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# THE VARIATIONAL PRINCIPLE OF FIXED POINT THEOREMS IN CERTAIN FUZZY TOPOLOGICAL SPACES<sup>1</sup>

P. BALASUBRAMANIAM AND S. MURALI SANKAR

The main purpose of this paper is to introduce the concept of  $F$ -type fuzzy topological spaces. Further variational principle and Caristi's fixed point theorem have been extended in the  $F$ -type fuzzy topological spaces.

## 1. INTRODUCTION

General topology can be regarded as a special case of fuzzy topology where all membership functions in question take values 0 and 1 only. The usual fuzzy metric spaces, fuzzy Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of  $F$ -type fuzzy topological spaces. Therefore, one would expect weaker results in the case of fuzzy topology. Recently several metric space fixed point theorems were extended to fuzzy topological spaces. Many authors introduced the concept of fuzzy metric spaces in different ways (see [4, 5, 11]). Grabiec [5] proved the contraction principle in the setting of fuzzy metric spaces introduced by Kramosil and Michálek [7]. The famous Ekeland's variational principle and Caristi's fixed point theorem are forceful tools in nonlinear analysis, control theory, economic theory and global analysis for details (see [1, 2, 3]).

In this paper, we establish a variational principle and Caristi's fixed point theorem in  $F$ -type fuzzy topological spaces and utilize the results to obtain a fixed point theorem for Menger probabilistic metric spaces. Our results generalize the previous results of [1, 2, 3].

## 2. PRELIMINARIES

In this section some necessary definitions are stated. Throughout this paper  $D = (D, <)$  denotes a direct set,  $<$  a partial order relation.

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**Definition 2.1.** (L. A. Zadeh [11]) Let  $X$  be any set. A fuzzy set  $A$  in  $X$  is a function with domain  $X$  and values in  $[0, 1]$ .

**Definition 2.2.** (B. Schweizer and A. Sklar [9]) A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if  $([0, 1], *)$  is an abelian topological monoid with unit 1 such that  $a * b \leq c * d$  whenever  $a \leq c$ ,  $b \leq d$ ,  $(a, b, c, d \in [0, 1])$ .

**Definition 2.3.** (A. George and P. Veeramani [4]) The 3-tuple  $(X, M, *)$  is said to be a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions:

1.  $M(x, y, t) > 0$ .
2.  $M(x, y, t) = 1$  if and only if  $x = y$  for all  $t > 0$ .
3.  $M(x, y, t) = M(y, x, t)$ .
4.  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ .
5.  $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous  
 $x, y, z \in X$  and  $t, s > 0$ .

**Definition 2.4.** (A. George and P. Veeramani [4]) A sequence  $\{x_n\}$  in a fuzzy metric space  $(X, M, *)$  is called Cauchy if  $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1$  for every  $t > 0$  and  $p > 0$ .  $(X, M, *)$  is complete if every Cauchy sequence in  $X$  converges in  $X$ . A sequence  $\{x_n\}$  in  $X$  is convergent to  $x \in X$  if  $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$  for each  $t > 0$ .

**Definition 2.5.** (R. Srivastava, S. N. Lal and A. K. Srivastava [10]) A fuzzy topology on a set  $X$  is a collection  $T$  of fuzzy sets in  $X$  such that

- i)  $\emptyset, X \in T$ .
- ii)  $A, B \in T \Rightarrow A \cap B \in T$ .
- iii)  $\{A_i / i \in I\} \subseteq T \Rightarrow \bigcup_i A_i \in T$ .

Members of  $T$  are called  $T$ -open fuzzy sets and the pair  $(X, T)$  is called a fuzzy topological space. Complements of open fuzzy sets are called closed fuzzy sets.

**Definition 2.6.** (A. K. Katsaras and D. B. Liu [6]) A fuzzy topological vector space  $E$  equipped with a fuzzy topology such that the two maps

- (a)  $\varphi : E \times E \rightarrow E$   
 $(x, y) \rightarrow x + y$
- (b)  $\psi : K \times E \rightarrow E$   
 $(\lambda, x) \rightarrow \lambda x$

are continuous when  $K$  has the usual topology and  $E \times E, K \times E$  are given product fuzzy topologies.

**Definition 2.7.** (R. Srivastava, S. N. Lal and A. K. Srivastava [10]) A fuzzy point  $p$  is said to belong to a fuzzy set  $A$  in  $X$  (notation:  $p \in A$ ) if and only if

$$\mu_p(x_p) < \mu_A(x_p) \quad \text{and} \quad \mu_p(x) \leq \mu_A(x) \quad \text{if } x \neq x_p.$$

Throughout this paper we shall use ' $p \in A$ ' in the sense of definition.

**Definition 2.8.** (R. Srivastava, S. N. Lal and A. K. Srivastava [10]) Let  $p$  be a fuzzy point in  $(X, T)$ . A fuzzy set  $N$  is called a fuzzy neighbourhood of  $p$  if and only if there exists a  $T$ -open fuzzy set  $A$  such that  $p \in A \subseteq N$ . If  $N$  is  $T$ -open it is called a fuzzy open neighbourhood of  $p$  in  $(X, T)$ .

**Definition 2.9.** (R. Srivastava, S. N. Lal and A. K. Srivastava [10]) A fuzzy topological space  $(X, T)$  is said to be fuzzy Hausdorff if and only if for any two distinct fuzzy points  $p, q \in X$ , there exists a disjoint  $U, V \in T$  with  $p \in U$  and  $q \in V$ .

**Definition 2.10.** A fuzzy topological space  $(X, T)$  is said to be  $F$ -type, if it is Hausdorff and for each  $x \in X$ , there exists a neighbourhood base

$$U_x = \{U_x(\lambda, r, t) \mid \lambda \in D, t > 0, 0 < r < 1\}$$

of  $x$  with the following properties:

(F.1) If  $y \in U_x(\lambda, r, t)$ , then  $x \in U_y(\lambda, r, t)$ .

(F.2)  $U_x(\lambda, r_1, t) \subset U_y(\mu, r_2, t)$  for  $\lambda < \mu, r_1 \leq r_2$ .

(F.3) For every  $\lambda \in D$  there exists  $\mu \in D$  with  $\lambda < \mu$  such that

$$y \in U_x(\lambda, r_1 + r_2, t) \text{ whenever } U_x(\mu, r_1, t) \cap U_y(\mu, r_2, t) \neq \emptyset. \quad (2.1)$$

(F.4)  $X = \bigcup_{0 < r < 1} U_x(\lambda, r, t)$  for each  $\lambda \in D$  and  $x \in X$ .

**Definition 2.11.** An  $F$ -type fuzzy topological space  $(X, T)$  is sequentially complete if and only if every Cauchy sequence is convergent.

**Definition 2.12.** The 3-tuple  $(X, M, *)$  is said to be a fuzzy quasi-metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm,  $D$  is a directed set and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions:

(Q.1)  $M_\lambda(x, y, t) = 1$  if and only if  $x = y$  for all  $t > 0$  and  $\lambda \in D$ .

(Q.2)  $M_\lambda(x, y, t) = M_\lambda(y, x, t)$ .

(Q.3)  $M_\lambda(x, y, t) \leq M_\mu(x, y, t)$  for  $\lambda < \mu$ .

(Q.4) For every  $\lambda \in D$  there exists  $\mu \in D$  with  $\lambda < \mu$  such that

$$M_\mu(x, y, t) * M_\mu(y, z, t) \leq M_\lambda(x, z, t) \quad \text{for all } x, y, z \in X, t > 0. \quad (2.2)$$

**Theorem 2.1.** Let  $Q = \{M_\lambda/\lambda \in D\}$  be a family of fuzzy quasi-metrics on  $X$ . Let

$$B_x(\lambda, r, t) = \{y \in X : M_\lambda(x, y, t) > 1 - r\} \quad (2.3)$$

where  $x \in X$  and  $t > 0$  and  $0 < r < 1$ . Then there exists a unique fuzzy Hausdorff topology  $T_Q$  on  $X$  such that  $(X, T_Q)$  is an  $F$ -type fuzzy topological space and

$$\mathcal{B}_x = \{B_x(\lambda, r, t) : \lambda \in D, t > 0 \text{ and } 0 < r < 1\}$$

is a neighbourhood base of  $x$  for  $T_Q$ . The fuzzy topology  $T_Q$  is called the fuzzy topology on  $X$  deduced by the family  $Q$  of fuzzy quasi-metrics.

**Proof.** Let  $\mathcal{N}_x = \{W \mid W \subset X, \exists B_x(\lambda, r, t) \subset W\}$  for each  $x \in X$ . It is easy to verify that  $\mathcal{N}_x$  satisfies the following conditions:

- (1) If  $W \in \mathcal{N}_x$ , then  $x \in W$ .
- (2) If  $W_1, W_2 \in \mathcal{N}_x$ , then  $W_1 \cap W_2 \in \mathcal{N}_x$ .
- (3) If  $W \in \mathcal{N}_x$  and  $W \subset V$ , then  $V \in \mathcal{N}_x$ .
- (4) If  $W \in \mathcal{N}_x$ , then  $\exists V \in \mathcal{N}_x$  such that  $V \subset W$  and  $W \in \mathcal{N}_y$  for each  $y \in V$ .

We prove only (4), since  $W \in \mathcal{N}_x$ ,  $B_x(\lambda, r, t) \subset W$  for some  $\lambda \in D$  and  $t > 0$  and  $0 < r < 1$ . By (Q.4), for above  $\lambda$  there exists an  $\mu \in D$  with  $\lambda < \mu$  such that (2.2) holds. Obviously,

$$V = B_x(\lambda, r/2, t) \in \mathcal{N}_x.$$

Moreover, for each  $y \in V$  we can prove that  $B_y(\mu, r/2, t) \subset W$ . In fact, if  $z \in B_y(\mu, r/2, t)$  then  $M_\mu(y, z, t) > 1 - (r/2)$ .

Note that  $M_\mu(x, y, t) > 1 - (r/2)$ .

Thus by (2.2) we have

$$\begin{aligned} M_\lambda(x, z, t) &\geq M_\mu(x, y, t) * M_\mu(y, z, t) \\ &> \{1 - r/2\} * \{1 - r/2\} \\ &> (1 - r) \text{ for all } 0 < r < 1 \end{aligned}$$

and so  $z \in B_x(\lambda, r, t) \subset W$ .

Hence  $B_y(\mu, r/2, t) \subset W$ . This implies that  $W \in \mathcal{N}_y$ .

Therefore, there exists a unique fuzzy topology on  $X$ , written  $T_Q$  such that  $\mathcal{N}_x$  is the neighbourhood system of  $x$  for  $T_Q$ . By the definition of  $\mathcal{N}_x$ , it is easy to know that  $\mathcal{B}_x$  is a neighbourhood base of  $x$ .

Now we prove that  $T_Q$  is fuzzy Hausdorff. Let  $x, y \in X$ ,  $x \neq y$ . By (Q.1), there exists some  $\lambda \in D$  such that  $M_\lambda(x, y, t) = 1 - r$ . By (Q.4), there exists  $\mu \in D$  with  $\lambda < \mu$  such that (2.2) holds. From this we can prove that  $B_x(\mu, r/2, t) \cap B_y(\mu, r/2, t) = \emptyset$ . Suppose  $z \in B_x(\mu, r/2, t) \cap B_y(\mu, r/2, t)$  then

$$\begin{aligned} 1 - r &= M_\lambda(x, y, t) \\ &\geq M_\mu(x, z, t) * M_\mu(z, y, t) \\ &> \{1 - r/2\} * \{1 - r/2\} \\ &> (1 - r) \text{ for all } 0 < r < 1 \end{aligned}$$

which is a contradiction. Hence  $T_Q$  is Hausdorff.

Moreover, it is not difficult to show that  $\mathcal{B}_x$  satisfies (F.1) to (F.4), we only verify (F.3) others are trivial. For  $\lambda \in D$ , by (Q.4) there exists  $\mu \in D$  with  $\lambda < \mu$  such that (2.2) holds.

If  $B_x(\mu, r_1, t) \cap B_y(\mu, r_2, t) \neq \emptyset$ , then there is  $z \in B_x(\mu, r_1, t) \cap B_y(\mu, r_2, t)$ , so that

$$M_\mu(x, z, t) > 1 - r_1 \quad \text{and} \quad M_\mu(y, z, t) > 1 - r_2.$$

Thus by (2.2) we have,

$$\begin{aligned} M_\lambda(x, y, t) &\geq M_\mu(x, z, t) * M_\mu(z, y, t) \\ &> (1 - r_1) * (1 - r_2) \\ &> (1 - r) \quad \text{for all } 0 < r < 1 \end{aligned}$$

which implies that  $y \in B_x(\lambda, r_1 + r_2, t)$ . Therefore  $(X, T_Q)$  is  $F$ -type. □

**Theorem 2.2.** Let  $(X, T)$  be an  $F$ -type fuzzy topological space. Then there exists a family  $Q = \{M_\lambda : \lambda \in D\}$  of fuzzy quasi-metrics on  $X$  satisfying (Q.1) to (Q.4) in Theorem 2.1 such that  $T_Q = T$ . In this case,  $Q$  is called the generating family of quasi-metrics for  $T$ .

*Proof.* Let  $U_x = \{U_x(\lambda, r, t) : \lambda \in D, t > 0, 0 < r < 1\}$  be a neighbourhood base of  $x$  for  $T$  satisfying (F.1) to (F.4). By (F.4), for each  $\lambda \in D$  we can define a mapping  $M_\lambda : X^2 \times (0, \infty) \rightarrow [0, 1]$  as

$$M_\lambda(x, y, t) = \inf\{0 < r < 1 : y \in U_x(\lambda, r, t)\}$$

Now, we prove that  $Q = \{M_\lambda : \lambda \in D\}$  satisfies (Q.1) to (Q.4). It is trivial that (F.1) implies (Q.2) and (F.2) implies (Q.3).

(Q.1) By the definition of  $M_\lambda$ , it is obvious that  $M_\lambda(x, x, t) = 1$  for all  $\lambda \in D$ . Conversely, if  $x, y \in X$  with  $x \neq y$  then by Hausdorff property of  $T$ , there exist  $U_x(\lambda, r_1, t)$  and  $U_y(\mu, r_2, t)$  such that

$$U_x(\lambda, r_1, t) \cap U_y(\mu, r_2, t) = \emptyset.$$

This implies that  $y \notin U_x(\lambda, r_1, t)$  and so  $M_\lambda(x, y, t) \leq 1 - r$ .

(Q.4) Let  $\lambda \in D$ . By (F.3) there exists  $\mu \in D$  with  $\lambda < \mu$  such that (2.1) holds. By the definitions of  $M_\mu(x, z, t)$  and  $M_\mu(z, y, t)$  for any given  $0 < \varepsilon < 1$  there exists  $0 < r_1, r_2 < 1$  with

$$1 - r_1 > M_\mu(x, z, t) * (1 - \varepsilon)$$

and

$$1 - r_2 > M_\mu(z, y, t) * (1 - \varepsilon)$$

such that  $z \in U_x(\mu, r_1, t)$  and  $z \in U_y(\mu, r_2, t)$  which implies

$$U_x(\mu, r_1, t) \cap U_y(\mu, r_2, t) \neq \emptyset.$$

Thus, it follows from (2.1) that  $y \in U_x(\lambda, r_1 + r_2, t)$  and so

$$\begin{aligned} M_\lambda(x, y, t) &\geq (1 - r_1) * (1 - r_2) \\ &> M_\mu(x, z, t) * M_\mu(z, y, t) * (1 - \varepsilon)^2. \end{aligned}$$

By the arbitrariness of  $\varepsilon$  we get

$$M_\lambda(x, y, t) \geq M_\mu(x, z, t) * M_\mu(z, y, t).$$

Lastly, we prove that  $T = T_Q$ . By Theorem 2.1 we know that

$$B_x = \{B_x(\lambda, r, t) : \lambda \in D, t > 0, 0 < r < 1\}$$

is a neighbourhood base of  $x$  to  $T_Q$ , where

$$B_x(\lambda, r, t) = \{y \in X : M_\lambda(x, y, t) > 1 - r\}.$$

It is obvious that  $B_x(\lambda, r, t) \subset U_x(\lambda, r, t)$  and  $U_x(\lambda, r/2, t) \subset B_x(\lambda, r, t)$ .

Hence  $T = T_Q$ . □

**Corollary 2.1.** Let  $(X, T)$  be an  $F$ -type fuzzy topological space and  $Q = \{M_\lambda : \lambda \in D\}$  be a generating family of fuzzy quasi-metrics for  $T$ ,  $\{x_n\} \subset X$  and  $x \in X$ . Then

1.  $x_n \rightarrow x$  if and only if  $\lim_{n \rightarrow \infty} M_\lambda(x_n, x, t) = 1$  for all  $\lambda \in D, t > 0$ .
2.  $\{x_n\}$  is a Cauchy sequence if and only if for each  $\lambda \in D$  and each  $0 < \varepsilon < 1$  there exists a positive integer  $N$  such that  $M_\lambda(x_m, x_n, t) > 1 - \varepsilon$  whenever  $m, n > N$ .

**Example 2.1.** Every fuzzy metric space  $(X, M, *)$  is an  $F$ -type fuzzy topological space. In fact, we can arbitrarily take a directed set  $D$ , say  $D = (0, 1)$  and define

$$M_\lambda(x, y, t) = M(x, y, t) \quad \text{for all } \lambda \in D, t > 0, x, y \in X.$$

Then it is easy to see that  $\{M_\lambda : \lambda \in D\}$  satisfies (Q.1) to (Q.4) in Theorem 2.1. Therefore  $(X, T_M)$  is an  $F$ -type fuzzy topological space where  $T_M$  is a fuzzy topology deduced by the metric  $M$  on  $X$ .

**Example 2.2.** Every fuzzy Hausdorff topological vector space  $X$  is an  $F$ -type fuzzy topological space. In fact, let  $\mathcal{U} = \{U_\alpha : \alpha \in D\}$  be a balanced neighbourhood base of 0. Define a partially order on  $D$  as  $\alpha < \beta \Leftrightarrow U_\beta \subset U_\alpha$ . Thus  $D = (D, <)$  is directed set. Let

$$U_\lambda(\lambda, r, t) = x + rU_\lambda \quad \text{for } x \in X, \lambda \in D, 0 < r < 1.$$

Obviously,  $\mathcal{U}_x = \{U_x(\lambda, r, t) : \lambda \in D, t > 0, 0 < r < 1\}$  is a neighbourhood base of  $x$ . Notice that  $\mathcal{U}$  has the following properties: every  $U_\alpha$  in  $\mathcal{U}$  is balanced and absorbing and for each  $U_\alpha \in \mathcal{U}$  there exists an  $U_\beta \in \mathcal{U}$  such that  $U_\beta + U_\beta \subset U_\alpha$ . Hence it is not difficult to show that  $\mathcal{U}_x (x \in X)$  satisfies (F.1) to (F.4). Hence  $X$  is  $F$ -type.

### 3. VARIATIONAL PRINCIPLE AND FIXED POINT THEOREM

**Lemma 3.1.** Let  $(X, T)$  be a sequentially complete  $F$ -type fuzzy topological space and  $\{M_\lambda : \lambda \in D\}$  be a generating family of fuzzy quasi-metrics for  $T$ . Let  $\{W_n\}$  be a sequence of nonempty subsets in  $X$  with the following properties:

1.  $W_n \supset W_{n+1}$ ,  $(n = 1, 2, \dots)$ .
2.  $\lim_{n \rightarrow \infty} \delta_\lambda(W_n) = 1$  for all  $\lambda \in D$  where  $\delta_\lambda(A) = \sup\{M_\lambda(x, y, t) : x, y \in A \text{ and } t > 0\}$ . Then

$$\lim_{n \rightarrow \infty} \delta_\lambda(\overline{W}_n) = 1 \quad \text{for all } \lambda \in D \tag{3.1}$$

and there exists a unique  $\xi \in \bigcap_{n=1}^\infty \overline{W}_n$ .

*Proof.* We first prove (3.1). For any given  $\lambda \in D$  and  $n \in N$ , by the definition of  $\delta_\lambda(\overline{W}_n)$  we can choose two sequences  $\{x_m\}$  and  $\{y_m\}$  in  $\overline{W}_n$  such that

$$\lim_{m \rightarrow \infty} M_\lambda(x_m, y_m, t) = \delta_\lambda(\overline{W}_n)$$

since  $x_m, y_m \in \overline{W}_n$ , there exists two sequences  $\{x_m^k\}_{k=1}^\infty, \{y_m^k\}_{k=1}^\infty$  in  $W_n$  such that  $x_m^k \rightarrow x_m, y_m^k \rightarrow y_m$  as  $k \rightarrow \infty$  so that,

$$\lim_{k \rightarrow \infty} M_\alpha(x_m^k, x_m, t) = 1$$

and

$$\lim_{k \rightarrow \infty} M_\alpha(y_m^k, y_m, t) = 1 \quad \text{for all } \alpha \in D, t > 0.$$

Notice that  $\{M_\lambda : \lambda \in D\}$  satisfies (Q.3) and (Q.4). Hence for the given  $\lambda \in D$  there exists  $\mu \in D$  with  $\lambda < \mu$  such that

$$\begin{aligned} M_\lambda(x_m, y_m, t) &\geq M_\mu(x_m, x_m^k, t) * M_\mu(x_m^k, y_m^k, t) * M_\mu(y_m^k, y_m, t) \\ &\geq M_\mu(x_m, x_m^k, t) * \delta_\mu(W_n) * M_\mu(y_m^k, y_m, t). \end{aligned}$$

Letting  $k \rightarrow \infty$  we get

$$M_\lambda(x_m, y_m, t) \geq \delta_\mu(W_n)$$

and so

$$\delta_\lambda(\overline{W}_n) = \lim_{m \rightarrow \infty} M_\lambda(x_m, y_m, t) \geq \delta_\mu(W_n).$$

Noting that,  $\lim_{n \rightarrow \infty} M_\mu(W_n) = 1$ , hence (3.1) holds.

Next we show that there exists a point  $\xi \in \bigcap_{n=1}^\infty \overline{W}_n$ . Arbitrary taking  $x_n \in \overline{W}_n$ ,  $(n = 1, 2, \dots)$  by (3.1), for each  $0 < \varepsilon < 1$  and  $\lambda \in D$  there exists  $N \in \mathbb{N}$ , the set of all natural numbers such that

$$\sup_{x, y \in \overline{W}_n} M_\lambda(x, y, t) > 1 - \varepsilon \quad \text{whenever } n > N.$$



Since  $x_m \in \overline{W}_m \subset \overline{W}_n$  for  $m, n \in N$  with  $m \geq n$  we have

$$M_\lambda(x_m, x_n, t) > 1 - \varepsilon \quad \text{whenever } m \geq n > N.$$

This implies that  $\{x_n\}$  is a Cauchy sequence in  $(X, T)$ . By the sequential completeness of  $X$ , there exists a point  $\xi \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = \xi.$$

Notice that  $x_m \in \overline{W}_n, m \geq n, (n = 1, 2, \dots)$ .

Hence we have  $\xi \in \overline{W}_n, (n = 1, 2, \dots)$ . Thus  $\xi \in \bigcap_{n=1}^\infty \overline{W}_n$ .

Lastly we prove the uniqueness of  $\xi$ . Assume that there exist  $\eta \in \bigcap_{n=1}^\infty \overline{W}_n$  with  $\eta \neq \xi$ . Since  $\{M_\lambda : \lambda \in D\}$  satisfies (Q.1), there exists  $\lambda \in D$  such that

$$M_\lambda(\xi, \eta, t) > 1 - r \quad \text{for all } 0 < r < 1.$$

By (3.1), we have

$$\delta_\lambda(\overline{W}_N) > 1 - r \quad \text{for some } N \in \mathbb{N}.$$

Let  $\xi, \eta \in \overline{W}_n$ , we get

$$\begin{aligned} M_\lambda(\xi, \eta, t) &\geq \delta_\lambda(\overline{W}_N) \\ &> M_\lambda(\xi, \eta, t) \end{aligned}$$

which is a contradiction. This completes the proof. □

**Lemma 3.2.** Let  $(X, T)$  be an  $F$ -type fuzzy topological space and  $\{M_\lambda : \lambda \in D\}$  be a generating family of fuzzy quasi-metrics for  $T$ . Let  $\varphi : X \rightarrow [0, 1]$  be a lower semi-continuous function, bounded from below and  $k : D \rightarrow (0, \infty)$  be non-increasing function. We define a relation  $<$  on  $X$  as  $x < y$  iff

$$M_\lambda(x, y, t) \leq M_\lambda(\varphi(x), \varphi(y), tk(\lambda)) \quad \text{for all } \lambda \in D, t > 0. \tag{3.2}$$

Then  $(X, <)$  is a partial order set and it has a maximal element.

*Proof.* We first prove that  $<$  is a partial order on  $X$ . It is obvious that  $x < x$  for each  $x \in X$ . If  $x, y \in X, x < y$  and  $y < x$  then by (3.2), we have  $\varphi(x) \geq \varphi(y)$ , and  $\varphi(x) \leq \varphi(y)$ , so that  $\varphi(x) = \varphi(y)$ . From this it follows that,

$$M_\lambda(x, y, t) = 1 \quad \text{for all } \lambda \in D \text{ and so } x = y.$$

If  $x, y, z \in X$  with  $x < y$  and  $y < z$  then we have

$$M_\alpha(x, y, t) \leq M_\alpha(\varphi(x), \varphi(y), tk(\alpha))$$

and

$$M_\alpha(y, z, t) \leq M_\alpha(\varphi(y), \varphi(z), tk(\alpha)) \quad \text{for all } \alpha \in D \text{ and } t > 0.$$

By (Q.3), for every  $\lambda \in D$  there exists  $\mu \in D, \lambda < \mu$  such that

$$\begin{aligned} M_\lambda(x, z, t) &\geq M_\mu(x, y, t) * M_\mu(y, z, t) \\ &\geq M_\mu(\varphi(x), \varphi(z), tk(\mu)). \end{aligned}$$

Note that  $k(\lambda)$  is non increasing. We have

$$M_\lambda(x, z, t) \leq M_\lambda(\varphi(x), \varphi(z), tk(\lambda)) \quad \text{for all } \lambda \in D \text{ and so } x < z.$$

Therefore  $<$  is a partial order on  $X$ . Next, we prove that the partially ordered set  $(X, <)$  has a maximal element.

Suppose that  $C$  is totally ordered subset of  $X$ . By (3.2), it is easy to know that  $\varphi$  is monotone decreasing on  $C$ . Note that  $\varphi$  is bounded from below. Hence we can let  $\gamma = \inf_{x \in C} \varphi(x)$  and can choose an increasing sequence  $\{x_n\}$  in  $C$  such that  $\varphi(x_n)$  decreasing  $\gamma$  as  $n \rightarrow \infty$ . Let

$$W_n = \{x \in C : \varphi(x) \leq \varphi(x_n)\}, \quad (n = 1, 2, \dots).$$

Obviously,  $W_n \neq \emptyset$ , since  $x_n \in W_n$  and  $W_n \supset W_{n+1}$  ( $n = 1, 2, \dots$ ).

Moreover, let  $y, z \in W_n$  then  $\gamma \leq \varphi(z)$  and  $\varphi(y) \leq \varphi(x_n)$ . Without loss of generality we assume  $y < z$ . By (3.2), we have

$$\begin{aligned} M_\lambda(y, z, t) &\leq M_\lambda(\varphi(y), \varphi(z), tk(\lambda)) \\ &\leq M_\lambda(\varphi(x_n), \gamma, tk(\lambda)) \quad \text{for all } \lambda \in D \end{aligned}$$

and so

$$\delta_\lambda(W_n) \leq M_\lambda(\varphi(x_n), \gamma, tk(\lambda)) \quad \text{for all } \lambda \in D.$$

Letting  $n \rightarrow \infty$  we get  $\lim_{n \rightarrow \infty} \delta_\lambda(W_n) = 1$  for all  $\lambda \in D$ .

Thus from Lemma 3.1, it follows that  $\lim_{n \rightarrow \infty} \delta_\lambda(\overline{W}_n)$  for all  $\lambda \in D$  and exists a unique  $\xi \in \bigcap_{n=1}^\infty \overline{W}_n$ . Since  $x_n \in W_n \subset \overline{W}_n$  ( $n = 1, 2, \dots$ ) we have

$$\lim_{n \rightarrow \infty} M_\lambda(x_n, \xi, t) = 1 \quad \text{for all } \lambda \in D.$$

Now we show that  $\xi$  is an upper bound of  $C$ . For  $x \in C$  consider the following two cases.

Case (1): Let  $x_n < x$  ( $n = 1, 2, \dots$ ). Then  $\varphi(x) \leq \varphi(x_n)$ , that is,  $x \in W_n$ , ( $n = 1, 2, \dots$ ). Notice that  $\bigcap_{n=1}^\infty \overline{W}_n = \{\xi\}$ , hence  $x = \xi$ .

Case (2): Let  $x < x_{n_0}$  for some  $n_0 \in N$ . Then  $x < x_n$  whenever  $n \geq n_0$ . Thus when  $n \geq n_0$

$$M_\alpha(x, x_n, t) \leq M_\alpha(\varphi(x), \varphi(x_n), tk(\alpha)) \quad \text{for all } \alpha \in D. \tag{3.3}$$

Because  $\{M_\lambda : \lambda \in D\}$  satisfies (Q.3) and  $x_n \rightarrow \xi$ . Hence for any  $\lambda \in D$  there exists  $\mu \in D$  with  $\lambda < \mu$  such that

$$M_\lambda(x, \xi, t) \leq \varliminf_{n \rightarrow \infty} M_\mu(x, x_n, t). \tag{3.4}$$

Since  $\varphi$  is lower semicontinuous, we have

$$\varphi(\xi) \leq \varliminf_{n \rightarrow \infty} \varphi(x_n).$$

Notice that  $k(\lambda)$  is nonincreasing, by (3.3) and (3.4) we obtain

$$M_\lambda(x, \xi, t) \leq M_\lambda(\varphi(x), \varphi(\xi), tk(\lambda)) \quad \text{for all } \lambda \in D.$$

Hence  $x < \xi$ . This implies that  $\xi$  is an upper bound of  $C$ . Thus by Zorn's lemma  $(X, <)$  has a maximal element.  $\square$

**Theorem 3.1.** Let  $(X, T)$  be a sequentially complete  $F$ -type fuzzy topological space and  $\{M_\lambda : \lambda \in D\}$  be a generating family of fuzzy quasi-metrics for  $T$ . Let  $\varphi : X \rightarrow [0, 1]$  be a lower semicontinuous function and  $k : D \rightarrow (0, \infty)$  be nonincreasing function. Suppose further that the mapping  $f : X \rightarrow X$  satisfies the conditions

$$M_\lambda(x, f(x), t) \leq M_\lambda(\varphi(x), \varphi(f(x)), tk(\lambda)) \quad \text{for all } \lambda \in D, t > 0. \quad (3.5)$$

Then  $f$  has a fixed point in  $X$ .

*Proof.* From Lemma 3.2,  $(X, <)$  is a partially ordered set, where  $<$  is defined by (3.2), and  $(X, <)$  has a maximal element say  $x_*$ . By (3.5), we have

$$M_\lambda(x_*, f(x_*), t) \leq M_\lambda(\varphi(x_*), \varphi(f(x_*)), tk(\lambda)) \quad \text{for all } \lambda \in D.$$

This implies that  $x_* < f(x_*)$ . Note that  $x_*$  is a maximal element in  $(X, <)$ .

Hence  $f(x_*) = x_*$ .  $\square$

The following theorem is an equivalent form of Theorem 3.1.

**Theorem 3.2.** Let  $(X, T)$  be a sequentially complete  $F$ -type fuzzy topological space and  $\mathcal{U}_x = \{U_x(\lambda, r, t) : \lambda \in D, t > 0, 0 < r < 1\}$  be a neighbourhood base of  $x$  in  $X$  with the properties (F.1)–(F.4). Let  $\varphi : X \rightarrow [0, 1]$  be a lower semicontinuous function and  $k : D \rightarrow (0, \infty)$  be non-increasing function. Suppose further that the mapping  $f : X \rightarrow X$  satisfies the following condition: For each  $x \in X$ ,  $\varphi(x) \geq \varphi(f(x))$  and

$$f(x) \in (U_x(\lambda, r, M_\lambda(\varphi(x), \varphi(f(x))), tk(\lambda)) + \varepsilon) \quad \text{for all } \lambda \in D, \varepsilon > 0. \quad (3.6)$$

Then  $f$  has a fixed point in  $x$ .

**Caristi's fixed point Theorem:** Let  $(X, T)$  be a complete fuzzy metric space and  $\varphi : X \rightarrow [0, 1]$  be a lower semicontinuous function and if the mapping  $f : X \rightarrow X$  satisfies the condition

$$M(x, f(x), t) \leq M(\varphi(x), \varphi(f(x)), t),$$

then  $f$  has a fixed point in  $X$ .

**Proof.** Arbitrarily take a directed set  $D$  and let

$$M_\lambda(x, y, t) = M(x, y, t) \quad \text{for all } x, y \in X \text{ and } \lambda \in D, t > 0.$$

Then by Example 2.1,  $(X, T_M)$  is an  $F$ -type fuzzy topological space and let  $\{M_\lambda : \lambda \in D\}$  be a generating family of fuzzy quasi-metrics for  $T_M$ .

Then by Theorem 3.1, we have

$$M_\lambda(x, f(x), t) = M_\lambda(\varphi(x), \varphi(f(x)), tk(\lambda)),$$

taking  $k(\lambda) = 1$  for all  $\lambda \in D$ , we have

$$M(x, f(x), t) = M(\varphi(x), \varphi(f(x)), t).$$

Hence  $f$  has a fixed point. □

#### 4. APPLICATIONS

In this section, we shall apply the results of Section 3 to obtain a fixed point theorem and a variational principle in Menger probabilistic metric spaces (briefly, a Menger PM-space [8]).

**Example 4.1.** Let  $(X, \mathfrak{F}, \Delta)$  be a complete Menger space with a  $t$ -norm  $\Delta$  satisfying  $\sup_{0 < t < 1} \Delta(t, t) = 1$ . Let  $\varphi : X \rightarrow [0, 1]$  be a lower semicontinuous function and  $k : (0, 1) \rightarrow (0, \infty)$  be non-increasing function. Suppose that the mapping  $f : X \rightarrow X$  satisfies the following condition: For each  $x \in X$ ,  $\varphi(x) \geq \varphi(f(x))$  and

$$F_{x, f(x)}(t) \geq H(t - (\varphi(x) - \varphi(f(x)))) \quad \text{for all } x \in X, \text{ and } t > 0. \quad (4.1)$$

Then  $f$  has a fixed point in  $X$ .

**Proof.** Let  $(X, \mathfrak{F}, \Delta)$  be a Menger PM-space with a  $t$ -norm satisfying  $\sup_{0 < t < 1} \Delta(t, t) = 1$ . Then  $X$  is an  $F$ -type fuzzy topological space in  $(\varepsilon, \lambda)$ -topology  $T$  of  $(X, \mathfrak{F}, \Delta)$ .

In fact,  $((0, 1), <)$  is a directed set, where the partial order  $<$  is defined by

$$\alpha < \beta \Leftrightarrow \alpha \geq \beta.$$

By a neighbourhood base of  $x$  for topology  $T$ ,  $U_x(\lambda, \varepsilon) = \{y \in X \mid F_{x,y}(\varepsilon) > 1 - \varepsilon\}$  and the definition of distribution function, it is easy to see that  $\mathcal{U}_x = \{U_x(\lambda, r, \varepsilon) / \lambda \in (0, 1), \varepsilon > 0, 0 < r < 1\}$  ( $x \in X$ ) satisfies (F.1), (F.2) and (F.4).

Now, we prove that (F.3) is also true. Since  $\sup_{0 < t < 1} \Delta(t, t) = 1$ , for any  $\lambda \in (0, 1)$  there exists  $\mu \in (0, \lambda]$  such that  $\Delta(1 - \mu, 1 - \mu) > 1 - \lambda$ . Assume that  $U_x(\mu, r_1, t_1) \cap U_y(\mu, r_2, t_2) \neq \emptyset$ , then we take  $z \in U_x(\mu, r_1, t_1) \cap U_y(\mu, r_2, t_2)$ , and so  $F_{x,z}(t_1) > 1 - \mu$  and  $F_{y,z}(t_2) = F_{z,y}(t_2) > 1 - \mu$ .

Thus from Menger PM-space, it follows that

$$\begin{aligned} F_{x,y}(t_1 + t_2) &\geq \Delta(F_{x,z}(t_1), F_{z,y}(t_2)) \\ &\geq \Delta(1 - \mu, 1 - \mu) > 1 - \lambda \end{aligned}$$

which implies  $y \in U_x(\lambda, t_1 + t_2)$ . Hence  $X$  is an  $F$ -type fuzzy topological space in the  $(\varepsilon, \lambda)$ -topology of  $(X, \mathfrak{F}, \Delta)$ .  $X$  is a sequentially complete  $F$ -type fuzzy topological space for the  $(\varepsilon, \lambda)$ -topology of  $(X, \mathfrak{F}, \Delta)$ . Moreover, it is easy to show that (4.1) implies (3.6). In fact, for any  $\varepsilon > 0$  we put  $t = \varphi(x) - \varphi(f(x)) + \varepsilon$ . Thus from Theorem 3.2  $f$  has a fixed point in  $X$ .

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