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## EXPONENTIAL RATES FOR THE ERROR PROBABILITIES IN SELECTION PROCEDURES

FRIEDRICH LIESE AND KLAUS J. MIESCKE

For a sequence of statistical experiments with a finite parameter set the asymptotic behavior of the maximum risk is studied for the problem of classification into disjoint subsets. The exponential rates of the optimal decision rule is determined and expressed in terms of the normalized limit of moment generating functions of likelihood ratios. Necessary and sufficient conditions for the existence of adaptive classification rules in the sense of Rukhin [11] are given. The results are applied to the problem of the selection of the best population. Exponential families are studied as a special case, and an example for the normal case is included.

### 1. INTRODUCTION

The Bayes and maximum error probabilities in the problem of testing a simple null hypothesis versus a simple alternative, or more generally in a multiple decision problem, tend to zero with an exponential rate of convergence for increasing sample size. Pioneering work has been done in the paper by Chernoff [3]. Here and in papers by Krafft and Plachky [6], Krafft and Puri [7] and several other authors the i. i. d. case is treated. A more general version of the so called Chernoff theorem can be found in Vajda [15].

The classification problem is a multiple decision problem with given distributions  $P_1, \dots, P_k$  of the  $k$  populations. In the first part of this paper we deal with a more general question. Suppose we are given a family of distributions  $Q_\vartheta$ ,  $\vartheta \in \Theta$ , and  $\Theta = \Theta_1 \cup \dots \cup \Theta_k$  is a partition of the set  $\Theta$ . After taking an observation we want to decide to which of the families  $(Q_\vartheta)_{\vartheta \in \Theta}$ , the distribution of the data belongs to. We are interested in asymptotic results. Therefore it is assumed that a sequence  $Q_{n,\vartheta}$  is given. The special case of i. i. d. observations corresponds to  $Q_{n,\vartheta} = Q_\vartheta \times \dots \times Q_\vartheta = Q_\vartheta^n$ . In general the asymptotic behavior of the  $Q_\vartheta^n$  will be described through the requirement that the normalized logarithms of the moment generating functions converge to some function which is automatically convex. In generalization of the results of Krafft and Puri [7] we calculate the exponential rate of the error probabilities of the minimax decision rule for the classification into subsets. For this problem the distributions  $Q_{n,\vartheta}$  are not completely known in many situations. In general they depend also on some nuisance parameter. The question

which naturally arises is whether there exists a decision rule not depending on  $\alpha$ , which achieves the same exponential rate of the error probabilities as given by the optimum decision rule for known  $\alpha$ . This problem of the existence of such adaptive decisions for the problem of classification is studied in the second part of the paper. The results there are generalizations of Rukhin [11].

Suppose  $P_{\eta_1}, \dots, P_{\eta_k}$  are distributions depending on some real valued parameter  $\eta$ , for populations  $\pi_1, \dots, \pi_k$ , respectively. The population with the largest  $\eta$ -value is called the best population. First we assume that the set  $\{\eta_1, \dots, \eta_k\}$  is known but we do not know which population belongs to which  $\eta$ -value. Taking observations from each population we want to select the population with the largest  $\eta$ -value. If  $\Theta$  is the set of all permutations of  $(1, \dots, k)$  and  $\Theta_i \subset \Theta$  the set of permutations such that the population which is at position  $i$  has the largest  $\eta$ -value, then we can see that the problem of selecting the best population can be reduced to the problem of classification into subsets. In this way we apply the results from Sections 2 and 3 to the selection problem. Especially we investigate the case where independent observations of size  $n_i$  are taken from population  $\pi_i$ . Explicit expressions are given for the exponential rate of the probability of selecting a false population. If the distributions  $P_{\eta_i}$  belong to some exponential family we compare the exponential rate of the best selection procedure for known  $\eta_i$  with the rate of the so called natural selection rule based on sample means of the statistic which generates the exponential family. It turns out that the natural selection rule is asymptotically optimal w. r. t. the exponential rates iff the two smallest sample sizes  $n_{i_1}$  and  $n_{i_2}$ , say, are asymptotically the same, i.e. it holds  $n_{i_1}/n_{i_2} \rightarrow 1$ . Furthermore, for populations from an exponential family this condition is necessary and sufficient for the existence of an adaptive selection rule which in this case is the natural rule based on the sample means.

## 2. CLASSIFICATION INTO SUBSETS

Let  $\Theta$  be a finite nonempty set with  $N$  elements, and let  $\Theta_1, \dots, \Theta_k$  be a decomposition into disjoint subsets. We write  $\vartheta_1 \equiv \vartheta_2$  if  $\vartheta_1$  and  $\vartheta_2$  belong to the same subset  $\Theta_i$ . For any  $\vartheta \in \Theta$  we set  $\langle \vartheta \rangle = i$  if  $\vartheta \in \Theta_i$ . Hence  $\vartheta_1 \equiv \vartheta_2$  iff  $\langle \vartheta_1 \rangle = \langle \vartheta_2 \rangle$ .

Suppose we are given a family of distributions  $Q_{\vartheta}, \vartheta \in \Theta$ , defined on the sample space  $(\mathcal{X}, \mathcal{A})$ . We will study the following problem. Taking an observation  $x \in \mathcal{X}$  we have to decide to which family  $(Q_{\vartheta})_{\vartheta \in \Theta_i}$  the distribution of the corresponding distribution belongs to.  $\mathcal{D} = \{1, \dots, k\}$  is our decision space and a randomized decision  $q = (q_1, \dots, q_k)$  is a vector of measurable functions  $q_i : \mathcal{X} \rightarrow [0, 1]$  such that  $\sum_{i=1}^k q_i(x) = 1$  for every  $x \in \mathcal{X}$ .  $q_i(x)$  is the probability to decide for the subset  $\Theta_i$  if  $x$  is observed. We use the 0-1 loss function  $L : \Theta \times \mathcal{D} \rightarrow [0, \infty)$  defined by

$$L(\vartheta, i) = \begin{cases} 0 & \text{if } \langle \vartheta \rangle = i \\ 1 & \text{else.} \end{cases}$$

The risk of decision  $q$  is then given by

$$R(\vartheta, q) = 1 - \int q_{\langle \vartheta \rangle} dQ_{\vartheta}.$$

We examine the asymptotic behavior of the risk for increasing sample size. To this end, we assume that a sequence of experiments  $E_n = (\mathcal{X}_n, \mathcal{A}_n, Q_{n,\vartheta}, \vartheta \in \Theta)$ ,  $n = 1, 2, \dots$  is given. Furthermore let  $\pi(\vartheta), \vartheta \in \Theta$ , be positive prior weights. Given a sequence  $q_n = (q_{n,1}, \dots, q_{n,k})$  of decisions we introduce the maximum risk  $R_m(q_n)$  by  $R_m(q_n) = \max_{\vartheta \in \Theta} R(\vartheta, q_n)$  and the Bayes risk by  $R_\pi(q_n) = \sum_{\vartheta \in \Theta} \pi(\vartheta)R(\vartheta, q_n)$ . Set  $R_n^m = \min_{q_n} R_m(q_n)$ ,  $R_n^\pi = \min_{q_n} R_\pi(q_n)$ . The question now is how fast both  $R_n^m$  and  $R_n^\pi$  tend to zero if  $n \rightarrow \infty$ .

This problem was studied in the i. i. d. case  $(\mathcal{X}^n, \mathcal{A}^n, Q_\vartheta^n)$  by Krafft and Puri [7] if the subsets  $\Theta_i$  are singletons. If furthermore  $k = 2$ , i. e.  $\Theta = \{\vartheta_1\} \cup \{\vartheta_2\}$  then we have a simple hypotheses testing problem.

In the i. i. d. case the exponential rate of convergence to zero of  $R_n^m, R_n^\pi$  was obtained by Chernoff [3, 4], by Krafft and Plachky [6] and other authors. A Chernoff type theorem for an increasing sequence of sub- $\sigma$ -algebras, i. e.  $E_n = (\mathcal{X}, \mathcal{A}_n, Q_{n,\vartheta}, \vartheta \in \Theta)$  where  $Q_{n,\vartheta}$  is the restriction of  $Q_\vartheta$  to  $\mathcal{A}_n$ , was obtained in Vajda [15].

Similar as in the above mentioned papers we use the concept of Hellinger integral to characterize relations between different distributions. Let  $P, Q$  be distributions defined on  $(\mathcal{X}, \mathcal{A})$  and suppose  $P$  and  $Q$  are equivalent ( $P \sim Q$ ), i. e.  $P \ll Q$  and  $Q \ll P$ . Then  $H_s(P, Q) = \int \left(\frac{dP}{dQ}\right)^s dQ$  is called *Hellinger integral of order  $s$* .

We summarize some well-known properties of Hellinger integrals important in the sequel. For proofs we refer to Liese and Vajda [9]. It holds  $0 \leq H_{s_1}(P, Q) \leq 1 \leq H_{s_2}(P, Q) < \infty$  if  $s_1 \in [0, 1], s_2 \notin [0, 1]$ . Furthermore  $H_0(P, Q) = H_1(P, Q) = 1$ .  $G(s) = \ln H_s(P, Q)$  is a convex function taking values in  $(-\infty, +\infty]$ .

We characterize the asymptotic behavior of the  $Q_{n,\vartheta}$  from the experiments  $E_r$  with the help of Hellinger integrals. The next assumptions are fundamental for all further investigations.

**Assumption 1.** All distributions in the sequence of experiments

$$E_n = (\mathcal{X}_n, \mathcal{A}_n, Q_{n,\vartheta}, \vartheta \in \Theta)$$

are equivalent ( $Q_{n,\vartheta_1} \sim Q_{n,\vartheta_2}$ , for every  $\vartheta_1, \vartheta_2 \in \Theta$ ) and there exists a sequence  $c_n \rightarrow \infty$  so that for every  $\vartheta_1, \vartheta_2 \in \Theta, -\infty < s < \infty$

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(Q_{n,\vartheta_1}, Q_{n,\vartheta_2}) = G(\vartheta_1, \vartheta_2, s) \tag{1}$$

exists, takes values in  $(-\infty, +\infty]$ , and for  $\vartheta_1 \neq \vartheta_2, G(\vartheta_1, \vartheta_2, s)$  is not identical zero in the interval  $0 \leq s \leq 1$ .

Note that by the convexity of  $G(\vartheta_1, \vartheta_2, s)$  the set  $\{s : G(\vartheta_1, \vartheta_2, s) < \infty\}$  is an interval with the interior, say,  $(a_1, a_2)$ . Furthermore, by  $0 \leq H_s(Q_{n,\vartheta_1}, Q_{n,\vartheta_2}) \leq 1$  for  $0 \leq s \leq 1$  it holds  $G(\vartheta_1, \vartheta_2, s) < \infty$  for  $0 \leq s \leq 1$  if Assumption 1 is fulfilled.

**Assumption 2.** For every  $\vartheta_1, \vartheta_2 \in \Theta, G(\vartheta_1, \vartheta_2, s)$  is continuous in  $[0, 1]$  and continuously differentiable in  $(0, 1)$ .

**Assumption 3.** For every  $\vartheta_1, \vartheta_2 \in \Theta$  it holds  $[0, 1] \subseteq (a_1, a_2)$  and  $G(\vartheta_1, \vartheta_2, s)$  is continuously differentiable in  $(a_1, a_2)$ . Furthermore  $\lim_{s \downarrow a_1} G'(\vartheta_1, \vartheta_2, s) = -\infty$ ,  $\lim_{s \uparrow a_2} G'(\vartheta_1, \vartheta_2, s) = \infty$ .

**Theorem 1.** Suppose that Assumptions 1 and 2 are fulfilled for the experiments  $E_n = (\mathcal{X}_n, \mathcal{A}_n, Q_{n,\vartheta}, \vartheta \in \Theta)$ . Then for any sequence  $q_n$  of randomized decisions

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln R_m(q_n) &= \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln R_\pi(q_n) \\ &\geq \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s). \end{aligned}$$

*Proof.* Fix  $\vartheta_1 \neq \vartheta_2$  and put  $\psi_n = 1 - q_{n,(\vartheta_1)}$ .  $\psi_n$  is a test for  $H_0 : Q_{n,\vartheta_1}$  versus  $H_A : Q_{n,\vartheta_2}$  with

$$\begin{aligned} R(\vartheta_1, q_n) &= \int (1 - q_{n,(\vartheta_1)}) dQ_{n,\vartheta_1} \\ &= \int \psi_n dQ_{n,\vartheta_1} \end{aligned} \tag{2}$$

and

$$\begin{aligned} R(\vartheta_2, q_n) &= \int (1 - q_{n,(\vartheta_2)}) dQ_{n,\vartheta_2} \\ &= \int (1 - \psi_n) dQ_{n,\vartheta_2}. \end{aligned} \tag{3}$$

Let  $\varphi_n$  denote the Bayes test with prior  $\frac{1}{2}, \frac{1}{2}$ . Then

$$\frac{1}{2}(R(\vartheta_1, q_n) + R(\vartheta_2, q_n)) \geq \frac{1}{2} \int \varphi_n dQ_{n,\vartheta_1} + \frac{1}{2} \int (1 - \varphi_n) dQ_{n,\vartheta_2}.$$

Consequently by Lemma 3 in the Appendix with  $a = 0$

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln [\max(R(\vartheta_1, q_n), R(\vartheta_2, q_n))] \\ \geq \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln \left[ \frac{1}{2}(R(\vartheta_1, q_n) + R(\vartheta_2, q_n)) \right] \\ \geq \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s). \end{aligned}$$

Hence

$$\liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln R_m(q_n) \geq \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s).$$

Put  $\pi' = \min\{\pi(\vartheta), \vartheta \in \Theta\}$  and  $\pi'' = \max\{\pi(\vartheta), \vartheta \in \Theta\}$ . Then

$$R_\pi(q_n) \leq k\pi'' R_m(q_n) \tag{4}$$

$$R_m(q_n) \leq \frac{1}{\pi'} R_\pi(q_n), \tag{5}$$

which completes the proof. □

Now we will construct decision rules which asymptotically attain the optimal rate in Theorem 1. To this end, we eliminate nuisance parameters by taking the maximum of likelihood, i. e. by using

$$L_{n,i} = \max_{\vartheta \in \Theta_i} f_{n,\vartheta},$$

where  $f_{n,\vartheta}$  is the density of  $Q_{n,\vartheta}$  w. r. t. some  $\sigma$ -finite measure  $\mu_n$  on  $(\mathcal{X}_n, \mathcal{A}_n)$ . Let  $A_n(x) \subseteq \{1, \dots, k\}$  with  $A_n(x) = \{i : L_{n,i}(x) = \max_{1 \leq j \leq k} L_{n,j}(x)\}$ , and denote by  $q_n^0(x) = (q_{n,1}^0(x), \dots, q_{n,k}^0(x))$  the uniform distribution on  $A_n(x)$ .

**Theorem 2.** Under the assumptions of Theorem 1 it holds that

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln R_m(q_n^0) = \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln R_\pi(q_n^0) = \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s).$$

*Proof.* In view of inequalities (4), (5), we have to consider only  $R_m(q_n^0)$ . For fixed  $\vartheta \in \Theta$  we have

$$\begin{aligned} R(\vartheta, q_n^0) &\leq \sum_{j \neq (\vartheta)} Q_{n,\vartheta}(L_{n,j} \geq L_{n,(\vartheta)}) \\ &\leq \sum_{j \neq (\vartheta)} Q_{n,\vartheta}(L_{n,j} \geq f_{n,\vartheta}) \\ &\leq \sum_{j \neq (\vartheta)} \sum_{\vartheta' \in \Theta_j} Q_{n,\vartheta}(f_{n,\vartheta'} \geq f_{n,\vartheta}) \\ &\leq kN \max_{\vartheta_1 \neq \vartheta_2} (Q_{n,\vartheta_1}(f_{n,\vartheta_1} \leq f_{n,\vartheta_2}) + Q_{n,\vartheta_2}(f_{n,\vartheta_2} < f_{n,\vartheta_1})) \end{aligned}$$

where  $N$  is the cardinality of  $\Theta$ . The rest follows from Lemma 3 in the Appendix with  $\pi_{1,n} = \pi_{2,n} = \frac{1}{2}$ . □

Now we investigate the asymptotic behavior of the risks if the weights are allowed to depend on  $n$ . To be more precise we suppose that  $\pi_n(\vartheta), \vartheta \in \Theta$ , are non-negative numbers so that  $a(\vartheta) = \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \pi_n(\vartheta)$  exists for every  $\vartheta \in \Theta$ . Set

$$R_{\pi_n}(q_n) = \sum_{\vartheta \in \Theta} R(\vartheta, q_n) \pi_n(\vartheta)$$

and

$$R_{\pi_n}^m(q_n) = \max_{\vartheta \in \Theta} [\pi_n(\vartheta) R(\vartheta, q_n)].$$

In order to construct a weighted maximum likelihood rule we set

$$L_{\pi_n,i} \doteq \max_{\vartheta \in \Theta_i} [\pi_n(\vartheta) f_{n,\vartheta}]$$

and introduce  $q_{\pi_n}^0$  analogously to  $q_n^0$  where  $L_{\pi_n,i}$  is used in  $A_n$  instead of  $L_{n,i}$ . Set  $\Gamma(\vartheta_1, \vartheta_2, s) = sa(\vartheta_1) + (1-s)a(\vartheta_2) + G(\vartheta_1, \vartheta_2, s)$ .

**Theorem 3.** If Assumptions 1 and 3 are fulfilled then for every sequence  $q_n$

$$\liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln R_{\pi_n}^m(q_n) = \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln R_{\pi_n}(q_n) \geq \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} \Gamma(\vartheta_1, \vartheta_2, s).$$

*Proof.* The proof of Theorem 3 is completely analogous to the proofs of Theorems 1 and 2 if we apply Lemma 4 instead of Lemma 3 and the inequalities

$$R_{\pi_n}(q_n) \leq N R_{\pi_n}^m(q_n)$$

and

$$R_{\pi_n}^m(q_n) \leq R_{\pi_n}(q_n)$$

instead of (4) and (5) respectively. □

### 3. ADAPTIVE CLASSIFICATION RULES

Suppose that we are given experiments  $E_n = (\mathcal{X}_n, \mathcal{A}_n, Q_\vartheta \in \Theta)$  and that we want to classify the distribution of the data into the sets  $\Theta_1, \dots, \Theta_k$ . In many situations the distributions  $Q_{n,\vartheta}$  depend on some unknown nuisance parameter  $\alpha \in \Lambda$ . We introduce the experiments

$$\mathcal{E}_n = (\mathcal{X}_n, \mathcal{A}_n, Q_{n,\vartheta}^\alpha, (\alpha, \vartheta) \in \Lambda \times \Theta).$$

Let  $\mu_n$  be a  $\sigma$ -finite dominating measure and denote by  $f_{n,\vartheta}^\alpha$  the density of  $Q_{n,\vartheta}^\alpha$  w. r. t.  $\mu_n$ . We suppose that for every  $\alpha, \beta \in \Lambda$  Assumption 1 is fulfilled for  $Q_{n,\vartheta_1}^\alpha, Q_{n,\vartheta_2}^\beta$  instead of  $Q_{n,\vartheta_1}, Q_{n,\vartheta_2}$ . Set

$$G(\alpha, \beta, \vartheta_1, \vartheta_2, s) = \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(Q_{n,\vartheta_1}^\alpha, Q_{n,\vartheta_2}^\beta),$$

and

$$G(\alpha, \beta) = \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1, \vartheta_2, s).$$

By Theorem 1 we get for every sequence of decision rules  $q_n$  and every  $\alpha \in \Lambda$

$$\liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln \left( \max_{\vartheta \in \Theta} R(\alpha, \vartheta, q_n) \right) \geq G(\alpha, \alpha). \tag{6}$$

If we fix  $\alpha \in \Lambda$  and apply the maximum likelihood rule then this rule attains equality in (6). But, unfortunately, the maximum likelihood rule depends on the unknown nuisance parameter  $\alpha$ . Similar as in Rukhin ([11, 12, 13]) we ask for the existence of such rules which do not depend on  $\alpha \in \Lambda$  but attain the lower bound in (6) for every fixed  $\alpha \in \Lambda$ .

**Definition 1.** A sequence of classification rules  $q_n$  is called *adaptive* if

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \left( \max_{\vartheta \in \Theta} R(\alpha, \vartheta, q_n) \right) = G(\alpha, \alpha)$$

for every  $\alpha \in \Lambda$ .

Similarly as in Rukhin and Vajda [14], where the case of subsets  $\Theta_i$  each consisting of one element have been treated, we now derive necessary and sufficient conditions for the existence of adaptive classification rules. Put

$$\Gamma_{\alpha, \beta}(a, b, \vartheta_1, \vartheta_2) = \inf_{0 < s < 1} [G(\alpha, \beta, \vartheta_1, \vartheta_2, s) + sa + (1 - s)b].$$

**Lemma 1.** Assume that Assumption 1 is fulfilled for every  $\alpha, \beta \in \Lambda$  and  $q_n$  is any sequence of selection rules. If Assumption 2 is fulfilled then

$$\liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln [\max(R(\alpha, \vartheta_1, q_n), R(\beta, \vartheta_2, q_n))] \geq \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1, \vartheta_2, s).$$

If even the stronger Assumption 3 is fulfilled and

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \pi_{1,n} = a, \quad \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \pi_{2,n} = b,$$

then

$$\liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln [\max(\pi_{1,n} R(\alpha, \vartheta_1, q_n), \pi_{2,n} R(\beta, \vartheta_2, q_n))] \geq \Gamma_{\alpha, \beta}(a, b, \vartheta_1, \vartheta_2).$$

**Remark.** If  $a = b$  then  $\Gamma_{\alpha, \beta}(a, b, \vartheta_1, \vartheta_2) = a + \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1, \vartheta_2, s)$ . Therefore the first and the second statement coincides in this case.

*Proof.* Denote by  $q_{n,i}$ ,  $1 \leq i \leq k$ , the components of the vector  $q_n$  and fix  $\vartheta_1 \neq \vartheta_2$ . Put  $\psi_n = 1 - q_{n,(\vartheta_1)}$ . Then  $\psi_n$  is a test for  $H_0 : Q_{n,\vartheta_1}^\alpha$  versus  $H_A : Q_{n,\vartheta_2}^\beta$  with

$$\begin{aligned} R(\alpha, \vartheta_1, q_n) &= \int [1 - q_{n,(\vartheta_1)}] dQ_{n,\vartheta_1}^\alpha = \int \psi_n dQ_{n,\vartheta_1}^\alpha, \\ R(\beta, \vartheta_2, q_n) &= \int [1 - q_{n,(\vartheta_2)}] dQ_{n,\vartheta_2}^\beta = \int \sum_{i \neq (\vartheta_2)} q_{n,i} dQ_{n,\vartheta_2}^\beta \\ &\geq \int q_{n,(\vartheta_1)} dQ_{n,\vartheta_2}^\beta = \int [1 - \psi_n] dQ_{n,\vartheta_2}^\beta. \end{aligned}$$

Let  $\varphi_n$  denote the Bayes test for  $H_0 : Q_{n,\vartheta_1}^\alpha$  versus  $H_A : Q_{n,\vartheta_2}^\beta$  with prior weights  $\pi_{1,n}, \pi_{2,n}$ . Then

$$\begin{aligned} &2 \max(\pi_{1,n} R(\alpha, \vartheta_1, q_n), \pi_{2,n} R(\beta, \vartheta_2, q_n)) \\ &\geq \pi_{1,n} \int \varphi_n dQ_{n,\vartheta_1}^\alpha + \pi_{2,n} \int (1 - \varphi_n) dQ_{n,\vartheta_2}^\beta. \end{aligned} \tag{7}$$



The first statement now follows from Lemma 3 in the Appendix by putting  $\pi_{1,n} = \pi_{2,n} = 1/2$ , where the second statement is a direct consequence of Lemma 4 in the Appendix.

Assume now that  $\Lambda$  is finite with  $L$  elements and eliminate the nuisance parameter  $\alpha \in \Lambda$  by taking the maximum over  $\alpha \in \Lambda$ . Set

$$M_{n,i}(x) = \max_{\alpha \in \Lambda, \vartheta \in \Theta_i} [f_{n,\vartheta}^\alpha(x) \exp\{-c_n G(\alpha, \alpha)\}],$$

$$A_n(x) = \left\{ i : M_{n,i}(x) = \max_{1 \leq j \leq k} M_{n,j}(x) \right\},$$

and denote by  $q_n^1(x) = (q_{n,1}^1(x), \dots, q_{n,k}^1(x))$  the uniform distribution on  $A_n$ .

**Lemma 2.** If the Assumptions 1 and 3 are fulfilled and  $\Lambda$  is finite then

$$\limsup_{n \rightarrow \infty} \left[ \frac{1}{c_n} \ln R(\alpha, \vartheta, q_n^1) - G(\alpha, \alpha) \right]$$

$$\leq \max_{\beta \in \Lambda} \max_{\vartheta, \vartheta': \vartheta' \neq \vartheta} \Gamma_{\alpha, \beta}(-G(\alpha, \alpha), -G(\beta, \beta), \vartheta, \vartheta').$$

*Proof.* By the definition of  $q_n^1$  we get

$$R(\alpha, \vartheta, q_n^1)$$

$$\leq \sum_{j \neq (\vartheta)} Q_{n,\vartheta}^\alpha(M_{n,(\vartheta)} \leq M_{n,j})$$

$$\leq \sum_{j \neq (\vartheta)} Q_{n,\vartheta}^\alpha(\exp\{-c_n G(\alpha, \alpha)\} f_{n,\vartheta}^\alpha \leq M_{n,j})$$

$$\leq \sum_{\beta \in \Lambda} \sum_{j \neq (\vartheta)} \sum_{\vartheta' \in \Theta_j} Q_{n,\vartheta}^\alpha(\exp\{-c_n G(\alpha, \alpha)\} f_{n,\vartheta}^\alpha \leq \exp\{-c_n G(\beta, \beta)\} f_{n,\vartheta'}^\beta)$$

$$\leq kNL \max_{\vartheta' \neq \vartheta} \max_{\beta \in \Lambda} Q_{n,\vartheta}^\alpha(\exp\{-c_n G(\alpha, \alpha)\} f_{n,\vartheta}^\alpha \leq \exp\{-c_n G(\beta, \beta)\} f_{n,\vartheta'}^\beta).$$

Set  $\pi_{1,n} = \exp\{-c_n G(\alpha, \alpha)\}$ ,  $\pi_{2,n} = \exp\{-c_n G(\beta, \beta)\}$  and denote by  $\varphi_n$  the Bayes test for  $H_0 : Q_{n,\vartheta}^\alpha$  versus  $H_1 : Q_{n,\vartheta'}^\beta$ , with relative weights  $\pi_{1,n}, \pi_{2,n}$ , respectively. Then

$$\exp\{-c_n G(\alpha, \alpha)\} Q_{n,\vartheta}^\alpha(\exp\{-c_n G(\alpha, \alpha)\} f_{n,\vartheta}^\alpha \leq \exp\{-c_n G(\beta, \beta)\} f_{n,\vartheta'}^\beta)$$

$$\leq \pi_{1,n} \int \varphi_n dQ_{n,\vartheta}^\alpha + \pi_{2,n} \int (1 - \varphi_n) dQ_{n,\vartheta'}^\beta.$$

To complete the proof we have only to apply Lemma 4 in the Appendix. □

Now we are ready to formulate a necessary and sufficient condition for the existence of adaptive classification rules. This condition corresponds to Theorem 2.1 in Rukhin [13].

**Theorem 4.** Suppose that Assumptions 1 and 3 are fulfilled for every  $\alpha, \beta \in \Lambda$ . If an adaptive classification rule exists then

$$\inf_{0 < s < 1} [G(\alpha, \beta, \vartheta_1, \vartheta_2, s) - sG(\alpha, \alpha) - (1 - s)G(\beta, \beta)] \leq 0 \tag{8}$$

for all  $\alpha, \beta \in \Lambda$ ,  $\vartheta_1, \vartheta_2 \in \Theta$  with  $\vartheta_1 \neq \vartheta_2$ . Conversely, if (8) is fulfilled and  $\Lambda$  is finite then the weighted maximum likelihood classification rule  $q_n^1$  is adaptive.

*Proof.* Suppose an adaptive rule  $q_n^a$  exists. Put  $q_n = q_n^a$ ,  $\pi_{1,n} = \exp\{-c_n G(\alpha, \alpha)\}$  and  $\pi_{2,n} = \exp\{-c_n G(\beta, \beta)\}$ . Then by the second statement in Lemma 1

$$\begin{aligned} & \inf_{0 < s < 1} [G(\alpha, \beta, \vartheta_1, \vartheta_2, s) - sG(\alpha, \alpha) - (1 - s)G(\beta, \beta)] \\ &= \Gamma_{\alpha, \beta}(-G(\alpha, \alpha), -G(\beta, \beta), \vartheta_1, \vartheta_2) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln [\max(\pi_{1,n} R(\alpha, \vartheta_1, q_n^a), \pi_{2,n} R(\beta, \vartheta_2, q_n^a))] \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{c_n} \max \left[ \max_{\vartheta_1 \in \Theta} \ln R(\alpha, \vartheta_1, q_n^a) - G(\alpha, \alpha), \max_{\vartheta_2 \in \Theta} \ln R(\beta, \vartheta_2, q_n^a) - G(\beta, \beta) \right] \\ &\leq 0, \end{aligned}$$

by the adaptivity of  $q_n^a$ . Conversely, let (8) be fulfilled. Then by Lemma 2,

$$\limsup_{n \rightarrow \infty} \left[ \frac{1}{c_n} \ln \max_{\vartheta \in \Theta} R(\alpha, \vartheta, q_n^1) \right] \leq G(\alpha, \alpha),$$

which completes the proof. □

**Remark.** Condition (8) has an intuitive interpretation. As the Hellinger integral  $H_s(P, Q)$  of order  $0 < s < 1$  of the distributions  $P$  and  $Q$  is between 0 and 1 the expression  $-\ln H_s(P, Q)$  is non-negative, and it can be shown that  $-\ln H_s(P, Q)$  is small iff the variational distance between  $P$  and  $Q$  is small. For details we refer to Liese and Vajda [9]. Hence  $-G(\alpha, \beta, \vartheta_1, \vartheta_2, s)$  is an asymptotic measure for the distance between  $Q_{n, \vartheta_1}^\alpha$  and  $Q_{n, \vartheta_2}^\beta$ . Note that condition (8) is equivalent to

$$\inf_{0 < s < 1} [-sG(\alpha, \alpha) - (1 - s)G(\beta, \beta) - (-G(\alpha, \beta, \vartheta_1, \vartheta_2, s))] \leq 0$$

for every  $\alpha, \beta \in \Lambda$ ,  $\vartheta_1, \vartheta_2 \in \Theta$ ,  $\vartheta_1 \neq \vartheta_2$ , which says that if an adaptive rule exists then asymptotically the distance between  $Q_{n, \vartheta_1}^\alpha$  and  $Q_{n, \vartheta_2}^\beta$  is not smaller than the minimum distances within the models  $Q_{n, \vartheta}^\alpha$ ,  $\vartheta \in \Theta$  and  $Q_{n, \vartheta}^\beta$ ,  $\vartheta \in \Theta$ .

#### 4. APPLICATION TO SELECTING THE BEST POPULATION

Suppose we have independent samples from populations  $\pi_1, \dots, \pi_k$ , where the distributions depend on a parameter. Let  $\pi_i$  have the parameter  $\eta_i \in (a, b)$ . We assume that the set  $\{\eta_1, \dots, \eta_k\}$  is known, but not which population belongs to which  $\eta$ -value. The task now is to select that population with the largest  $\eta$ -value. For every

$1 \leq i \leq k, n = 1, 2, \dots$  let  $(R_{i,n}, \mathcal{R}_{i,n})$  be the sample space of a sample from population  $\pi_i$  with distribution  $P_{i,n,\eta_i}$ . The samples from  $\pi_1, \dots, \pi_k$  are assumed to be independent, but we do not assume that the observations from  $\pi_i$  are independent. Moreover the sample sizes are allowed to be different for different populations. Let  $\Theta$  denote the set of all permutations of  $(1, \dots, k)$  where for  $\vartheta \in \Theta, \vartheta : (1, \dots, k) \mapsto (i_1, \dots, i_k)$ , we set  $\vartheta(\eta_j) = \eta_{i_j}$  and  $Q_{n,\vartheta} = P_{1,n,\vartheta(\eta_1)} \times \dots \times P_{k,n,\vartheta(\eta_k)}$ . We may assume w.l.g. that  $\eta_1 \leq \dots \leq \eta_k$ . In addition we assume that  $\eta_{k-1} < \eta_k$ . Put  $\Theta_j = \{\vartheta : \vartheta(j) = k\}$ . For  $\vartheta \in \Theta_j$  the population  $\pi_j$  has the largest  $\eta$ -value. Introduce the experiment  $E_n$  by

$$E_n = \left( \prod_{i=1}^k R_{i,n}, \bigotimes_{i=1}^k \mathcal{R}_{i,n}, Q_{n,\vartheta}, \vartheta \in \Theta \right). \tag{9}$$

The problem of selecting the best population, i.e. with the largest  $\eta$ -value, is thus reduced to the decision for one of the subsets  $\Theta_1, \dots, \Theta_k$ . We assume that for every  $\eta_i, \eta_j$  the distributions  $P_{i,n,\eta_i}, P_{j,n,\eta_j}$  satisfy the Assumptions 1 and 2 if we substitute  $Q_{n,\vartheta_1}$  and  $Q_{n,\vartheta_2}$  by  $P_{i,n,\eta_i}$  and  $P_{j,n,\eta_j}$ , respectively. Put

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(P_{i,n,\eta}, P_{i,n,\eta'}) = G_i(\eta, \eta', s),$$

where  $\eta, \eta' \in \{\eta_1, \dots, \eta_k\}$ . In the sequel we will apply repeatedly the following well known property of Hellinger integrals:

$$H_s(P_1 \times \dots \times P_m, Q_1 \times \dots \times Q_m) = \prod_{i=1}^m H_s(P_i, Q_i). \tag{10}$$

Let  $\vartheta_1, \vartheta_2 \in \Theta$  with  $\vartheta_1(l) = i_l$  and  $\vartheta_2(l) = j_l$ . Then this property implies that the family  $Q_{n,\vartheta} = P_{1,n,\vartheta(\eta_1)} \times \dots \times P_{k,n,\vartheta(\eta_k)}$  satisfies Assumptions 1 and 2 and it holds

$$G(\vartheta_1, \vartheta_2, s) = \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(Q_{n,\vartheta_1}, Q_{n,\vartheta_2}) = \sum_{l=1}^k G_l(\eta_{i_l}, \eta_{j_l}, s). \tag{11}$$

In order to apply the earlier results we have to find  $\max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s)$ . Since  $G_i(\eta, \eta', s) \leq 0$  for  $0 < s < 1$  we have to take the maximum only over such permutations which coincide at a maximum number of  $k - 2$  arguments. For any  $\vartheta_1 \neq \vartheta_2$  the indices of populations with parameter  $\eta_k$  are different. Hence we have to take into consideration only permutations  $(i_1, \dots, i_k), (j_1, \dots, j_k)$  for which there are  $l, m \in \{1, \dots, k\}, l \neq m$  with  $i_l = j_m = k, i_m = j_l, i_r = j_r, r \neq l, r \neq m$ . Put  $t = i_m = j_l$ . Then by the above arguments

$$\max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s) = \max_{l,m:l \neq m; t:t < k} \inf_{0 < s < 1} [G_l(\eta_k, \eta_t, s) + G_m(\eta_t, \eta_k, s)]. \tag{12}$$

Let us study the rate of convergence to zero of the risks of decisions for the sequence of experiments  $E_n$  from (9). By construction,  $\vartheta^{-1}(k)$  is the index of the population with the largest  $\eta$ -value. We set  $L(\vartheta, i) = 0$  if  $i = \vartheta^{-1}(k)$ , and  $L(\vartheta, i) = 1$ , else. Let

$q_{i,n} : \prod_{i=1}^k (R_{i,n}, \mathcal{R}_{i,n}) \mapsto [0, 1]$  be measurable with  $\sum_{i=1}^k q_{i,n} = 1$ , where  $q_{i,n}$  is the probability to decide in favor of population  $\pi_i$ . The risk is given by

$$R(\vartheta, q_n) = 1 - \int q_{\vartheta^{-1}(k),n} dQ_{n,\vartheta} = 1 - Q_{n,\vartheta}(CS, q_n),$$

where CS stands for correct selection and  $q_n = (q_{1,n}, \dots, q_{k,n})$ . Denote by  $g_{i,n,\eta}$  the density of  $P_{i,n,\eta}$  w. r. t. to some  $\sigma$ -finite measure  $\nu_{i,n}$ , and put  $\mu_n = \nu_{1,n} \times \dots \times \nu_{k,n}$ . Then

$$f_{n,\vartheta} = \frac{dQ_{n,\vartheta}}{d\mu_n} = \prod_{i=1}^k g_{i,n,\vartheta(\eta_i)}.$$

Consider again the maximum likelihood selection rule  $q_n^0$  introduced in Theorem 2. The following Theorem is a direct consequence of Theorems 1 and 2.

**Theorem 5.** Suppose that for every  $\eta, \eta' \in \{\eta_1, \dots, \eta_k\}$  the distributions  $P_{i,n,\eta}, P_{i,n,\eta'}, \eta \neq \eta'$ , satisfy Assumptions 1 and 2. Then for every sequence  $q_n$  of selection rules

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{c_n} \ln \left( \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, q_n)) \right) \\ \geq \max_{l,m:l \neq m; t:t < k} \inf_{0 < s < 1} [G_l(\eta_k, \eta_t, s) + G_m(\eta_t, \eta_k, s)], \end{aligned} \tag{13}$$

where the lower bound is attained by the maximum likelihood selection rule  $q_n^0$ .

We now assume that the samples from the populations consist of independent observations distributed according to  $P_{\eta_1}, \dots, P_{\eta_k}$ . Let the sample sizes be  $n_1, \dots, n_k$ .

Put  $n = n_1 + \dots + n_k$  and suppose  $\frac{n_i}{n} \xrightarrow{n \rightarrow \infty} \gamma_i$ , where  $0 < \gamma_i < 1$ . We have  $P_{i,n,\eta_i} = P_{\eta_i}^{n_i}$ . Set  $G(\eta, \eta', s) = \ln H_s(P_\eta, P_{\eta'})$ . If  $c_n = n$  is chosen, then by (10)

$$G_i(\eta, \eta', s) = \gamma_i \ln(H_s(P_\eta, P_{\eta'})) = \gamma_i G(\eta, \eta', s), \tag{14}$$

$$G(\vartheta_1, \vartheta_2, s) = \sum_{l=1}^k \gamma_l G(\eta_{i_l}, \eta_{j_l}, s) \tag{15}$$

and by (12)

$$\max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s) = \max_{l \neq m; t:t < k} \inf_{0 < s < 1} [\gamma_l G(\eta_k, \eta_t, s) + \gamma_m G(\eta_t, \eta_k, s)]. \tag{16}$$

Consider the case of asymptotically equal sample sizes, i.e.  $\gamma_1 = \dots = \gamma_k = \frac{1}{k}$ . Because of  $H_s(P, Q) = H_{1-s}(Q, P)$  we have  $G(\eta, \eta', s) = G(\eta', \eta, 1 - s)$ . The convex function  $G(\eta, \eta', s) + G(\eta', \eta, s)$  is symmetric w. r. t.  $s = 1/2$ . Hence

$$\max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s) = \frac{2}{k} \max_{r < k} G\left(\eta_k, \eta_r, \frac{1}{2}\right). \tag{17}$$

If  $P_\eta \neq P_{\eta'}$  for every  $\eta \neq \eta'$ , then the family  $P_{i,n,\eta} = P_{\eta_i}^{n_i}$  satisfies the assumptions of Theorem 5.

**Corollary.** Suppose  $P_{\eta_i} \neq P_{\eta_j}$  for  $i \neq j$ ,  $\frac{n_i}{n} \mapsto \gamma_i$ , and the observations are taken independently from the populations. Then for every sequence of selection rules  $q_n$

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \left( \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, q_n)) \right) \\ & \geq \max_{l \neq m; t < k} \inf_{0 < s < 1} [\gamma_l G(\eta_k, \eta_t, s) + \gamma_m G(\eta_t, \eta_k, s)], \end{aligned} \tag{18}$$

where the maximum likelihood rule  $q_n^0$  attains the lower bound. If  $\gamma_1 = \dots = \gamma_k = \frac{1}{k}$ , then the right hand term in (18) is simplified to  $\frac{2}{k} \max_{t < k} G(\eta_k, \eta_t, \frac{1}{2})$ .

### 5. SELECTIONS UNDER UNEQUAL SAMPLE SIZES IN EXPONENTIAL FAMILIES

We now restrict our considerations to exponential families. Let  $(R, \mathcal{R})$  be a measurable space,  $T : R \mapsto \mathbb{R}_1$  a measurable function and  $\nu$  a  $\sigma$ -finite measure on  $(R, \mathcal{R})$ . The set

$$I_T = \left\{ \eta : \int \exp\{\eta T\} d\nu < \infty \right\}$$

is an interval. We assume that the interior of  $I_T$  is nonempty and denote it by  $(a_1, a_2)$ . Put  $K(\eta) = \ln \int e^{\eta T} d\nu$  and

$$P_\eta(A) = \int_A \exp\{\eta T - K(\eta)\} d\nu. \tag{19}$$

Suppose the family  $(P_\eta)_{\eta \in I_T}$  is nontrivial in the sense that  $\nu \circ T^{-1}$  is not concentrated at one point. Then  $P_\eta \neq P_{\eta'}$  for  $\eta \neq \eta'$  and  $K$  is strictly convex. The definition of  $H_s$  yields for  $\eta_1, \eta_2 \in I_T$

$$H_s(P_{\eta_1}, P_{\eta_2}) = \exp\{-D(\eta_1, \eta_2, s)\} \tag{20}$$

where  $D(\eta_1, \eta_2, s) = sK(\eta_1) + (1-s)K(\eta_2) - K(s\eta_1 + (1-s)\eta_2)$ . Suppose now that the populations  $\pi_1, \dots, \pi_k$  have distributions  $P_{\eta_1}, \dots, P_{\eta_k}$ ,  $a_1 < \eta_1 \leq \dots \leq \eta_{k-1} < \eta_k < a_2$  and  $n_i$  independent observations are taken from  $\pi_i$  where  $\frac{n_i}{n} \mapsto \gamma_i$ . Denote by  $\gamma' \leq \gamma''$  the two smallest  $\gamma_i$ -values which will be very crucial in the following. Since  $K$  is convex the function  $\eta \mapsto D(\eta, \eta', s)$  is nonincreasing for  $\eta < \eta'$  and nondecreasing for  $\eta > \eta'$ . Hence  $G(\eta, \eta', s) = \ln H_s(P_\eta, P_{\eta'})$  and

$$\begin{aligned} & \max_{l \neq m; t < k} \inf_{0 < s < 1} [\gamma_l G(\eta_k, \eta_t, s) + \gamma_m G(\eta_t, \eta_k, s)] \\ & = - \min_{l \neq m; t < k} \sup_{0 < s < 1} [\gamma_l D(\eta_k, \eta_t, s) + \gamma_m D(\eta_t, \eta_k, s)] \\ & = - \min_{l \neq m} \sup_{0 < s < 1} [\gamma_l D(\eta_k, \eta_{k-1}, s) + \gamma_m D(\eta_{k-1}, \eta_k, s)] \\ & = - \sup_{0 < s < 1} [\gamma' D(\eta_k, \eta_{k-1}, s) + \gamma'' D(\eta_{k-1}, \eta_k, s)] \\ & =: -M(\gamma', \gamma'', \eta_{k-1}, \eta_k). \end{aligned} \tag{21}$$

**Proposition 1.** For independent observations from populations, whose distributions belong to the exponential family (19) and have parameters  $\eta_1 \leq \dots \leq \eta_{k-1} < \eta_k$ , it holds

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \left( \max_{\vartheta} (1 - Q_{n,\vartheta}(CS, q_n)) \right) \geq -M(\gamma', \gamma'', \eta_{k-1}, \eta_k)$$

for any sequence of selection rules  $q_n$ , and the maximum likelihood rule  $q_n^0$  attains the lower bound.

The statement of Proposition 1 is a direct consequence of the Corollary of Theorem 5.

Suppose the r. v.  $X_{i,j}, j = 1, \dots, n_i, i = 1, \dots, k$  with values in  $(R, \mathcal{R})$  are independent and  $X_{i,j} \sim P_{\eta_i}$ . Put  $\bar{T}_{i,n} = \frac{1}{n_i} \sum_{j=1}^{n_i} T(X_{i,j})$  and set  $A_n = \{i : \bar{T}_{i,n} = \max_{1 \leq l \leq k} \bar{T}_{l,n}\}$ . Then the natural selection rule, which selects in terms of the largest of the values  $\bar{T}_{i,n}$ , is given by the uniform distribution  $\tilde{q}_n = (\tilde{q}_{n,1}, \dots, \tilde{q}_{n,k})$  on  $A_n$ . If  $Q_{n,\vartheta}$  is the true distribution then  $\pi_{\vartheta^{-1}(k)}$  has the largest  $\eta$ -value  $\eta_k$  and  $\pi_{\vartheta^{-1}(l)}$  has parameter  $\eta_l$ . Hence

$$\begin{aligned} & \max_{l \neq k} Q_{n,\vartheta} (\bar{T}_{\vartheta^{-1}(k),n} < \bar{T}_{\vartheta^{-1}(l),n}) \\ & \leq Q_{n,\vartheta} \left( \bar{T}_{\vartheta^{-1}(k),n} < \max_{l \neq k} \bar{T}_{\vartheta^{-1}(l),n} \right) \\ & \leq 1 - Q_{n,\vartheta}(CS, \tilde{q}_n) \\ & \leq Q_{n,\vartheta} \left( \bar{T}_{\vartheta^{-1}(k),n} \leq \max_{l \neq k} \bar{T}_{\vartheta^{-1}(l),n} \right) \\ & \leq (k-1) \max_{l \neq k} Q_{n,\vartheta} (\bar{T}_{\vartheta^{-1}(k),n} \leq \bar{T}_{\vartheta^{-1}(l),n}). \end{aligned} \tag{22}$$

Lemma 5 yields

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \ln Q_{n,\vartheta} (\bar{T}_{\vartheta^{-1}(k),n} < \bar{T}_{\vartheta^{-1}(l),n}) \\ & = \lim_{n \rightarrow \infty} \frac{1}{n} \ln Q_{n,\vartheta} (\bar{T}_{\vartheta^{-1}(k),n} \leq \bar{T}_{\vartheta^{-1}(l),n}) \\ & = (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}) D \left( \eta_l, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(l)} + \gamma_{\vartheta^{-1}(k)}} \right). \end{aligned}$$

Consequently,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \ln \max_{\vartheta} (1 - Q_{n,\vartheta}(CS, \tilde{q}_n)) \\ & = \max_{\vartheta} \lim_{n \rightarrow \infty} \frac{1}{n} \ln (1 - Q_{n,\vartheta}(CS, \tilde{q}_n)) \\ & = - \min_{\vartheta} \min_{l \neq k} (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}) D \left( \eta_l, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}} \right). \end{aligned}$$

To evaluate the term on the right hand side we note that

$$D(\eta_l, \eta_k, s) \geq D(\eta_{k-1}, \eta_k, s)$$

for every  $0 \leq s \leq 1$ . Hence

$$\begin{aligned} & \min_{\vartheta} \min_{l \neq k} (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}) D \left( \eta_l, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}} \right) \\ & \geq \min_{\vartheta} \min_{l \neq k} (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}) D \left( \eta_{k-1}, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}} \right) \\ & = \min_{i \neq j} (\gamma_i + \gamma_j) D \left( \eta_{k-1}, \eta_k, \frac{\gamma_i}{\gamma_i + \gamma_j} \right) \\ & = \min_{\vartheta} (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(k-1)}) D \left( \eta_{k-1}, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(k-1)}} \right) \\ & \geq \min_{\vartheta} \min_l (\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}) D \left( \eta_{k-1}, \eta_k, \frac{\gamma_{\vartheta^{-1}(k)}}{\gamma_{\vartheta^{-1}(k)} + \gamma_{\vartheta^{-1}(l)}} \right) \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \ln \max_{\vartheta} (1 - Q_{n,\vartheta}(CS, \tilde{q}_n)) \\ & = - \min_{i \neq j} (\gamma_i + \gamma_j) D \left( \eta_{k-1}, \eta_k, \frac{\gamma_i}{\gamma_i + \gamma_j} \right) \\ & =: - \min_{i \neq j} L(\gamma_i, \gamma_j, \eta_{k-1}, \eta_k). \end{aligned}$$

Thus we obtained

**Theorem 6.** If  $X_{i,j} \sim P_{\eta_i}$ , and all  $P_{\eta_i}$  belong to the exponential family (19), then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, \tilde{q}_n)) = - \min_{i \neq j} L(\gamma_i, \gamma_j, \eta_{k-1}, \eta_k).$$

Consider again  $\gamma', \gamma''$ , the two smallest  $\gamma$ -values. From the first statement in Lemma 7 in the Appendix we get

$$M(\gamma', \gamma'', \eta_{k-1}, \eta_k) \geq \min_{i \neq j} L(\gamma_i, \gamma_j, \eta_{k-1}, \eta_k), \tag{23}$$

which says that for known  $\eta_i$  the maximum likelihood selection rule is at least as good as the natural rule. Furthermore, by the second statement in Lemma 7, equality holds in (23) iff  $\gamma' = \gamma''$ . Thus we have proved the following statement.

**Proposition 2.** Suppose the conditions of Proposition 1 are fulfilled and denote by  $\tilde{q}_n$  the natural selection rule. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, \tilde{q}_n)) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \ln \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, q_n^0))$$

where equality holds iff  $\gamma' = \gamma''$ .

Now we examine whether for populations from an exponential family and non-equal sample sizes adaptive selection procedures do exist. Suppose  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\beta = (\beta_1, \dots, \beta_k)$  and assume  $\vartheta_1, \vartheta_2$  are permutations with  $\vartheta_1(l) = i_l$  and  $\vartheta_2(l) = j_l$ . Set  $Q_{n,\vartheta_1}^\alpha = P_{\alpha_{i_1}}^{n_1} \times \dots \times P_{\alpha_{i_k}}^{n_k}$  and  $Q_{n,\vartheta_2}^\beta = P_{\beta_{j_1}}^{n_1} \times \dots \times P_{\beta_{j_k}}^{n_k}$ , where each  $P_\eta$  is from the family (19). Then by (20), analogously to (15),

$$G(\alpha, \beta, \vartheta_1, \vartheta_2, s) = - \sum_{l=1}^k \gamma_l D(\alpha_{i_l}, \beta_{j_l}, s). \tag{24}$$

Next, the parameters will be specialized. Put for  $a_1 < t, x, \eta_0 < a_2$  and sufficiently small  $h > 0$ ,  $\alpha = (\eta_0, \dots, \eta_0, t, t+h)$  and  $\beta = (\eta_0, \dots, \eta_0, x, x+h)$ . Denote by  $\gamma' \leq \gamma''$  the two smallest  $\gamma_i$ -values. Then by formula (24), with the same arguments which led to (12) and (21), we get

$$\begin{aligned} G(\alpha, \alpha) &= - \sup_{0 < s < 1} [\gamma' D(t, t+h, s) + \gamma'' D(t+h, t, s)] \\ G(\beta, \beta) &= - \sup_{0 < s < 1} [\gamma' D(x, x+h, s) + \gamma'' D(x+h, x, s)]. \end{aligned}$$

Let  $l, m$  be two indices such that  $\gamma_l = \gamma', \gamma_m = \gamma''$  and choose two permutations  $\vartheta_1^0$  and  $\vartheta_2^0$  such that  $\alpha_{i_l} = t+h, \alpha_{i_m} = t$  and  $\beta_{j_l} = x, \beta_{j_m} = x+h$ . Then by (24)

$$\begin{aligned} &\max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1, \vartheta_2, s) \\ &\geq \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1^0, \vartheta_2^0, s) \\ &= - \sup_{0 < s < 1} [\gamma' D(t+h, x, s) + \gamma'' D(t, x+h, s)]. \end{aligned}$$

Suppose now an adaptive selection rule  $q_n^a$  exists. Then by the definition of the adaptivity

$$\limsup_{n \rightarrow \infty} \frac{1}{c_n} \ln R(\alpha, \vartheta_1^0, q_n^a) \leq \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \left( \max_{\vartheta \in \Theta} R(\alpha, \vartheta, q_n^a) \right) = G(\alpha, \alpha)$$

and analogously

$$\limsup_{n \rightarrow \infty} \frac{1}{c_n} \ln R(\beta, \vartheta_2^0, q_n^a) \leq G(\beta, \beta).$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{c_n} \ln [\max(R(\alpha, \vartheta_1^0, q_n^a), R(\beta, \vartheta_2^0, q_n^a))] \leq \max(G(\alpha, \alpha), G(\beta, \beta))$$



and by the first statement in Lemma 1

$$\max(G(\alpha, \alpha), G(\beta, \beta)) \geq \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1^0, \vartheta_2^0, s).$$

Hence

$$\begin{aligned} & \min \left( \sup_{0 < s_1 < 1} [\gamma' D(t, t + h, s_1) + \gamma'' D(t + h, t, s_1)], \right. \\ & \quad \left. \sup_{0 < s_2 < 1} [\gamma' D(x, x + h, s_2) + \gamma'' D(x + h, x, s_2)] \right) \\ & \leq \sup_{0 < s < 1} [\gamma' D(t + h, x, s) + \gamma'' D(t, x + h, s)]. \end{aligned}$$

Consequently, by  $D(x, y, \frac{1}{2}) = D(y, x, \frac{1}{2})$  we get with  $\kappa = \frac{\gamma'}{\gamma' + \gamma''}$ ,

$$\begin{aligned} & \min \left( D(t, t + h, \frac{1}{2}), D(x, x + h, \frac{1}{2}) \right) \\ & \leq \sup_{0 < s < 1} [\kappa D(t + h, x, s) + (1 - \kappa) D(t, x + h, s)]. \end{aligned}$$

Put  $t = x + (1 - 2\kappa)h$ . Then the last inequality for sufficiently small  $h > 0$  implies together with Lemma 6 that  $\kappa = \frac{1}{2}$ , i.e.  $\gamma' = \gamma''$ . Conversely, if  $\gamma' = \gamma''$ , then by Proposition 2 the natural selection rule attains for all parameter configurations the optimal exponential rate and is adaptive. To summarize, we have proved the following theorem.

**Theorem 7.** Under the assumptions of Proposition 1 an adaptive selection rule exists for the family  $(P_\eta)_{a_1 < \eta < a_2}$  iff the smallest two sample sizes are asymptotically equal, i.e.  $\gamma' = \gamma''$ . In this case the natural selection rule is adaptive.

**Example.** We now study the special case of normal populations and suppose that the r.v.  $X_{i,j} \sim N(\mu_i, \sigma_i^2)$   $1 \leq i \leq n_i$ ;  $i = 1, \dots, k$ , are independent. The variances  $\sigma_1^2, \dots, \sigma_k^2$  are supposed to be known and without loss of generality we assume  $\mu_1 \leq \dots \leq \mu_{k-1} < \mu_k$ .  $(\bar{X}_1, \dots, \bar{X}_k)$  is a sufficient statistic with  $\bar{X}_i \sim N(\mu_i, p_i)$ ,  $p_i = \frac{\sigma_i^2}{n_i}$ . To study the problem of selection of the best population it is therefore enough to consider the experiment

$$E = (\mathcal{R}_k, \mathcal{B}_k, Q_\vartheta, \vartheta \in \Theta)$$

where  $Q_\vartheta = \prod_{i=1}^k N(\mu_{\vartheta(i)}, p_i)$  and  $\Theta$  is the set of all permutations of  $\{1, \dots, k\}$ . The set  $\{\mu_1, \dots, \mu_k\}$  is known and fixed. The asymptotic  $p_i \rightarrow 0$  includes both the asymptotic for large sample sizes and the small variance asymptotic. To be more precise we assume that  $p_{i,n} > 0$   $i = 1, \dots, k$ , are sequences such that  $\lim_{n \rightarrow \infty} \frac{p_{i,n}}{p_n} = \rho_i > 0$  exist, where  $p_n = \sum_{i=1}^k p_{i,n}$ . Denote by  $\rho' \geq \rho''$  the two largest  $\rho_i$ -values. We set  $Q_{n,\vartheta} = \prod_{i=1}^k N(\mu_{\vartheta(i)}, p_{i,n})$  and study the asymptotic behavior of the error probability of selection procedures in the experiments

$$E_n = (\mathcal{R}_k, \mathcal{B}_k, Q_{n,\vartheta}, \vartheta \in \Theta). \tag{25}$$

To apply the results from Chapter 4 we need the Hellinger integral for two normal distributions. A simple calculation shows

$$H_s(N(\mu_1, \sigma^2), N(\mu_2, \sigma^2)) = \exp \left\{ -\frac{1}{2}s(1-s) \frac{(\mu_1 - \mu_2)^2}{\sigma^2} \right\}. \tag{26}$$

If  $\vartheta_1(l) = i_l, \vartheta_2(l) = j_l$  then by (10)

$$\ln H_s(Q_{n,\vartheta_1}, Q_{n,\vartheta_2}) = -\frac{1}{2}s(1-s) \sum_{l=1}^k \frac{(\mu_{i_l} - \mu_{j_l})^2}{p_{l,n}}. \tag{27}$$

Consequently, with  $c_n = \frac{1}{p_n}$

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(Q_{n,\vartheta_1}, Q_{n,\vartheta_2}) = -\frac{1}{2}s(1-s) \sum_{l=1}^k \frac{(\mu_{i_l} - \mu_{j_l})^2}{\rho_l} = G(\vartheta_1, \vartheta_2, s)$$

and

$$\begin{aligned} \max_{\vartheta_1 \neq \vartheta_2} \inf_{0 < s < 1} G(\vartheta_1, \vartheta_2, s) &= -\frac{1}{8} \frac{(\mu_{k-1} - \mu_k)^2}{\rho'} - \frac{1}{8} \frac{(\mu_k - \mu_{k-1})^2}{\rho''} \\ &= -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (\mu_{k-1} - \mu_k)^2. \end{aligned}$$

Analogously to Theorem 5 we get from Theorem 1 and Theorem 2 that for every sequence of selection rules  $q_n$  for the experiments  $E_n$  in (25)

$$\liminf_{n \rightarrow \infty} p_n \ln \left( \max_{\vartheta \in \Theta} (1 - Q_{n,\vartheta}(CS, q_n)) \right) \geq -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (\mu_{k-1} - \mu_k)^2 \tag{28}$$

where the lower bound is attained by the maximum likelihood selection rule  $q_n^0$ .

In order to study the asymptotic of the error probabilities of the natural selection rule we recall to the well known fact that

$$\lim_{t \rightarrow \infty} P(Z > t) \left[ \frac{1}{\sqrt{2\pi t}} e^{-\frac{t^2}{2}} \right]^{-1} = 1$$

if  $Z \sim N(0, 1)$ . This implies

$$\lim_{x \rightarrow \infty} \frac{1}{x} \ln P(Z > a\sqrt{x}) = -\frac{1}{2}a^2. \tag{29}$$

Denote by  $Y_i : \mathcal{R}_k \rightarrow \mathcal{R}_1$  the projections to the coordinates. Then  $Y_i \sim N(\mu_{\vartheta(i)}, p_{i,n})$  in the experiments  $E_n$ . The natural selection rule  $\tilde{q}_n$  selects the population  $\pi_i$  with  $Q_{n,\vartheta}$ -probability one iff  $Y_i > \max_{j \neq i} Y_j$ . Hence

$$\begin{aligned} \max_{1 \leq l \leq k-1} Q_{n,\vartheta}(Y_{\vartheta(k)} < Y_{\vartheta(l)}) &\leq 1 - Q_{n,\vartheta}(CS, \tilde{q}_n) \\ &\leq (k-1) \max_{1 \leq l \leq k-1} Q_{n,\vartheta}(Y_{\vartheta(k)} < Y_{\vartheta(l)}) \end{aligned}$$

and by (29) and similar considerations as before Theorem 6

$$\lim_{n \rightarrow \infty} p_n \ln \max_{\vartheta} (1 - Q_{n,\vartheta}(CS, \hat{q}_n)) = -\frac{1}{2} \frac{1}{\rho' + \rho''} (\mu_{k-1} - \mu_k)^2. \tag{30}$$

To compare the exponential rates of error probabilities of the maximum likelihood rule and the natural rule we note that

$$(\rho' \rho'')^{1/2} \leq \frac{1}{2}(\rho' + \rho'')$$

and consequently

$$\frac{1}{2} \frac{1}{\rho' + \rho''} \leq \frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} \tag{31}$$

where the sign of equality holds iff  $\rho' = \rho''$ . This means that in view of (28) and (30) the maximum likelihood selection rule is strictly better than the natural selection rule w. r. t. the exponential rates, except for  $\rho' = \rho''$  where the rates coincide. To relate this statement to Theorem 6 we consider the large sample asymptotic  $p_{i,n} = \frac{\sigma^2}{n_i}$  and note that for known  $\sigma^2$  the family  $N(\mu, \sigma^2), -\infty < \mu < \infty$ , is an exponential family with  $K(\mu) = \frac{1}{2} \frac{\mu^2}{\sigma^2}$ . Then  $D(\mu_1, \mu_2, s) = \frac{1}{2} s(1 - s) \frac{(\mu_1 - \mu_2)^2}{\sigma^2}$ .

Note that

$$\rho_i = \lim_{n \rightarrow \infty} \frac{p_{i,n}}{p_n} = \lim_{n \rightarrow \infty} \frac{n}{n_i} \left( \sum_{i=1}^k \frac{n}{n_i} \right)^{-1} = \frac{\Gamma}{\gamma_i}$$

where  $\Gamma = \sum_{i=1}^k \frac{1}{\gamma_i}$ . From this we can see that

$$M(\gamma', \gamma'', \mu_{k-1}, \mu_k) = \frac{\Gamma}{8} \frac{\rho' \rho''}{\rho' + \rho''} \frac{(\mu_{k-1} - \mu_k)^2}{\sigma^2}$$

and

$$\min_{i \neq j} L(\gamma_i, \gamma_j, \mu_{k-1}, \mu_k) = \frac{\Gamma}{2} \frac{1}{\rho' + \rho''} \frac{(\mu_{k-1} - \mu_k)^2}{\sigma^2}.$$

Therefore inequality (31) implies (23) and from the relations (28) and (30) we get the statements of Theorem 6 and Proposition 2 for normal populations.

Now we examine under which conditions there exist asymptotically optimal selection rules which do not use the knowledge of  $\{\mu_1, \dots, \mu_k\}$  in the experiments (25). To study the problem of the existence of adaptive selection rules we assume that  $\vartheta_1, \vartheta_2$  are permutations with  $\vartheta_1(l) = i_l, \vartheta_2(l) = j_l$  and set  $\alpha = (\alpha_1, \dots, \alpha_k), \alpha_1 \leq \dots \leq \alpha_{k-1} < \alpha_k, \beta = (\beta_1, \dots, \beta_k), \beta_1 \leq \dots \leq \beta_{k-1} < \beta_k,$

$$Q_{n,\vartheta_1}^\alpha = N(\alpha_{i_1}, p_{1,n}) \times \dots \times N(\alpha_{i_k}, p_{k,n})$$

$$Q_{n,\vartheta_2}^\beta = N(\beta_{j_1}, p_{1,n}) \times \dots \times N(\beta_{j_k}, p_{k,n}),$$

$c_n = \frac{1}{p_n}$ . Then by (26)

$$G(\alpha, \beta, \vartheta_1, \vartheta_2, s) = \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(Q_{n,\vartheta_1}^\alpha, Q_{n,\vartheta_2}^\beta) = - \sum_{l=1}^k \frac{1}{2} s(1 - s) \frac{(\alpha_{i_l} - \beta_{j_l})^2}{\rho_l}$$

and

$$G(\alpha, \alpha) = -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (\alpha_k - \alpha_{k-1})^2$$

$$G(\beta, \beta) = -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (\beta_k - \beta_{k-1})^2.$$

Put for  $h > 0$  and  $x > t$ ,  $\alpha = (\mu_0, \dots, \mu_0, t, t + h)$ ,  $\beta = (\mu_0, \dots, \mu_0, x, x + h)$ . Let  $l, m$  be two indices such that  $\rho_l = \rho', \rho_m = \rho''$  and choose two permutations  $\vartheta_1^0, \vartheta_2^0$  such that  $\alpha_{i_l} = t + h, \alpha_{i_m} = t, \beta_{j_l} = x, \beta_{j_m} = x + h$ .

Then

$$G(\alpha, \alpha) = G(\beta, \beta) = -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} h^2$$

and

$$\begin{aligned} \inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1^0, \vartheta_2^0, s) &= \inf_{0 < s < 1} -\frac{1}{2} s(1-s) \left[ \frac{(t+h-x)^2}{\rho'} + \frac{(t-x-h)^2}{\rho''} \right] \\ &= -\frac{1}{8} \left[ \frac{(t+h-x)^2}{\rho'} + \frac{(t-x-h)^2}{\rho''} \right] \\ &= -\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (h^2 + (t-x)^2) - \frac{1}{4} h \left( \frac{t-x}{\rho'} - \frac{t-x}{\rho''} \right). \end{aligned}$$

Suppose an adaptive selection rule exists. Then by the first statement in Lemma 1 and Definition 1

$$\inf_{0 < s < 1} G(\alpha, \beta, \vartheta_1^0, \vartheta_2^0, s) \leq \max(G(\alpha, \alpha), G(\beta, \beta)).$$

Hence

$$-\frac{1}{8} \frac{\rho' + \rho''}{\rho' \rho''} (t-x)^2 \leq -\frac{1}{4} h \left( \frac{t-x}{\rho'} - \frac{t-x}{\rho''} \right).$$

Because of  $x > t$  we get

$$\frac{\rho' + \rho''}{\rho' \rho''} (x-t) \geq 2h \frac{\rho' - \rho''}{\rho' \rho''}.$$

As  $\rho'' \leq \rho'$  this inequality for every  $h > 0$  and every  $x, t$  with  $x > t$  implies  $\rho' = \rho''$ .

As we know already that for  $\rho' = \rho''$  the natural rule is asymptotically optimal, we conclude that an adaptive selection rule for the experiments (25) exist iff  $\rho' = \rho''$ . For the large sample asymptotic  $n_i \rightarrow \infty$  this statement is also a consequence of Theorem 7.

### APPENDIX

In this Appendix, we collect and modify large deviation results for the log-likelihood. Although our statements below are presented for arbitrary sequences of distributions, some of the proofs are straightforward modifications of the proofs in the i. i. d. case,

since only some asymptotic behavior of the Hellinger integral is used. Therefore, complete proofs are omitted for brevity, but they can be found in Liese and Miescke [8].

Let  $P$  and  $Q$  be distributions on  $(\mathcal{X}, \mathcal{A})$  with densities  $f$  and  $g$ , resp., w. r. t. some  $\sigma$ -finite  $\mu$ . Suppose  $\pi_1, \pi_2 > 0$ . Let  $\varphi$  be a Bayes test for  $H_0 : P$  versus  $H_A : Q$  with weights  $\pi_1, \pi_2$ . Then  $\varphi = 1$  on  $\{\pi_1 f < \pi_2 g\}$ ,  $\varphi = 0$  on  $\{\pi_1 f \geq \pi_2 g\}$ , and  $0 \leq \varphi \leq 1$  be any value on  $\{\pi_1 f = \pi_2 g\}$ . If  $0 < s < 1$ , then

$$\pi_1 \int \varphi dP + \pi_2 \int (1 - \varphi) dQ \leq \pi_1^s \pi_2^{1-s} H_s(P, Q). \tag{32}$$

This inequality has been established in Krafft and Plachky [6]. Suppose now we are given sequences  $P_n$  and  $Q_n$  of distributions, and  $\pi_{i,n}$ ,  $i=1,2$ , with

$$\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \pi_{1,n} = a, \quad \lim_{n \rightarrow \infty} \frac{1}{c_n} \ln \pi_{2,n} = b.$$

Let  $\varphi_n$  be a sequence of Bayes tests for  $H_0 : P_n$  versus  $H_A : Q_n$  with weights  $\pi_{1,n}$  and  $\pi_{2,n}$ . Then by (32), with, say  $G(s) = \lim_{n \rightarrow \infty} (1/c_n) \ln H_s(P_n, Q_n)$ ,

$$\limsup_{n \rightarrow \infty} \left[ \pi_{1,n} \int \varphi_n dP_n + \pi_{2,n} \int (1 - \varphi_n) dQ_n \right]^{\frac{1}{c_n}} \leq e^{s a + (1-s) b + G(s)}. \tag{33}$$

Under some mild conditions equality holds in (33). This statement is known as Chernoff's theorem which was originally proved for the i. i. d. case and for  $\pi_{1,n} = \pi_1$ ,  $\pi_{2,n} = \pi_2$  which implies  $a = b = 0$ . The case where  $P_n, Q_n$  are restrictions to an increasing sequence of sub- $\sigma$ -algebras has been considered in Vajda [15]. An inspection of the proofs of Chernoff's theorem in Krafft and Plachky [6] and Vajda [15] shows that their arguments up to small modifications also work in the general situation. This leads to the following statement.

**Lemma 3.** Assume  $P_n \sim Q_n$  and  $\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(P_n, Q_n) = G(s)$  exists for every  $0 \leq s \leq 1$ , where  $G$  is continuously differentiable in  $(0, 1)$ , continuous in  $[0, 1]$  and not identical zero. If  $a = b$  then

$$\lim_{n \rightarrow \infty} \left[ \pi_{1,n} \int \varphi_n dP_n + \pi_{2,n} \int (1 - \varphi_n) dQ_n \right]^{\frac{1}{c_n}} = \exp \left\{ a + \inf_{0 < s < 1} G(s) \right\}.$$

The restriction to equivalent distributions ( $P_n \sim Q_n$ ) is not necessary. But for simplicity we consider only this situation everywhere throughout the paper.

To deal with the case where  $\pi_{1,n}$  and  $\pi_{2,n}$  have different asymptotic behavior in the sense of exponential rates one needs additional assumptions on  $G(s)$ . Then arguments of the paper by Krafft and Plachky [6] may be applied.

**Lemma 4.** Suppose  $\lim_{n \rightarrow \infty} \frac{1}{c_n} \ln H_s(P_n, Q_n) = G(s)$  exists for every  $-\infty < s < \infty$ , and assume that the interior  $I^0 = (a_1, a_2)$  of  $I = \{s : G(s) < \infty\}$  contains  $[0, 1]$ . If  $G$  is continuously differentiable in  $I^0$ , and if  $\lim_{s \downarrow a_1} G'(s) = -\infty$  and  $\lim_{s \uparrow a_2} G'(s) = \infty$ , then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[ \pi_{1,n} \int \varphi_n dP_n + \pi_{2,n} \int (1 - \varphi_n) dQ_n \right]^{\frac{1}{c_n}} \\ &= \exp \left\{ \inf_{0 < s < 1} [as + (1 - s)b + G(s)] \right\}. \end{aligned}$$

Now we formulate a large deviation result for the arithmetic mean of r. v. from an exponential family. To be more precise let  $P_\eta, \eta \in I$ , be the nontrivial exponential family (19) with natural parameter set  $I_T$ . Let  $m_i \rightarrow \infty, i = 1, 2$  so that  $\kappa = \lim_{m \rightarrow \infty} \frac{m_1}{m}$  exists and satisfies  $0 \leq \kappa < 1$ , where  $m = m_1 + m_2$ .

Suppose  $\eta_i \in I_T^0 = (a_1, a_2)$  and that  $X_{i,j}, j = 1, \dots, m_i, i = 1, 2$  are independent r. v. with  $X_{i,j} \sim P_{\eta_i}$ . Set  $\bar{T}_{i,m} = \frac{1}{m_i} \sum_{j=1}^{m_i} T(X_{i,j})$  and

$$D(\eta_1, \eta_2, s) = sK(\eta_1) + (1 - s)K(\eta_2) - K(s\eta_1 + (1 - s)\eta_2).$$

The following large deviation result may be proved by the method of exponential centering which can be found in Bahadur [1] and Rüschemdorf [10].

**Lemma 5.** If the r. v.  $X_{i,j}$  are independent with  $X_{i,j} \sim P_{\eta_i}, \eta_1 < \eta_2, \eta_i \in I_T^0$ ,

$\frac{m_1}{m} \xrightarrow{m \rightarrow \infty} \kappa$  and  $0 < \kappa < 1$ , then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{1}{m} \ln P(\bar{T}_{1,m} - \bar{T}_{2,m} > 0) \\ &= \lim_{m \rightarrow \infty} \frac{1}{m} \ln P(\bar{T}_{1,m} - \bar{T}_{2,m} \geq 0) = -D(\eta_1, \eta_2, \kappa). \end{aligned}$$

Next we formulate two technical lemmas on convex functions. Suppose  $K$  is a convex function defined on the interval  $I$  and  $K$  is twice continuously differentiable in the interior  $(a, b)$  of  $I$ . An application of l'Hospital's rule yields, for every fixed  $x \in (a, b)$ , and for every  $t_1, t_2$ ,

$$\begin{aligned} \lim_{h \downarrow 0} \frac{2}{h^2} D(x + t_1 h, x + t_2 h, s) &= \lim_{h \downarrow 0} \frac{2}{h^2} [sK(x + t_1 h) + (1 - s)K(x + t_2 h) \\ &\quad - K(s(x + t_1 h) + (1 - s)(x + t_2 h))] \\ &= K''(x)s(1 - s)(t_1 - t_2)^2, \end{aligned} \tag{34}$$

where the limit is uniform for  $0 \leq s \leq 1$ .

**Lemma 6.** Suppose  $x \in (a, b)$ ,  $K''(x) > 0$ ,  $h_n > 0$ , and  $h_n \downarrow 0$  for  $n \rightarrow \infty$ .

If  $0 < \kappa \leq \frac{1}{2}$  and

$$\begin{aligned} & \min \left( D\left(x, x + h_n, \frac{1}{2}\right), D\left(x + (1 - 2\kappa)h_n, x + 2(1 - \kappa)h_n, \frac{1}{2}\right) \right) \\ & \leq \sup_{0 < s < 1} [\kappa D(x, x + 2(1 - \kappa)h_n, s) + (1 - \kappa) D(x + h_n, x + (1 - 2\kappa)h_n, s)] \end{aligned}$$

for every  $n$ , then it follows that  $\kappa = \frac{1}{2}$ .

*Proof.* The relation (34) yields, uniformly in  $0 \leq s \leq 1$

$$\begin{aligned} & \frac{2}{h_n^2} [\kappa D(x, x + 2(1 - \kappa)h_n, s) + (1 - \kappa) D(x + h_n, x + (1 - 2\kappa)h_n, s)] \\ & = 4\kappa(1 - \kappa) K''(x)s(1 - s) + o(1), \end{aligned}$$

and

$$\begin{aligned} & \frac{2}{h_n^2} \left[ \min \left( D\left(x, x + h_n, \frac{1}{2}\right), D\left(x + (1 - 2\kappa)h_n, x + 2(1 - \kappa)h_n, \frac{1}{2}\right) \right) \right] \\ & = \frac{1}{4} K''(x) + o(1). \end{aligned}$$

Hence, by the assumed inequalities, as  $n \rightarrow \infty$ ,

$$\frac{1}{4} K''(x) \leq \sup_{0 < s < 1} 4\kappa(1 - \kappa) K''(x)s(1 - s) = \kappa(1 - \kappa) K''(x),$$

which implies  $\kappa = \frac{1}{2}$ , since  $K''(x) > 0$ . □

**Lemma 7.** Let  $K : (a, b) \rightarrow \mathbb{R}$ , be a convex function and assume that  $a < x < y < b$  are fixed. If  $\gamma_1, \dots, \gamma_k$  are positive numbers, and  $\gamma' \leq \gamma''$  are the two smallest  $\gamma_i$ -values, then

$$\begin{aligned} & \min_{i \neq j} (\gamma_i + \gamma_j) D\left(x, y, \frac{\gamma_i}{\gamma_i + \gamma_j}\right) \\ & \leq (\gamma' + \gamma'') \sup_{0 < s < 1} \left[ \frac{\gamma'}{\gamma' + \gamma''} D(x, y, s) + \frac{\gamma''}{\gamma' + \gamma''} D(x, y, 1 - s) \right]. \end{aligned} \tag{35}$$

If  $\gamma' = \gamma''$  then equality holds. Conversely, if for a strictly convex  $K$  equality holds in the above inequality, then  $\gamma' = \gamma''$ .

*Sketch of proof:* We have

$$\begin{aligned} & \min_{i \neq j} (\gamma_i + \gamma_j) D\left(x, y, \frac{\gamma_i}{\gamma_i + \gamma_j}\right) \\ & \leq (\gamma' + \gamma'') \min \left[ D\left(x, y, \frac{\gamma'}{\gamma' + \gamma''}\right), D\left(x, y, \frac{\gamma''}{\gamma' + \gamma''}\right) \right] \\ & \leq (\gamma' + \gamma'') \left[ \frac{\gamma'}{\gamma' + \gamma''} D\left(x, y, \frac{\gamma'}{\gamma' + \gamma''}\right) + \frac{\gamma''}{\gamma' + \gamma''} D\left(x, y, \frac{\gamma''}{\gamma' + \gamma''}\right) \right] \\ & \leq (\gamma' + \gamma'') \sup_{0 < s < 1} \left[ \frac{\gamma'}{\gamma' + \gamma''} D(x, y, s) + \frac{\gamma''}{\gamma' + \gamma''} D(x, y, 1 - s) \right]. \end{aligned}$$

Hence (35) is established. It is easy to see that  $D(x, y, \alpha) \geq 2\alpha D(x, y, \frac{1}{2})$  for  $x < y, \alpha \leq \frac{1}{2}$ . Consequently, for  $\gamma' = \gamma''$  and  $\gamma_i \leq \gamma_j$

$$\begin{aligned} & (\gamma_i + \gamma_j) D\left(x, y, \frac{\gamma_i}{\gamma_i + \gamma_j}\right) \\ & \geq 2\gamma_i D\left(x, y, \frac{1}{2}\right) \\ & \geq 2\gamma' D\left(x, y, \frac{1}{2}\right) \\ & = (\gamma' + \gamma'') \sup_{0 < s < 1} \left[ \frac{\gamma'}{\gamma' + \gamma''} D(x, y, s), \frac{\gamma''}{\gamma' + \gamma''} D(x, y, 1 - s) \right] \end{aligned}$$

as  $D(x, y, s) + D(x, y, 1 - s)$  is symmetric w. r. t.  $s_0 = \frac{1}{2}$ . Hence for  $\gamma' = \gamma''$  we have equality in (35).

Conversely, an analysis of the stability of the above inequalities shows that equality occurs only for  $\gamma' = \gamma''$ .

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