Omega})^3$ for every $x \in B_{\delta_1}(x_0)$ such that for almost every $t \in (t_0-\varepsilon, t_0+\varepsilon)$

$$p_2(x,t) + p_3(x,t) = -\int_\Omega \nabla \mathcal{P}(y,t) \cdot \eta_x(y) \,\mathrm{d}y \tag{29}$$

(the functions p_2, p_3 were defined in (9)) and therefore

$$\mathcal{P}(x,t) = \varphi \mathcal{P}(x,t) = p_1(x,t) - \int_\Omega \nabla \mathcal{P}(y,t) \cdot \eta_x(y) \,\mathrm{d}y. \tag{30}$$

Moreover, we will show that there exists $K > 0$ such that

$$\|\eta_x\|_{W^{3,2}(\Omega)^3} \leqslant K \tag{31}$$

for every $x \in B_{\delta_1}(x_0)$.

Let us recall the following lemma (see [9]).

Lemma 7. *Let $D \subset \mathbb{R}^3$ be an open bounded domain, $\partial D \in C^\beta$, where $\beta = \max(m+2, 2)$ and assume that $m \geqslant -1$ is an integer. Let $\Phi \in W^{m+1,2}(D)$ and $\int_D \Phi \,\mathrm{d}x = 0$. Then there exist $\eta \in W^{m+2,2}(D)^3$ and $q \in W^{m+1,2}(D)$ such that*

$$\begin{aligned} -\Delta \eta + \nabla q &= 0 \quad \text{in} \quad D, \\ \operatorname{div} \eta &= \Phi \quad \text{in} \quad D, \\ \eta &= 0 \quad \text{on} \quad \partial D. \end{aligned} \tag{32}$$

Moreover,

$$\|\eta\|_{W^{m+2,2}(D)} \leqslant c \|\Phi\|_{W^{m+1,2}(D)},$$

where $c = c(D, m)$.

Take φ defined in (8). Denote for every $x \in B_{\delta_1}(x_0)$

$$\Psi_x(y) = \frac{1}{4\pi} \frac{1}{|x-y|} \Delta \varphi(y) + \frac{1}{2\pi} \nabla_y \frac{1}{|x-y|} \nabla \varphi(y) - m(\Omega)^{-1},$$

where $m(\Omega)$ denotes the Lebesgue measure of Ω. Then $\int_\Omega \Psi_x(y) \,\mathrm{d}y = 0$ and $\|\Psi_x\|_{2,2} \leqslant K < \infty$ for every $x \in B_{\delta_1}(x_0)$ and for some $K > 0$. Using Lemma 7, we obtain a function $\eta_x \in W^{3,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3$, $\operatorname{div} \eta_x = \Psi_x$ and $\|\eta_x\|_{W^{3,2}(\Omega)} \leqslant cK < \infty$ for every $x \in B_{\delta_1}(x_0)$. Obviously, $\eta_x \in C^\infty(\overline{\Omega})^3$, since $\Psi_x \in C^\infty(\overline{\Omega})^3$. Therefore, using the equality $\int_\Omega \mathcal{P}(y,t) \,\mathrm{d}y = 0$, we can write

$$p_2(x,t) + p_3(x,t) = \int_\Omega (\Psi_x(y) + m(\Omega)^{-1}) \mathcal{P}(y,t) \,\mathrm{d}y = -\int_\Omega \nabla \mathcal{P}(y,t) \cdot \eta_x(y) \,\mathrm{d}y. \tag{33}$$

Thus, (29)–(31) are verified. We know from the proof of Theorem 2 that $p_1 \in L^\infty(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0)))$. Due to (30) this means that to prove

$$\mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))) \tag{34}$$

for some $\beta \in \langle 2, \infty\rangle$, it is sufficient to show that $\int_\Omega \nabla\mathcal{P}(y,t)\cdot\eta_x(y)\,\mathrm{d}y$ as a function of (x,t) is from the space $L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0)))$. We use this idea in the the proof of the two next lemmas, where we present two examples of global conditions imposed on u under which (34) is satisfied. Recall that (34) then implies (according to Remark 3) that Theorem 2 holds for $\alpha = \beta \geqslant 2$. For the sake of simplicity, we omit in the proof of Lemma 8 and Lemma 9 the dependence of functions on y and t.

Lemma 8. *Let g be a function for which $\nabla g = P_3(\nu\Delta u)$ and suppose that $g|_{\partial\Omega} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^1(\partial\Omega))$, $\beta \in \langle 2,\infty\rangle$. Then (34) holds.*

P r o o f. Let $x \in B_{\delta_1}(x_0)$. Suppose that η_x is the function from (29). Using (1) and (23)–(25) we get for almost every $t \in (t_0-\varepsilon, t_0+\varepsilon)$ that

$$\begin{aligned}\int_\Omega \nabla\mathcal{P}\eta_x\,\mathrm{d}y = \int_\Omega \nabla\mathcal{P}P_{23}(\eta_x)\,\mathrm{d}y &= -\int_\Omega (u\cdot\nabla)uP_{23}(\eta_x)\,\mathrm{d}y \\ &+ \int_\Omega \nu\Delta uP_{23}(\eta_x)\,\mathrm{d}y - \int_\Omega \frac{\partial u}{\partial t}P_{23}(\eta_x)\,\mathrm{d}y.\end{aligned}$$

Obviously, $\int_\Omega \partial u/\partial t P_{23}(\eta_x)\,\mathrm{d}y = 0$ and $\int_\Omega \nu\Delta uP_{23}(\eta_x)\,\mathrm{d}y = \int_\Omega \nu P_3(\Delta u)P_3(\eta_x)\,\mathrm{d}y$. Thus,

$$\int_\Omega \nabla\mathcal{P}\eta_x\,\mathrm{d}y = -\int_\Omega (u\cdot\nabla)uP_{23}(\eta_x)\,\mathrm{d}y + \int_\Omega \nu P_3(\Delta u)P_3(\eta_x)\,\mathrm{d}y. \tag{35}$$

It follows from Lemma 6, (31) and the fact that $\|P_{23}\varphi_x\|_{W^{3,2}(\Omega)^3} \leqslant c\|\varphi_x\|_{W^{3,2}(\Omega)^3}$ that

$$\int_\Omega (u\cdot\nabla)uP_{23}(\eta_x)\,\mathrm{d}y \leqslant C \tag{36}$$

for some $C > 0$, every $x \in B_{\delta_1}(x_0)$ and almost every $t \in (t_0-\varepsilon, t_0+\varepsilon)$.

Denote by h the function for which $\nabla h = P_3(\eta_x)$. Then $\Delta h = 0$ and

$$\begin{aligned}\int_\Omega P_3(\nu\Delta u)\cdot P_3(\eta_x)\,\mathrm{d}y &= \int_\Omega \nabla g\cdot\nabla h\,\mathrm{d}y \\ &= \int_{\partial\Omega} g\frac{\partial h}{\partial n}\,\mathrm{d}_yS - \int_\Omega g\Delta h\,\mathrm{d}y = \int_{\partial\Omega} g\frac{\partial h}{\partial n}\,\mathrm{d}_yS.\end{aligned} \tag{37}$$

From $\eta_x \in W_0^{1,2}(\Omega)^3$ we get that $\partial h/\partial n = -P_2(\eta_x)\cdot n$ on $\partial\Omega$ and this implies $\|\partial h/\partial n\|_{L^\infty(\partial\Omega)} \leqslant c\|\eta_x\|_{W^{3,2}(\Omega)^3}$. Therefore,

$$\left|\int_{\partial\Omega} g\frac{\partial h}{\partial n}\,\mathrm{d}_yS\right| \leqslant \|g\|_{L^1(\partial\Omega)}\left\|\frac{\partial h}{\partial n}\right\|_{L^\infty(\partial\Omega)} \leqslant c\|g\|_{L^1(\partial\Omega)}\|\eta_x\|_{W^{3,2}(\Omega)^3} \leqslant cK\|g\|_{L^1(\partial\Omega)}.$$

From this inequality and from (33), (35), (36) and (37) we get (34) and Lemma 8 is proved. □

Lemma 9. *Suppose that $\partial u/\partial n \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^1(\partial\Omega)^3)$, $\beta \in \langle 2,\infty\rangle$. Then (34) holds.*

P r o o f. First, we proceed similarly as we did in Lemma 8. Instead of (37) we estimate ($\nabla h = P_3(\eta_x)$)

$$\begin{aligned}\nu\int_\Omega \Delta u\cdot P_3(\eta_x)\,\mathrm{d}y &= \int_{\partial\Omega}\frac{\partial u}{\partial n}\cdot\nabla h\,\mathrm{d}_yS - \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial^2 h}{\partial x_i\partial x_j}\,\mathrm{d}y\\ &= \int_{\partial\Omega}\frac{\partial u}{\partial n}\cdot\nabla h\,\mathrm{d}_yS - \int_{\partial\Omega} u_in_j\frac{\partial^2 h}{\partial x_i\partial x_j}\,\mathrm{d}_yS + \int_\Omega u_i\frac{\partial}{\partial x_i}(\Delta h)\,\mathrm{d}y\\ &= \int_{\partial\Omega}\frac{\partial u}{\partial n}\cdot\nabla h\,\mathrm{d}_yS.\end{aligned}$$

By virtue of Lemma 6, we have

$$\left|\int_{\partial\Omega}\frac{\partial u}{\partial n}\cdot\nabla h\,\mathrm{d}_yS\right| \leqslant \left\|\frac{\partial u}{\partial n}\right\|_{L^1(\partial\Omega)^3}\|\eta_x\|_{W^{3,2}(\Omega)^3}$$

and we obtain (34). Lemma 9 is proved. □

Now let us formulate one local condition (see Theorem 10) under which (34) is satisfied. The following considerations can be done for an arbitrary weak solution of (1)–(4). Let $2r<\delta$ and $x\in B_r(x_0)$. Let further $\zeta\in C^\infty(\langle 0,\infty))$, $\zeta\equiv 1$ in $\langle 0, r/2\rangle$ and $\zeta\equiv 0$ in $\langle 3r/4,\infty)$. Thus, ζ is independent of $t\in(t_0-\varepsilon, t_0+\varepsilon)$ and $x\in B_r(x_0)$. Put $\varphi_x(y) = \zeta(|x-y|)$ for every $x\in B_r(x_0)$ and every $y\in\mathbb{R}^3$. For almost every $t\in(t_0-\varepsilon,t_0+\varepsilon)$ and for every $x\in B_r(x_0)$ it is possible to write

$$\begin{aligned}\mathcal{P}(x,t) &= \frac{1}{4\pi}\int_{B_r(x)}\frac{1}{|x-y|}\varphi_x(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y\\ &\quad+\frac{1}{4\pi}\int_{B_r(x)}\frac{1}{|x-y|}\Delta_y\varphi_x(y)\mathcal{P}(y,t)\,\mathrm{d}y\\ &\quad+\frac{1}{2\pi}\int_{B_r(x)}\frac{(x-y)}{|x-y|^3}\cdot\nabla_y\varphi_x(y)\mathcal{P}(y,t)\,\mathrm{d}y.\end{aligned}\tag{38}$$

It is possible to show that

$$\int_{B_r(x)} \left[\frac{1}{|x-y|} \Delta_y \varphi_x(y) + 2\nabla_y \left(\frac{1}{|x-y|} \right) \cdot \nabla_y \varphi_x(y) - \frac{3}{r^3} \right] \mathrm{d}y = 0. \tag{39}$$

Lemma 7 yields that there exist $\eta_x \in C^\infty(\overline{B_r(x)})^3 \cap W_0^{1,2}(B_r(x))^3$ such that

$$\nabla \cdot \eta_x(y) = \frac{1}{|x-y|} \Delta_y \varphi_x(y) + 2\nabla_y \left(\frac{1}{|x-y|} \right) \cdot \nabla_y \varphi_x(y) - \frac{3}{r^3} \quad \text{on } B_r(x) \tag{40}$$

and $\|\eta_x\|_{W^{m+1,2}(B_r(x))^3}$ are bounded on $B_r(x_0)$ for every $m \in N$. Decompose η_x on $L^2(B_r(x))^3$ as $P_1(\eta_x)+P_2(\eta_x)+P_3(\eta_x)$ and let $\nabla \mathcal{Q}_{2,x} = P_2(\eta_x)$ and $\nabla \mathcal{Q}_{3,x} = P_3(\eta_x)$. Then $|\mathcal{Q}_{2,x}(y)| \leqslant c$, where c is a positive constant independent of $x \in B_r(x_0)$ and $y \in B_r(x)$. This follows from (40) and the fact that $\mathcal{Q}_{2,x}$ is a solution of the system

$$\begin{aligned} \Delta \mathcal{Q}_{2,x} &= \nabla \cdot \eta_x \quad && \text{in } B_r(x), \\ \mathcal{Q}_{2,x} &= 0 && \text{on } \partial B_r(x). \end{aligned}$$

We can see from (40) that $\nabla \cdot \eta_x$ is spherically symmetric around x and the same must be true for $\mathcal{Q}_{2,x}$. Consequently, $\partial \mathcal{Q}_{2,x}/\partial n = \text{const.}$ on $\partial B_r(x)$ (n is a normal vector on $\partial B_r(x)$) and since

$$\int_{\partial B_r(x)} \frac{\partial \mathcal{Q}_{2,x}}{\partial n} \, \mathrm{d}_y S = \int_{B_r(x)} \Delta \mathcal{Q}_{2,x} \, \mathrm{d}y = \int_{B_r(x)} \nabla \cdot \eta_x \, \mathrm{d}y = \int_{\partial B_r(x)} \eta_x \cdot n \, \mathrm{d}_y S = 0,$$

one gets that $\partial \mathcal{Q}_{2,x}/\partial n = 0$ on $\partial B_r(x)$. Further, $\mathcal{Q}_{3,x}$ is a solution of the system

$$\begin{aligned} \Delta Q_{3,x} &= 0 && \text{in } B_r(x), \\ \frac{\partial \mathcal{Q}_{3,x}}{\partial n} &= -\frac{\partial \mathcal{Q}_{2,x}}{\partial n} = 0 \quad && \text{on } \partial B_r(x), \end{aligned}$$

which gives $\mathcal{Q}_{3,x} = \text{const.}$ and therefore $\nabla \mathcal{Q}_{3,x} = 0$ on $B_r(x)$. Thus, we can write

$$\begin{aligned} \int_{B_r(x)} \nabla \cdot \eta_x(y) \mathcal{P}(y,t) \, \mathrm{d}y &= -\int_{B_r(x)} P_2(y\eta_x)(y) \cdot \nabla \mathcal{P}(y,t) \, \mathrm{d}y \\ = \int_{B_r(x)} \mathcal{Q}_{2,x}(y) \Delta \mathcal{P}(y,t) \, \mathrm{d}y &= -\int_{B_r(x)} \mathcal{Q}_{2,x}(y) \frac{\partial^2 (u_i u_j)}{\partial y_i \partial y_j}(y,t) \, \mathrm{d}y. \end{aligned} \tag{41}$$

Now it follows from (38), (40) and (41) that

$$\begin{aligned} \mathcal{P}(x,t) = \frac{1}{4\pi} \int_{B_r(x)} &\left[\frac{1}{|x-y|} \varphi_x(y) - \mathcal{Q}_{2,x}(y) \right] \frac{\partial u_i}{\partial y_j}(y,t) \frac{\partial u_j}{\partial y_i}(y,t) \, \mathrm{d}y \\ &+ \frac{3}{4\pi r^3} \int_{B_r(x)} \mathcal{P}(y,t) \, \mathrm{d}y, \end{aligned} \tag{42}$$

which can be written as

$$
(43)\qquad 0 = \frac{1}{4\pi}\int_{B_r(x)}\left[\frac{1}{|x-y|}\varphi_x(y) - \mathcal{Q}_{2,x}(y)\right]\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y + \frac{3}{4\pi r^3}\int_{B_r(x)}(\mathcal{P}(y,t)-\mathcal{P}(x,t))\,\mathrm{d}y.
$$

The first integral on the right-hand side of (43) is from $L^\infty(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_r(x_0)))$. Let $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$ for every $a \in \mathbb{R}$. We can conclude:

Theorem 10. *Let the assumptions of Theorem 2 be fulfilled. Let further $2r < \delta$ and $x \in B_r(x_0)$. If $(\mathcal{P}-\mathcal{P}(x,t))^+$ or $(\mathcal{P}-\mathcal{P}(x,t))^-$ is from the space $L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_r(x)))$ for some $\beta \in \langle 2,\infty\rangle$, then $\mathcal{P}-\mathcal{P}(x,t) \in L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_r(x)))$. Consequently, there exists a sufficiently small θ such that $\mathcal{P}-\mathcal{P}(x,t) \in L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_\theta(x_0)))$. Considering now the pressure $\mathcal{P}-\mathcal{P}(x,t)$ instead of the pressure $\mathcal{P}$, we obtain, using Remark 3, that $D_x^\gamma \partial u/\partial t \in L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^\infty(B_{\theta_1}(x_0))^3)$ and $D_x^\gamma \mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^\infty(B_{\theta_1}(x_0)))$ for every multi-index γ, $|\gamma| \geqslant 1$ and $\theta_1 \in (0,\theta)$.*

Let us now present two consequences of Theorem 10. First, let the assumptions of Theorem 2 be satisfied. If $2r < \delta$ and there exists $x \in B_r(x_0)$ such that $\mathcal{P}-\mathcal{P}(x,t)$ is bounded from below (above) in $B_r(x) \times (t_0-\varepsilon, t_0+\varepsilon)$ then $\mathcal{P}-\mathcal{P}(x,t)$ and all space derivatives of $\mathcal{P}$ are in the space L^∞ in a space-time neighbourhood of (x_0,t_0).

Secondly, suppose that $\Omega = B_R(x_0)$, where $x_0 \in \mathbb{R}^3$ and $R > 0$, and x_0 is a regular point of u. We will show that if $\mathcal{P}^+$ or $\mathcal{P}^-$ is from the space $L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_{\delta_1}(x_0)))$ for some $\beta \in \langle 2,\infty\rangle$, $\delta_1 \in (0,\delta)$, then $\mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^1(B_{\delta_1}(x_0)))$. We can suppose that $\int_\Omega \mathcal{P}(y,t)\,\mathrm{d}y = 0$. It is possible to write

$$
\begin{aligned}
\mathcal{P}(x_0,t) = {}& \frac{1}{4\pi}\int_{B_r(x_0)}\frac{1}{|x_0-y|}\varphi_{x_0}(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y \\
&+ \frac{1}{4\pi}\int_{B_r(x_0)}\frac{1}{|x_0-y|}\Delta_y\varphi_{x_0}(y)\mathcal{P}(y,t)\,\mathrm{d}y \\
&+ \frac{1}{2\pi}\int_{B_r(x_0)}\frac{(x_0-y)}{|x_0-y|^3}\cdot\nabla_y\varphi_{x_0}(y)\mathcal{P}(y,t)\,\mathrm{d}y.
\end{aligned}
$$

Consequently,

$$
(44)\qquad \mathcal{P}(x_0,t) = \frac{1}{4\pi}\int_{B_r(x_0)}\frac{1}{|x_0-y|}\varphi_{x_0}(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y - \frac{1}{4\pi}\int_\Omega \mathcal{Q}_{2,x_0}(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y,
$$

where $\mathcal{Q}_{2,x_0} \in C^\infty(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$ is constructed on Ω in the same way as in the paragraph preceding Theorem 10. Since

$$\frac{1}{4\pi}\int_\Omega \mathcal{Q}_{2,x_0}(y)\frac{\partial u_i}{\partial y_j}(y,t)\frac{\partial u_j}{\partial y_i}(y,t)\,\mathrm{d}y = \frac{1}{4\pi}\int_\Omega \frac{\partial^2\mathcal{Q}_{2,x_0}}{\partial y_i\partial y_j}(y)u_i(y,t)u_j(y,t)\,\mathrm{d}y, \tag{45}$$

(44) gives immediately that $\mathcal{P}(x_0,\cdot) \in L^\infty(t_0-\varepsilon, t_0+\varepsilon)$. Using now Theorem 10 we can conclude that if $\mathcal{P}^+$ or $\mathcal{P}^-$ is from the space $L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_{\delta_1}(x_0)))$ for some $\beta \in \langle 2,\infty\rangle$, then $\mathcal{P} \in L^\beta(t_0-\varepsilon, t_0+\varepsilon, L^1(B_{\delta_1}(x_0)))$ and therefore $D_x^\gamma\mathcal{P}, D_x^\gamma\partial u/\partial t \in L^\beta(t_0-\varepsilon, t_0+\varepsilon; L^\infty(B_{\delta_1}(x_0))^3)$ as follows from Remark 3.

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Authors' addresses: *Zdeněk Skalák*, Institute of Hydrodynamics ASCR, Pod Paťankou 5, 166 12 Prague 6, Czech Republic, e-mail: `skalak@ih.cas.cz`; *Petr Kučera*, Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Prague 6, Czech Republic, e-mail: `kucera@mat.fsv.cvut.cz`.