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Jean-Michel Prou; Edouard Wagneur

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CONTROLLABILITY IN THE MAX–ALGEBRA[1]

JEAN–MICHEL PROU AND EDOUARD WAGNEUR

We are interested here in the reachability and controllability problems for DEDS in the max-algebra. Contrary to the situation in linear systems theory, where controllability (resp observability) refers to a (linear) subspace, these properties are essentially discrete in the max-linear dynamic system. We show that these problems, which consist in solving a max-linear equation lead to an eigenvector problem in the min-algebra. More precisely, we show that, given a max-linear system, then, for every natural number $k \geq 1$, there is a matrix Γ_k whose min-eigenspace associated with the eigenvalue 1 (or min-fixed points set) contains all the states which are reachable in k steps. This means in particular that if a state is not in this eigenspace, then it is not controllable. Also, we give an indirect characterization of Γ_k for the condition to be sufficient. A similar result also holds by duality on the observability side.

1. INTRODUCTION

An important stream of the literature on Discrete Event Dynamic Systems deals with the so-called max-*algebra* model. The theory developed so far, aims at the derivation of results which parallel those of classical deterministic automatic control theory, when the dynamics of the system is given by a set of *linear* equations, with parameters in the idempotent semiring $\underline{\mathbb{R}}= (\mathbb{R} \cup \{-\infty\}, \max, +)$. The aim is the development of a theory which, expectedly, will play a role similar to that of module theory for classical linear systems. The reader is referred to [2, 3, 4, 6], and [9] for basic issues and results.

We recall that a discrete event dynamic system is usually written in the max-algebra as:

$$x(k+1) = A \cdot x(k) \oplus B \cdot u(k+1), \tag{1}$$

where:

$x(k) \in \underline{\mathbb{R}}^n$ whose ith row $x_i(k)$ stands for the time when event i occurred for the kth time,

$u(k) \in \underline{\mathbb{R}}^m$ is the kth input (control) into the system,

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A, and B are matrices of the appropriate sizes,

$\oplus$ is the max operator, and

matrix multiplication is meant in the max-algebra sense:
$\sum_{j=1}^{n} a_{ij}x_j = \max_{1 \le j \le n}\{a_{ij} + x_j\}$ in the classical notation.

Since we are also going to deal with the min operator, we will use a notation which conforms more to that of classical lattice theory, than that of linear algebra. Thus we rewrite (1) as:

$$x(k+1) = A \cdot x(k) \vee B \cdot u(k+1), \tag{2}$$

where:
- $\vee$ stands for the max operator, and
- $A \cdot x$ is now expended as $\bigvee_{j=1}^{n} a_{ij}x_j$.

The aim of this paper is to show how the reachability problem in the max-algebra gives raise to an eigenvector problem in the min-algebra, and also to show how this problem may have a simple solution in some cases.
In section two below, we recall the basic definitions: independence, basis, and also some well-known results necessary to the understanding of the paper. In section three, we show how, for a given $m \times n$ matrix A, the semimodule of those vectors z such that $A \cdot x = z$ has a solution is a subset of the set of min-eigenvectors of a square matrix of size m which is easily computed from A. In Section 4, these results are applied to the controllability of a max-linear system such as (1) above, and, in Section 5, to their observability.

2. NOTATIONS, BASIC DEFINITIONS AND RESULTS

We assume that the semiring of scalars is the set of real numbers enlarged to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. The max and min operators, written $\vee$, and $\wedge$ respectively define a complete lattice structure on $\overline{\mathbb{R}}$. The least element $-\infty$ is written $\underline{0}$. It is the neutral element of $\vee$. Similarly, the largest element ∞, written $\overline{0}$ is the neutral element of $\wedge$. Usual addition, written multiplicatively, makes $(\overline{\mathbb{R}}, \vee, \wedge, \cdot)$ a l-group (lattice ordered group, [1]). The neutral element of $\cdot$ is written 1 (this corresponds to the real number 0) and the 'multiplication' symbol $\cdot$ will usually be omitted. Also $\forall a \in \mathbb{R}$, $a\underline{0} = \underline{0}$, $a\overline{0} = \overline{0}$, and $\underline{0}\,\overline{0} = \underline{0}$, $\overline{0}\,\underline{0} = \overline{0}$ by convention. In order to avoid any ambiguity, we will write δ for the number 1.

The operations $\vee$ and $\wedge$ are extended componentwise in a natural way to $\overline{\mathbb{R}}^n$, and, similarly, the external multiplication $\overline{\mathbb{R}} \times \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, $(\lambda, x) \mapsto \lambda x$. The universal bounds $(\underline{0}, \ldots, \underline{0})$, and $(\overline{0}, \ldots, \overline{0})$ will also be written $\underline{0}$ and $\overline{0}$, respectively. Ambiguity will generally be eliminated from the context.
Then $(\overline{\mathbb{R}}^n, \vee)$, and $(\overline{\mathbb{R}}^n, \wedge)$ are semimodules over the ℓ-group $\overline{\mathbb{R}}$. These structures are easily extended to more general ℓ-group semimodules (subsemimodules of $\overline{\mathbb{R}}^n$).

Let $X \subset \overline{\mathbb{R}}^n$. We write $M_X^{\vee}$ for the max-linear span of X, i.e. $M_X^{\vee}$ is the set of all finite $\vee$-linear combinations $x = \bigvee_{i=i_1}^{i_k} \lambda_i x_i$, $x_i \in X$, $\lambda_i \in \mathbb{R}$.

A similar definition holds for the min-linear span $M_X^{\wedge}$ of X.

Also X is $\vee$-*independent* if

$$\forall Y \subset X\,,\ x \in X \setminus Y \ \Rightarrow\ M_Y^\vee \bigcap M_x^\vee = \{0\}$$

$\wedge$-independence is defined similarly. A set which is $\vee$-independent will generally not be $\wedge$-independent, and conversely.

If X is $\vee$-independent, then X is a *basis* of $M_X^\vee$.

There are two matrix multiplications, defined as follows:

$$A \cdot B = \bigvee_{j=1}^{n} a_{ij} b_{jk}\,, \quad \text{and} \quad A \cdot' B = \bigwedge_{j=1}^{n} a_{ij} b_{jk}. \tag{3}$$

We will write $A^{\vee k}$, and $A^{\wedge k}$ for the kth power of A in the max and min algebra, respectively. Also $A^{\vee *} = \bigvee_{k \geq 0} A^{\vee k}$, and $A^{\wedge *} = \bigwedge_{k \geq 0} A^{\wedge k}$, where $A^{\vee 0} = \underline{I}$ (resp., $A^{\wedge 0} = \overline{I}$) the identity matrix in the max(min)-algebra.

As a linear operator, and w.r. to the $\cdot$ multiplication, an $m \times n$ matrix A is a map $\overline{\underline{I\!R}}^n \to \overline{\underline{I\!R}}^m$. Its columns generate a $\vee$-semimodule $\mathrm{Im}^\vee A$, whose dimension is given by the number of $\vee$-independent columns of A. A similar statement holds when A operates on $x \in \overline{\underline{I\!R}}^n$ by (left) $\cdot'$ multiplication ($\mathrm{Im}^\wedge A$ is a $\wedge$-semimodule). Since $\mathrm{Im}^\vee A$ ($\mathrm{Im}^\wedge A$) is the set of max(min)-linear combinations of the columns of A, we may also write $\mathrm{Im}^\vee A = M_A^\vee$ ($\mathrm{Im}^\wedge A = M_A^\wedge$).

The free $\vee$-semimodule over n generators is generated by $(e_i)_{i=1}^n$, with $e_i = (\delta_{i1}, \ldots, \delta_{in})$, where δ_{ij} is the Kronecker symbol. $(e_i)_{i=1}^n$ will be referred to as the canonical basis of $\overline{\underline{I\!R}}^{\vee n}$.

Given a basis X, the $\vee$-semimodule $M_X^\vee$ (in particular $\mathrm{Im}^\vee A$, for any matrix A) is partially ordered by $\leq$, where $x \leq y \iff x \vee y = y$. Also it is well-known that a semimodule morphism φ is isotone, i. e. $x \leq y \ \Rightarrow \varphi(x) \leq \varphi(y)$.

There is a duality between $\vee$-semimodules, and $\wedge$-semimodules. For any statement about $\vee$-semimodules, there is a similar statement for $\wedge$-semimodules.

3. SOLUTION TO EQUATIONS AND EIGENVECTORS

For a given $m \times n$ matrix $A = (a_{ij})$, let A^- stand for the matrix (a_{ij}^{-1}), and A^{-T} for its transpose. Then A^{-T} is called the residuated matrix of A. Consider the equation

$$A \cdot y = x \tag{4}$$

We have: $\bigvee_{j=1}^n a_{ij} y_j = x_i\,,\ i = 1\,, \ldots, m.$
Hence $\quad a_{ij} y_j \leq x_i\,,\ i = 1, \ldots, m\,,\ j = 1, \ldots, n, \quad$ i.e

$$y_j \leq a_{ij}^{-1} x_i\,,\ i = 1, \ldots, n\,,\ j = 1, \ldots, m, \text{ or, } y_j \leq \bigwedge_{i=1}^{n} a_{ij}^{-1} x_i\,,\ j = 1, \ldots, n.$$

Clearly $A^{-T} \cdot' x$ is a solution to $A \cdot y = x$ iff $A \cdot (A^{-T} \cdot' x) = x$. Let $A^\clubsuit$ stand for $(A \cdot A^{-T})$. We have:

$$(A^\clubsuit \cdot' x)_i = \bigwedge_{k=1}^{n} \Big(\bigvee_{j=1}^{m} a_{ij} a_{kj}^{-1} x_k \Big), \quad \text{and} \quad \Big(A \cdot (A^{-T} \cdot' x) \Big)_i = \bigvee_{j=1}^{m} \Big(\bigwedge_{k=1}^{n} a_{ij} a_{kj}^{-1} x_k \Big)$$

$$= \Big(\bigwedge_{k_1=1}^{n} a_{i1}a_{k_11}^{-1}x_{k_1}\Big) \vee \ldots \vee \Big(\bigwedge_{k_m=1}^{n} a_{im}a_{k_mm}^{-1}x_{k_m}\Big) = \bigwedge_{k_1,k_2\ldots k_m} \Big(\bigvee_{j=1}^{m} a_{ij}a_{k_jj}^{-1}x_{k_j}\Big)$$

by distributivity.

We distinguish the case where all the k_j take simultaneously the same value. Let then $\mathcal{N} = \{1, 2, \ldots n\}$ and write Δ_m for the diagonal in $\mathcal{N}^m$, and Σ_m for $\mathcal{N}^m \setminus \Delta_m$. When all the k_j have the same value, then the vector $\overline{k} \in \mathcal{N}^m$ belongs to Δ_m. Otherwise, we have $\overline{k} \in \Sigma_m$. Then, we may write:

$$\bigvee_{j=1}^{m}\Big(\bigwedge_{k=1}^{n} a_{ij}a_{kj}^{-1}x_k\Big) = \Big(\bigwedge_{\overline{k}\in\Delta_m}\Big(\bigvee_{j=1}^{m} a_{ij}a_{k_jj}^{-1}x_{k_j}\Big)\Big) \wedge \Big(\bigwedge_{\overline{k}\in\Sigma_m}\Big(\bigvee_{j=1}^{m} a_{ij}a_{k_jj}^{-1}x_{k_j}\Big)\Big)$$

But $\bigwedge_{\overline{k}\in\Delta_m}\big(\bigvee_{j=1}^{m} a_{ij}a_{k_jj}^{-1}x_{k_j}\big) = \bigwedge_{k=1}^{n}\big(\bigvee_{j=1}^{m} a_{ij}a_{kj}^{-1}x_k\big) = A^{\clubsuit} \cdot' x$.

Let $(R_A(x))_i$ stand for $\bigwedge_{\overline{k}\in\Sigma_m}\big(\bigvee_{j=1}^{m} a_{ij}a_{k_jj}^{-1}x_{k_j}\big)$. Then we have:

$$A \cdot (A^{-T} \cdot' x) = A^{\clubsuit} \cdot' x \wedge R_A(x) \tag{5}$$

The first part of the following statement is well-known (cf. [2] for instance), the proof of the second part is straightforward:

Proposition 3.1. $A \cdot (A^{-T} \cdot' x) = x \;\Rightarrow\; x$ is a $\wedge$-eigenvector of $A^{\clubsuit}$. Moreover, if $R_A(x) \geq A^{\clubsuit} \cdot' x$, then we have:
$A^{-T} \cdot' x$ is a solution to $A \cdot y = x$ iff x is a $\wedge$-eigenvector of $A^{\clubsuit}$.

The following example shows that the condition $R_A(x) \geq A^{\clubsuit} \cdot' x$ is necessary.

Example 3.2. Let $A = \begin{pmatrix} 1 & 1 & 1 & \underline{0} \\ \underline{0} & 1 & \underline{0} & 1 \\ \underline{0} & \underline{0} & 1 & 1 \end{pmatrix}$. Then $A^{-T} = \begin{pmatrix} 1 & \overline{0} & \overline{0} \\ 1 & 1 & \overline{0} \\ 1 & \overline{0} & 1 \\ \overline{0} & 1 & 1 \end{pmatrix}$.

Let $x = (2 \quad 6 \quad 8)^T$. It is easy to see that $A \cdot (A^{-T} \cdot' x) = (2 \quad 6 \quad 6)^T < A^{\clubsuit} \cdot' x = x$.

Note that the diagonal entries of $A^{\clubsuit}$ are all equal to 1. The following statement is well-known (cf. [3], pp. 63–64, for instance). An elementary proof is provided below (cf. also [8]).

Theorem 1. $(A^{\clubsuit}) \cdot' A = A$, and $(A^{\clubsuit})^{\wedge *} = A^{\clubsuit}$.

P r o o f. For the first statement, we have to show that $(A^{\clubsuit} \cdot' A)_{i\ell} = (A)_{i\ell} = a_{i\ell}$. For the second statement, it suffices to show that $((A^{\clubsuit})^{\wedge 2})_{i\ell} = (A^{\clubsuit})_{i\ell} = a_{i\ell}^{\clubsuit}$. The proofs are similar. We show the first statement, and leave the second to the reader.

We have:

$$(A^{\clubsuit} \cdot' A)_{i\ell} = \bigwedge_{k} a_{ik}^{\clubsuit} \cdot a_{k\ell} = \bigwedge_{k}\Big(\bigvee_{j} a_{ij}a_{kj}^{-1}\Big)a_{k\ell}$$

$$= \bigvee_j (\bigwedge_{k_j} a_{ij} a_{k_j j}^{-1} a_{k_j \ell}) = \bigvee_j (a_{i\ell} \wedge \bigwedge_{k_j \neq i} a_{ij} a_{k_j j}^{-1} a_{k_j \ell})$$

$$= a_{i\ell} \wedge \Big(\bigvee_j (\bigwedge_{k_j \neq i} a_{ij} a_{k_j j}^{-1} a_{k_j \ell}) \Big) = a_{i\ell} \wedge \Big(a_{i\ell} \vee \bigvee_{j \neq \ell} (\bigwedge_{k_j \neq i} a_{ij} a_{k_j j}^{-1} a_{k_j \ell}) \Big) = a_{i\ell}. \quad \square$$

From the matrix equality $A \cdot B = C$, we have, in particular, $M_C^{\vee} = \mathrm{Im}^{\vee} C \subset \mathrm{Im}^{\vee} A = M_A^{\vee}$. A similar interpretation also holds for the matrix equality $A \cdot' B = C$. Then by the definition of $A^{\clubsuit}$, we have $\mathrm{Im}^{\vee} A^{\clubsuit} \subset \mathrm{Im}^{\vee} A$. We have the following statement.

Proposition 3.3. $\mathrm{Im}^{\vee} A^{\clubsuit} \subset \mathrm{Im}^{\vee} A \subset \mathrm{Im}^{\wedge} A^{\clubsuit}$.

Proof. For the second inclusion, we first note that, by (5) above, we have $A \cdot (A^{-T} \cdot' x) \leq A^{\clubsuit} \cdot' x$. Also, $A^{\clubsuit} \cdot' x \leq x$, since the main diagonal of $A^{\clubsuit}$ consists of 1's. Now $x \in \mathrm{Im}^{\vee} A \iff [\exists\, y \text{ s.t. } A \cdot y = x] \Rightarrow x \leq A \cdot (A \cdot' x) \leq A^{\clubsuit} \cdot' x \leq x$. Hence $x \in \mathrm{Im}^{\vee} A \Rightarrow A \cdot (A \cdot' x) = A^{\clubsuit} \cdot' x = x \Rightarrow x \in \mathrm{Im}^{\wedge} A^{\clubsuit}$. $\square$

It is well-known that, for an irreducible matrix A the set of $\wedge$-eigenvectors of A w.r. to the eigenvalue 1 (or the set of $\wedge$-fixed points of A) is generated by the columns of $A^{\wedge *}$. Theorem 1 says more, namely that, since $A^{\clubsuit} \cdot' A^{\clubsuit} = A^{\clubsuit}$, then the set $X(A^{\clubsuit})$ of columns of $A^{\clubsuit}$ belongs to the set of $\wedge$-fixed points of this matrix. Moreover, by the first statement, we also have that $X(A)$ belongs to this set. Clearly, the equation $A \cdot y = x$ has a solution iff $x \in \mathrm{Im}^{\vee} A$. In particular, the first inclusion in Proposition 3.3 says that this holds true for all $x \in X(A^{\clubsuit})$, hence for all $x \in \mathrm{Im}^{\vee} A^{\clubsuit}$. The second inclusion states that, for every solution to $A \cdot y = x$, then $x \in \mathrm{Im}^{\wedge} A^{\clubsuit}$, i.e. x is a $\wedge$-fixed point of $A^{\clubsuit}$.

Note that the second inclusion in Proposition 3.3 is meant in the set-theoretic sense. $\mathrm{Im}^{\wedge} A^{\clubsuit}$ is an *inf*-semimodule, while $\mathrm{Im}^{\vee} A$ is a *sup*-semimodule.

Note also that our results also hold for the system $\bigvee_{i=1}^{p} (A_i \cdot y_i) = x$. Indeed let A stand for the concatenation of the matrices A_i : $A = [A_1 \ \ldots \ A_p]$, and $y = [y_1 \ \ldots \ y_p]^T$, then the system is exactly the same as in (4).

Also, all the above statements may be dualized, starting from $A \cdot' y = x$.

The interest of the second inclusion in Proposition 3.3 is that it substitutes an idenfication problem (here in the max-algebra) which may have a large number of parameters, say $n > m$ to a simpler one (here in the min-algebra) with m parameters. Of course, what we gain in complexity is lost in accuracy. More precisely, the kind of answer expected from the use of Proposition 3.3, is of negative type, e.g.
$x \notin \mathrm{Im}^{\wedge} A^{\clubsuit} \Rightarrow x \notin \mathrm{Im}^{\vee} A$, i.e. $A \cdot y = x$ has no solution.

The following examples illustrate the statement in Proposition 3.3.

Example 3.4. Let $x_i = (1,\, i,\, i^2)'$, $i = 1, 2, \ldots,$. It is easy to see that the x_i are $\vee$-independent.

Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 6 & 8 & 10 \end{pmatrix}$. Then $\dim(\mathrm{Im}^\vee A) = 5$, and $A^\clubsuit = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 10 & 5 & 1 \end{pmatrix}$.
By Theorem 1 $A^\clubsuit \cdot' A = A$, i. e, each of the 5 columns of A is a $\wedge$-linear combination of the 3 columns of $A^\clubsuit$. For example,

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \wedge 2 \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \wedge 4 \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}.$$

Now $X(A^\clubsuit)$ is $\vee$-dependent, since

$$\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \vee 5^{-1} \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}.$$

This illustrates the inclusion $\mathrm{Im}^\vee A^\clubsuit \subset \mathrm{Im}^\vee A$.

For the second inclusion in Proposition 3.3, let $x = \bigvee_i \lambda_i x_i \in \mathrm{Im}^\vee A$, with $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 2$, $\lambda_4 = 1$, $\lambda_5 = \delta$ (the integer 1). Then $x = \begin{pmatrix} 6 \\ 7 \\ 11 \end{pmatrix}$. Since x is a $\wedge$-fixed point of $A^\clubsuit$, we have $A^\clubsuit \cdot' x = x$, i. e.

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 10 & 5 & 1 \end{pmatrix} \cdot' \begin{pmatrix} 6 \\ 7 \\ 11 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} \wedge 7 \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \wedge 11 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \wedge 7 \wedge 11 \\ 11 \wedge 7 \wedge 11 \\ 16 \wedge 12 \wedge 11 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 11 \end{pmatrix}.$$

From Theorem 1, and Proposition 3.3 we would like to be able to recognize when $A^\clubsuit \cdot' x = x \Longrightarrow A \cdot (A^{-T} \cdot' x) = x$. We owe to Max plus [7] the following statement (however, since the result was stated without proof, and in a slightly different context, the authors assume full responsibility for any error in the proof). Let A be an arbitrary matrix.

Theorem 2. $\mathrm{Im}^\vee A^\clubsuit = \mathrm{Im}^\wedge A^\clubsuit \iff \mathrm{Im}^\vee A^\clubsuit$ is a lattice.

Proof. Clearly $\mathrm{Im}^\vee A^\clubsuit = \mathrm{Im}^\wedge A^\clubsuit \Longrightarrow \mathrm{Im}^\vee A^\clubsuit$ is a lattice.
Conversely, assume $\mathrm{Im}^\vee A^\clubsuit$ is a lattice. We show that $\mathrm{Im}^\wedge A^\clubsuit \subset \mathrm{Im}^\vee A^\clubsuit$.
Let $x \in \mathrm{Im}^\wedge A^\clubsuit$, then $x = A^\clubsuit \cdot' y$ for some $y \in \overline{\mathbb{R}}^n$, i. e. $x = \bigwedge_{j=1}^n a_j^\clubsuit y_j$, where $a_j^\clubsuit$ is column j of $A^\clubsuit$. Let $A_y^\clubsuit$ stand for the matrix with column j equal to $y_j a_j^\clubsuit$. Clearly, $\mathrm{Im}^\vee A_y^\clubsuit = \mathrm{Im}^\vee A^\clubsuit$, and, since this is a lattice, we have $x \in \mathrm{Im}^\vee A^\clubsuit$.
Since $\mathrm{Im}^\vee A^\clubsuit \subset \mathrm{Im}^\wedge A^\clubsuit$ by Proposition 3.3, this completes the proof. □

From Proposition 3.3 and Theorem 2, we have the following statements.

Corollary 3.5. If $\mathrm{Im}^{\vee} A^{\clubsuit}$ is a lattice, then $\mathrm{Im}^{\vee} A$ and $\mathrm{Im}^{\wedge} A^{\clubsuit}$ are lattices.

Corollary 3.6. Assume $\mathrm{Im}^{\vee} A^{\clubsuit}$ is a lattice then $A \cdot y = x$ has a solution iff x is a $\wedge$-fixed point of $A^{\clubsuit}$.

Theorem 2 gives an interesting property that the matrix $A^{\clubsuit}$ has to verify. But this property is not immediately recognizable just by looking at A. However, there are two interesting particular cases where this is possible. First recall that A is said to be $\vee$-invertible iff there exists a matrix B such that $B \cdot A = A \cdot B = \underline{I}$, the $\vee$ identity matrix (i. e. $\underline{I}$ has no entry $\overline{0}$. Note however that $\underline{I} \cdot \overline{0} = \overline{0}$).

Proposition 3.7. If A is a $\vee$-invertible matrix, then $\mathrm{Im}^{\vee} A = \mathrm{Im}^{\wedge} A^{\clubsuit} = \overline{\underline{I\!R}}^{m}$.

Proof. Clearly, if A is invertible, then $\mathrm{Im}^{\vee} A = \overline{\underline{I\!R}}^{m}$.
It is well-known that an invertible matrix A has the form $\Lambda \cdot P$, where Λ is a diagonal matrix, and P is a permutation matrix. Alternatively,
$a_{ij} \neq \underline{0} \Longrightarrow [\forall\, k \neq i\,,\ a_{kj} = \underline{0}, \text{and}\ \forall\, \ell \neq j\,,\ a_{i\ell} = \underline{0}]$. Then $a_{ij} = \underline{0} \Longrightarrow (A^{-T})_{ji} = \overline{0}$, and $a_{ij} \neq \underline{0} \Longrightarrow (A^{-T})_{ji} = a_{ij}^{-1}$. Hence $a^{\clubsuit}_{ik} = \bigvee_j a_{ij} a_{kj}^{-1} = \overline{\delta}_{ik}$, where $\overline{\delta}_{ik}$ is the Kronecker symbol in the *inf*-algebra, i. e.

$$\overline{\delta}_{ik} = \begin{cases} 1\,, & k = i\,, \\ \overline{0}\,, & \text{otherwise.} \end{cases}$$

It follows that $A^{\clubsuit} = \overline{I}$, the $\wedge$-identity matrix. Since $\overline{I} \cdot' \underline{0} = \underline{0}$, and $\overline{I}$ is the identity matrix in the min-algebra, we have $\mathrm{Im}^{\wedge} A^{\clubsuit} = \overline{\underline{I\!R}}^{m}$. It follows in particular that $\forall x \in \overline{\underline{I\!R}}^{m}\ :\ A \cdot (A^{-T} \cdot' x) = A \cdot' x = x$. □

Another case, where $\mathrm{Im}^{\vee} A = \mathrm{Im}^{\wedge} A^{\clubsuit}$ is when $n = 2$. More precisely, we have the following statement.

Proposition 3.8. (cf. [8]) For A of size $2 \times m$, we have $\mathrm{Im}^{\vee} A^{\clubsuit} = \mathrm{Im}^{\wedge} A^{\clubsuit}$.

Proof. We show that for A as in the statement of the Proposition, and $x \in \overline{\underline{I\!R}}^{2}$ arbitrary, we have

$$A^{\clubsuit} \cdot' x = A \cdot \left(A^{-T} \cdot' x\right).$$

$$A^{\clubsuit} = \begin{pmatrix} 1 & \bigvee_{j=1}^{m} a_{1j} a_{2j}^{-1} \\ \bigvee_{j=1}^{m} a_{2j} a_{1j}^{-1} & 1 \end{pmatrix} \quad \text{and} \quad A^{\clubsuit} \cdot' x = \begin{bmatrix} x_1 \wedge (\bigvee_{j=1}^{m} a_{1j} a_{2j}^{-1} x_2) \\ (\bigvee_{j=1}^{m} a_{2j} a_{1j}^{-1} x_1) \wedge x_2 \end{bmatrix}.$$

$$A^{-T} \cdot' x = \begin{bmatrix} a_{11}^{-1} x_1 \wedge a_{21}^{-1} x_2 \\ \cdots \\ a_{1j}^{-1} x_1 \wedge a_{2j}^{-1} x_2 \\ \cdots \\ a_{1m}^{-1} x_1 \wedge a_{2m}^{-1} x_2 \end{bmatrix} \quad \text{and} \quad \left(A \cdot (A^{-T} \cdot' x)\right)_i = \bigvee_{j=1}^{m} a_{ij} (a_{1j}^{-1} x_1 \wedge a_{2j}^{-1} x_2).$$

$$\text{Hence} \quad A \cdot (A^{-T} \cdot' x) = \begin{bmatrix} \bigvee_{j=1}^{m} (x_1 \wedge a_{1j} a_{2j}^{-1} x_2) \\ \bigvee_{j=1}^{m} (a_{2j} a_{1j}^{-1} x_1 \wedge x_2) \end{bmatrix} = \begin{bmatrix} x_1 \wedge (\bigvee_{j=1}^{m} a_{1j} a_{2j}^{-1} x_2) \\ (\bigvee_{j=1}^{m} a_{2j} a_{1j}^{-1} x_1) \wedge x_2 \end{bmatrix}.$$ □

Note that Proposition 3.8 states that, whatever the value of m, we investigate a square matrix of size two.

In the next section, we apply the statements above to the reachability and controllability problems in max-plus systems.

4. REACHABILITY AND CONTROLLABILITY IN DEDS

Consider a system (A, B, C) given by:

$$\begin{aligned} x(k+1) &= A \cdot x(k) \vee B \cdot u(k+1), \\ y(k+1) &= C \cdot x(k+1)\,, \end{aligned} \tag{6}$$

where $x_i(k)$ is the date when the kth occurrence of event i begins, $i = 1, \ldots, m$, and $u(k)$ that of the kth control (input) vector applied to the system.
The first equation means that, the $(k+1)$th occurrence of event i begins when all the kth occurrences of the events have been completed ($\bigvee_{j=1}^{m} a_{ij} x_i(k)$, with a_{ij} the duration of an event j which immediately precedes i) and the input sequence which affects i, $\bigvee_{\ell=1}^{n} b_{i\ell} u_\ell$ have been completed (where u_ℓ is the date of the release of input ℓ, and $b_{i\ell}$ is the time delay it takes before i can be affected, also, $b_{i\ell} = \underline{0}$ if ℓ does not immediately precede i). When the system is represented by a timed event graph, the state refers to the internal transitions, whereas the control refers to the input transitions. In both cases, the delays stand for the time of the transition firings (cf. [2]).

In condensed notation, the first equation may be written as:

$$x(k+1) = [B|A] \cdot \begin{pmatrix} u(k+1) \\ x(k) \end{pmatrix} \tag{7}$$

and expended as:

$$x(k+1) = [\Gamma_{k-i} | A^{\vee(k+1-i)}] \cdot \begin{pmatrix} u_{k-i} \\ x(i) \end{pmatrix} \tag{7i}$$

In particular, for $i = 1$:

$$x(k+1) = [\Gamma_{k-1} | A^{\vee k}] \cdot \begin{pmatrix} u_{k-1} \\ x(1) \end{pmatrix}, \tag{8}$$

where $\Gamma_{k-i} = [B | A \cdot B | \ldots | A^{\vee(k-i)} \cdot B]$, and $u_{k-i} = (u(k+1)\, u(k) \ldots u(i+1))^t$.

Assume that (7i) has a solution, i.e. given $x = x(k+1)$, there is an admissible initial condition $x(i)$ (the "initial condition" at which the sequence of events occurred for the ith time) and an input sequence $u(i+1)$, $u(i+2), \ldots u(k+1)$, such that (7i) holds. This means in particular that, starting from $x(i)$ we can reach $x(k+1)$ in $k+1-i$ steps. But then, starting from $x(i+1) = Ax(i) \vee Bu(i+1)$, we can reach $x(k+1)$ in $k-i$ steps, a.s.o. i.e $x(i)$, $x(i+1), \ldots, x(k)$ are all admissible initial conditions. Clearly, $x(i) \leq x(i+1)$, hence the set AIC of admissible initial conditions is a poset. Also the set of input control vectors ICV is a poset.

Following [5], (and with a slight modification), we state the reachability problem as:
Given $x \in \overline{\mathbb{R}}^m$, can we find a set $AIC(x) \neq \emptyset$ of admissible initial conditions, and a set $ICV(x) \neq \emptyset$ of input control vectors such that (7i) holds for some i $(0 \leq i \leq k)$?

Note that, for $i = 0$, (7i) yields an initial condition $x(0)$ and a sequence of control vectors u_k, with first control $u(1)$. The interpretation for $x(0)$ is the date at which no transition has been fired. Since $t = 0$ (the real number 0) may be the date of the first firing of the transitions, it is reasonable to assume $x(0) = \underline{0}$. In this case, since $A^{\vee k+1}(\underline{0}) = A(\underline{0}) = \underline{0}$, the system becomes

$$x(k+1) = \Gamma_k \cdot u_k. \tag{9}$$

In classical linear system theory, controllability and reachability coincide. Clearly, in the max algebra setting, the controllability problem as it is usually stated in linear system theory does not make sense. Indeed $A \cdot x = \underline{0} \Rightarrow a_{ij}x_j = \underline{0},\ \forall i,\ j$.

Definition 4.1. 1. We say that the state x is *reachable* if $AIC(x) \setminus \{x\} \neq \emptyset$.
2. We say that the state x is *controllable* if $\underline{0} \in AIC(x)$.

Note that every controllable state is reachable, whereas the converse doesn't hold. Also, the chains in $AIC(x)$ need not all have the same length. The properties of this set are not known, and a systematic study would be appropriate.

Remark 4.2. Note that, in classical system theory, controllability is meant as *system* controllability, wereas here we can only talk about *state* controllability, since controllability is a *discrete* property of max-linear systems, while it is a property of a *subspace* in classical linear systems theory.

From Proposition 3.3 above, we have the following.

Theorem 3. A necessary condition for x to be controllable is that it is a fixed point of

$$\Gamma_k^{\clubsuit} : \underline{\overline{\mathbb{R}}}^{m(k+1)} \longrightarrow \underline{\overline{\mathbb{R}}}^{m(k+1)},\ x \mapsto \Gamma_k^{\clubsuit} \cdot' x\,,$$

Moreover, if $\mathrm{Im}^{\vee}\Gamma_k^{\clubsuit}$ is a lattice, then this condition is also sufficient.

Proof. x controllable $\Rightarrow \underline{0} \in AIC(x) \Rightarrow AIC(x) \setminus \{x\} \neq \emptyset \Rightarrow \exists\, u_k = u$ s.t. $x = \Gamma_k \cdot u \Rightarrow x \in \mathrm{Im}^{\vee}_{\Gamma_k} \Rightarrow x \in \mathrm{Im}^{\wedge}\Gamma_k^{\clubsuit} \Rightarrow \exists\, y$ s.t. $x = \Gamma_k^{\clubsuit} \cdot' y \Rightarrow \Gamma_k^{\clubsuit} \cdot' x = \Gamma_k^{\clubsuit} \cdot' (\Gamma_k^{\clubsuit} \cdot' y) = (\Gamma_k^{\clubsuit} \cdot' \Gamma_k^{\clubsuit}) \cdot' y = \Gamma_k^{\clubsuit} \cdot' y = x$.
The sufficient condition is given by Corollary 3.6. □

The following example illustrates our discussion on initial conditions.

Example 4.3. $A = \begin{pmatrix} \delta & \underline{0} \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} \delta \\ 3 \end{pmatrix}$, $C = (\underline{0} \;\; \delta)$, where δ stands for the real number 1. We want to know if $x = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$ is reachable.

We have: $[\Gamma_o|A] = [B|A] = \begin{pmatrix} \delta & \delta & \underline{0} \\ 3 & 3 & 2 \end{pmatrix}$, $[\Gamma_o|A]^\clubsuit = \begin{pmatrix} \delta & \delta & \underline{0} \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} \delta^{-1} & 3^{-1} \\ \delta^{-1} & 3^{-1} \\ \overline{0} & 2^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 2^{-1} \\ \overline{0} & 1 \end{pmatrix}$.

We have $11\begin{pmatrix} 1 \\ \overline{0} \end{pmatrix} \wedge 14\begin{pmatrix} 2^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$, hence $x \in \mathrm{Im}^\wedge[\Gamma_o|A]^\clubsuit$.

But for $m = 2$, by Proposition 3.8, we have $\mathrm{Im}^\vee[\Gamma_o|A] = \mathrm{Im}^\wedge[\Gamma_o|A]^\clubsuit$. Thus x is reachable, and $\begin{pmatrix} u(2) \\ x(1) \end{pmatrix} = \begin{pmatrix} \delta^{-1} & 3^{-1} \\ \delta^{-1} & 3^{-1} \\ \overline{0} & 2^{-1} \end{pmatrix} \cdot' \begin{pmatrix} 11 \\ 14 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 12 \end{pmatrix}$, i.e. x is reachable from $x(1) = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$ in one step with control $u(2) = 10$.

Now $B^{-T} \cdot' x(1) = 9$, and with $u_1 = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$, we have:

$$\Gamma_1 \cdot u_1 = [B|A \cdot B] \cdot u_1 = \begin{pmatrix} \delta & 2 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 9 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix},$$

and x is reachable in two steps from $\underline{0}$, with controls $u(0) = 9$, $u(1) = 10$, i.e. x is controllable.

Our last example shows that existence of a solution to (8) does not imply that of a solution to (9).

Example 4.4. We consider the same problem except that now $B = \begin{pmatrix} \delta \\ 2 \end{pmatrix}$. We get the same reachability result with initial condition $x(1) = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$, and $u_1 = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$. However $B \cdot u(1) = \begin{pmatrix} 10 \\ 11 \end{pmatrix} \neq x(1)$, and $\Gamma_1 \cdot u_1 = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$.

5. OBSERVABILITY

The observability problem for DEDS may be stated just dually to the reachability problem. That is, given a sequence of observed states $y(0), y(1), \ldots, y(k)$ is it possible to recover the initial state $x(0)$ together with the sequence of controls $u(1), \ldots u(k) \in \overline{\mathbb{R}}^p$, such that, as in (6) above, we have: $y(0) = C \cdot x(0)$, $y(1) = C \cdot x(1) = C \cdot A \cdot x(0) \vee C \cdot B \cdot u(1)$, $\ldots$ $y(k) = C \cdot A \cdot x(k-1) \vee C \cdot B \cdot u(k)$
In matrix form:

$$\begin{bmatrix} y(0) \\ y(1) \\ \cdot \\ \cdot \\ y(k) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{k-1} \end{bmatrix} \cdot x(0) \vee \begin{pmatrix} \underline{0} & \ldots & \ldots & \underline{0} \\ CB & \underline{0} & \ldots & \underline{0} \\ \ldots & \ldots & \ldots & \ldots \\ CA^{k-2}B & CA^{k-3}B & \ldots & CB \end{pmatrix} \cdot \begin{bmatrix} u(1) \\ u(2) \\ \cdot \\ \cdot \\ u(k) \end{bmatrix} \tag{10}$$

Let

$$Y_k = \begin{bmatrix} y(0) \\ y(2) \\ \cdot \\ \cdot \\ y(k) \end{bmatrix}, \quad \mathcal{O}_k = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{k-1} \end{bmatrix},$$

$$\mathcal{K}_k = \begin{pmatrix} \underline{0} & \dots & \dots & \underline{0} \\ CB & \underline{0} & \dots & \underline{0} \\ \dots & \dots & \dots & \dots \\ CA^{k-2}B & CA^{k-3}B & \dots & CB \end{pmatrix}, \quad U_k = \begin{bmatrix} u(1) \\ u(2) \\ \cdot \\ \cdot \\ u(k) \end{bmatrix},$$

then (10) may be rewritten as:

$$Y_k = \mathcal{O}_k \cdot x(0) \vee \mathcal{K}_k \cdot U_k$$

or

$$Y_k = [\mathcal{O}_k | \mathcal{K}_k] \cdot \begin{bmatrix} x(0) \\ U_k \end{bmatrix}. \tag{11}$$

Following the general ideas of controllability (see also [5]), a system will be said to be observable if, for any given observed sequence of outputs, and knowing the associated sequence of inputs, we can find the state (initial condition) which, for this sequence of inputs yields the observed sequence of outputs.

Definition 5.1. We say that the system (A, B, C) is *observable* in k steps if, for given input and output sequences U_k , Y_k respectively, equation (11) above has a solution.

Reinterpreting the results of Section 3 above, we may state our necessary condition for observability.

Theorem 4. In order that Y_k be observable, it is necessary that it is a fixed point of

$$[\mathcal{O}_k, \mathcal{K}_k]^{\clubsuit} : \overline{\underline{\mathbb{R}}}^{mk} \longrightarrow \overline{\underline{\mathbb{R}}}^{mk}, \; Y_k \mapsto [\mathcal{O}_k, \mathcal{K}_k]^{\clubsuit} \cdot' Y_k ,$$

Moreover if $\mathrm{Im}^{\vee} [\mathcal{O}_k, \mathcal{K}_k]^{\clubsuit}$ is a lattice, then this conditon is also sufficient.

6. CONCLUSION

In this paper we have shown that the reachability problem in max-linear systems may be stated either as a problem of finding $\mathrm{Im}^{\vee}\Gamma_k$ in the max-algebra, with Γ_k an $m \times n$ matrix with $n \gg m$, or as an eigenvector problem in the min-algebra, like in [5], where the authors have to check, for each k if $n \gg m$ equations hold. Here, we reduce this problem to an eigenvector problem of size m for each k, as long as we get a negative answer.

Note that the reachability/controllability problem is a discrete problem in the max- (or min-)algebra, since from two given states, only one may be reachable (or controllable). Also, we state a necessary and sufficient condition for a matrix A , in order that:

$$\forall x\,,\; A\cdot(A^{-T})\cdot' x = A\cdot(A^{-T}\cdot' x),$$

which in turns yields a necessary and sufficient condition for the system $A\cdot y = x$ to have a solution in terms of the columns of the matrix $A^{\clubsuit}$.

All our reachability/controllability results are dually interpreted on the observability side with small changes.

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Jean-Michel Prou, IRCyN-UMR659 CNRS, Ecole Centrale, Université and Ecole des Mines de Nantes, Institut de Recherche en Cybernétique de Nantes BP 92101, 44321, Nantes, Cedex 03. France.
e-mail: prou@lan10.ec-nantes.fr

Edouard Wagneur, Ecole des Mines de Nantes, 4 rue Alfred Kastler, BP 20722, 44307, Nantes, Cedex 3. France.
e-mail: wagneur@auto.emn.fr